Pl. Kannappan

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Functional Equations and Inequalities with Applications

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Functional Equations and Inequalities with Applications

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Dedication

To my wife Renganayaki
and grandchildren
Kanna, Anand, Nila,
Sarasvathi, Senthil and Sathiya

Preface

It is natural to ask what a functional equation is. But there is no easy satisfactory answer to this question. While such concepts as element, relation, mapping, operation, etc., are well defined in set theory, the principal concept *set* is an undefined term. It is not considered as an impediment for the growth, development, and usefulness of set theory. We are going to follow this idea and consider functional equations as an undefined concept. This is not going to be a severe roadblock to understanding, appreciating, and contributing to the growth and development of this fascinating area. As in set theory, we hope the reader will get a general insight of what this theory is about.

Functional equations occur practically everywhere. Their influence and applications are felt in every field, and all fields benefit from their contact, use, and technique. The growth and development used to be influenced by their spectacular application in several areas—not only in mathematics but also in other disciplines. Applications can be found in a wide variety of fields—analysis, applied science, behavioural and social science, biology, combinatorics, computers, economics, engineering, geometry, inequalities, information theory, inner product space, physics, polynomials, psychology, reproducing scoring system, statistics, taxation, etc. Functional equations are being used with vigor in ever-increasing numbers to investigate problems in the above-mentioned areas and other fields.

Even though many eminent mathematicians—including Abel (1823), Banach (1920), Cauchy (1821), Darboux (1895), Euler (1768), Ostrowski (1929), Pexider (1903), and Poisson (1804)—since the time of d'Alembert (1769) have contributed to this field, no systematic presentation appeared until 1966 [12], 1989 [44], 1992 [144], and 1985 [558]. To remedy this situation, some books, monographs, and survey articles that treat and discuss many aspects and methods of functional equations are available. Notable are:

- J. Aczél, *Lectures on Functional Equations and Their Applications* [12]
- J. Aczél and Z. Daróczy, *On Measures of Information and Their Characterizations* [43]
- J. Aczél and J. Dhombres, *Functional Equations in Several Variables* [44]

- E. Castillo and M.R. Ruiz-Cobo, *Functional Equations and Modelling in Science and Engineering* [144]
- S. Czerwik, *Functional Equations and Inequalities in Several Variables* [187]
- J. Dhombres, *Some Aspects of Functional Equations* [218]
- B.R. Ebanks, P.K. Sahoo, and W. Sander, *Characterization of Information Measures* [245]
- W. Eichhorn, *Functional Equations in Economics* [250]
- M. Ghermanescu, *Ecuatti Functionale* [315]
- D.H. Hyers, G. Isac, and Th.M. Rassias, *Stability of Functional Equations in Several Variables* [385]
- S.M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*[404]
- Pl. Kannappan, Theory of functional equations [426]
- M. Kuczma, *Functional Equations in a Single Variable* [554]
- M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities: Cauchy's Equation and Jensen's Inequality* [558]
- M. Kuczma, A survey of the theory of functional equations [552]
- M. Kuczma, B. Choczewski, and R. Ger, *Iterative Functional Equations* [560]
- Th.M. Rassias, *Functional Equations and Inequalities* [681]
- Th.M. Rassias, Inner product spaces and applications [683]
- P.K. Sahoo and T. Riedel, *Mean Value Theorems and Functional Equations* [722]
- J. Smital, *On Functions and Functional Equations* [757]
- L. Székelyhidi, *Convolution Type Functional Equations on Topological Abelian Groups* [796]

To highlight the field's usefulness, applications, and impact on multiple fields and growth, we cite extracts from two books [44, 144]. From [44]:

From their very beginnings, functional equations arose from applications, were developed mostly for the sake of applications and, indeed, were applied quite intensively as soon as they were developed. Such a course of development is not typical in all theoretical aspects of mathematics.

This brings us to the natural question of motivation: *To what kind of problems can functional equations be applied* (and not only within mathematics). Mathematicians often construct new notions, partly based on older ones, which may come from mathematics or from the natural, behavioural and social sciences, for instance from physics or economics. Thus the notion of affine vectors evolved in order to represent forces, the (homogeneous) linear function to describe proportionality, the logarithmic function to transform geometric sequences into algebraic ones, and the trigonometric functions to determine unknown parts of triangles from known ones. The next task of the mathematician is to describe the properties of these new objects, that is, to establish their relation to other objects in order to include them in a systematic and orderly way into an existing or new theory. ... The vectors, for instance, form the foundation of linear algebra, the trigonometric functions that of trigonometry. ...

If we are lucky, the properties deduced from the definitions of mathematical objects can be quite numerous, even too rich. According to the principle of economy

of reasoning and to a legitimate desire for elegance, it is desirable to check whether the newly introduced objects are the *only ones* which have some of *the most important properties* that we have just established. It is here where functional equations may enter. One takes these properties as points of departure and tries to determine all objects satisfying them and, in particular, to find conditions under which there is unicity. ... This is the fundamental procedure which motivates functional equations. We see that is axiomatic in nature. But we see also that the axioms are not arbitrarily chosen, we do not generalize just for the sake of generalization. In fact, one prefers to take as points of departure those which are considered the most useful for applications.

After such a uniqueness result is established, for instance that concerning the equation of the cosine (d'Alembert's equation), one also tries to deduce (using the equation, not the solutions) the classical properties of the cosine directly. One thus arrives at an elegant way of developing trigonometry which, with some changes, can also be used to introduce and develop elliptic and, especially, hyperbolic trigonometry.

As in all parts of mathematics, different authors' viewpoints and levels of discussion of functional equations are different. Also, sometimes the methods of reduction of one functional equation to another or the proof of their equivalence or discussion and/or reduction of regularity conditions, extension of domains, etc. give rise to further intrinsic developments. This is a further sign of the increasing maturity of this field of mathematics. On the other hand, the characterization of functions by their equations, as described above, involves many branches of pure and applied mathematics in the development of the theory of functional equations.

From [144]:

... E. Castillo and A. Fernández-Cantelli ... were trying to model the influence of length and stress range on the fatigue life of longitudinal elements and, when analyzing the inconsistencies of some tentative models, ... found a compatibility equation which was written in terms of a functional equation and [encountered for the first time the field of functional equations]. ... Since then, we have completely changed our minds and incorporated the functional equations' philosophy and techniques in [our] daily procedures. Even though many years were required to find our first functional equation, many functional equations have appeared since then in our work, and, in fact, we would not think of building models or stating problems today without using functional equations.

Our experience is that model building in science or engineering is usually performed based on an arbitrary selection of simple and easily tractable equations that seem to reproduce reality to a given quality level. However, on many occasions these models exhibit technical failures or inconsistencies, such as those we discovered in our fatigue models when we got the compatibility equation, which make them unacceptable. We have found that functional equations are one of the main tools that avoid arbitrariness and allow a rigorous and consistent selection of models. In fact, conditions required by many models to be adequate replicas of reality can be written as functional equations.

Though the theory of functional equations is very old, not only technicians but *many mathematicians are still unaware of the power of this important field of mathematics*. However, most of the recent advances in the theory of functional equations have been published in mathematical journals which are not written in a language that many engineers and scientists can easily understand. This fact, which is common to many other areas of Mathematics, has been the reason why many engineers and applied scientists are still unaware of the long list of these advances and, consequently, they have not incorporated common functional equation techniques into their daily procedures.

Our experience with functional equations was so positive and relevant for applications that we became engrossed in this, not well known, field of Mathematics. Impressed by its importance and desiring to share with others this discovery, we decided to write the present book.

One of the aims of this book is to provide engineers and applied scientists with some selected results of functional equations which can be useful in applications. We are aware that this is not an easy task, and that any effort to bring together mathematicians and engineers, as experience shows, has many associated difficulties. ... However, we wish to go even further, trying to change the mentality of the readers and offer them a new way of thinking in mathematical modelling. Traditionally, engineers and scientists are used to stating practical problems in terms of derivatives or integrals, which lead to differential or integral equations, respectively. With this book we want to offer them the possibility of using functional equations too, as one more alternative which is at least as powerful as either of the other two.

Applications of functional equations were found much earlier than any systematic presentation could develop and brought about investigations to treat many more new functional equations. Hence the development of the theory at first (and to some extent now) was closely related to its applications to various branches of mathematics and other areas. Now studies for their own sake and not influenced by applications are also being pursued vigorously. The large number of papers appearing on this subject in various journals indicate the growth of this field and the ever-increasing interest of mathematicians and others. Functional equations used to be solved by assuming rich properties such as differentiability and reducing to differential equations. Certain functional equations can be reduced to integral equations. Sometimes it is possible to find the solution of a functional equation on a dense set, and sometimes it is possible to reduce the functional equation to another functional equation that is already known. Hilbert, in 1900, in connection with his fifth problem (see Chapter 11), highlighted that, while the theory of differential equations provides elegant and powerful techniques for solving functional equations, the differentiability assumptions are not inherently required, and so, to solve functional equations under general conditions, he stressed the general solutions. The trend now adopted by researchers in functional equations is to solve (treat) functional equations without any (or mild) regularity assumptions. This effort has given rise to the modern theory of functional equations. The theory of functional equations forms a modern mathematical discipline that has developed very rapidly in the last four decades. Functional equations are studied and solved for their own sake without

assuming any regularity conditions. The reader will encounter this throughout the book.

At present, there are exponentially ever-increasing numbers of mathematicians (functional equationists) and attractors (from other fields) from more than thirty countries. Researchers from Australia, Austria, Canada, China, the Czech Republic, Denmark, France, Germany, Greece, Hungary, India, Italy, Japan, Korea, Morocco, Poland, Romania, Russia, Spain, Switzerland, the United States, and others are engaged vigorously and passionately in studying functional equations and their applications and determining their solutions. Several schools at Waterloo, Katowice, Krakow, Graz, and Debrecen actively pursue and contribute to the study and growth of functional equations. Primary journals catering to, contributing to, influencing, and encouraging this area are: *Aequationes Mathematicae* (from 1968), *Publicationes Mathematicae Debrecen* (from 1953), *Glasnik Mathematicki*, *Annales Mathematicae Silesianae*, *Demonstratio Mathematica*, *Reznik Nauk Dydakt*, *Prace*, *Cracowiensis*, and *Results in Mathematics*. Annual international symposia have been held in Canada, various European countries, and the United States. with the number of participants increasing to new heights. Periodic special conferences organized in Poland and Hungary devoted to functional equations and inequalities contribute greatly to the development of this area and attract new researchers. *Aequationes Mathematicae* publishes the reports of the meetings and exhaustive bibliographies on a regular basis. Professor Janos Aczél has been instrumental in these and the annual symposia, thereby promoting and spreading information and attracting and recruiting new researchers. These meetings help individuals to come into contact with and get to know many researchers in this field.

These developments are sources of inspiration, attraction, and influence to bring more and more people to this interesting and ever-growing branch of mathematics, who in turn will enrich its study and scope. Hopefully interest in this field will continue forever.

This book has many new features in addition to the usual expected ones. Its aim is to introduce and cover as many important equations, areas, and methods of solution as possible. Some of the proofs follow the same lines as in other books, some are modified, some are direct and simple, and some are presented with new elegant proofs. Different methods of proof are presented. I have exploited *associativity* in many places. No general method of solution is possible. No one or dozen methods are enough to tackle functional equations, which could have been a discouragement to the study of solving them. Hopefully this book will remedy this problem and encourage many new workers to come study and develop this field.

The literature in this field is voluminous beyond imagination. It is impossible to include all the references (even some relevant ones). The list is getting longer by the day. I had the difficult and unpalatable task of choosing better representative (encyclopedic) references. No offense is intended to anyone if their works are not cited. The reader is encouraged to refer to as many of the works as possible—both the books mentioned previously and the references in the bibliography. Many results have been discovered and rediscovered several times. This book, like the others before it, will remedy this to some extent. For historical details of many of the

equations treated in this book, refer to J. Aczél and J. Dhombres [44]. Writing this book gave me an opportunity to listen, to learn, to better understand, and to educate myself in this fascinating area. My motivation is to give the reader a bird's eye view of what this field is about, its development, and its achievements by focusing on examples chosen from diverse areas and applications.

Basic facts of general and linear algebra and calculus (analysis) are presumed. We do not distinguish functional equations of one and several variables. We emphasize that we deal with functional equations with many unknown functions and determine their general solutions. The reader will encounter this in many places. Determining several unknown functions in one equation is a salient feature unique to functional equations. This book does not treat existence and uniqueness results, iteration theory, operator equations, difference equations, etc. The book has been written by drawing on a wealth of information from many publications—from other books, journals, and papers from different sources.

The Organization of this Book

This book contains seventeen chapters. The first chapter, as usual, contains the basic equations of Cauchy and Pexider and many of their generalizations, extensions, and direct applications. This chapter serves as a foundation for other chapters. Chapters 3 and 10 are devoted to trigonometric functional equations and functional equations from information theory and treat these subjects in greater detail than the other chapters. Chapter 4 is devoted entirely to the quadratic functional equation and its generalizations. It is followed in Chapter 5 by its connected interest, inner product spaces, highlighting functional equations' connection to i.p.s. Chapter 7 is devoted to the characterization of polynomials, Chapter 8 to nondifferentiable functions, and Chapter 9 to the characterization of groups, loops, and closure conditions, bringing out Thomsen and Reidemeister closure conditions, Bol and Moufang identities, and their functional equation generalizations. Chapter 11, Abel Equations and Generalizations, is devoted to Hilbert's fifth problem and its connection to information theory. A surprising link between the functional equation of Abel and the well-known fundamental equation of information (FEI) is another special feature. Chapter 2, Matrix Equations, is brief. Chapter 6, Stability, is also brief since many monographs and books [385, 404] are already available. Chapter 12, Regularity Conditions—Christensen Measurability, covers this material for the first time in book form. Chapter 13, Difference Equations, is devoted to the familiar Cauchy and Pexider differences. Chapter 14 covers the characterization of special functions such as gamma and beta functions. Chapter 15, Miscellaneous Equations, is devoted to many methods and approaches to determine solutions of equations and will be of interest. Chapter 16, General Inequalities, is also an interesting chapter, including both applications and general discussion. Chapter 17, devoted to multiple applications in various fields that are not already covered in the other chapters, will also be of great interest.

For easy reference and convenience, there are sections at the end of the book on notation and symbols, an author index, a subject index, and a large bibliography that will prove helpful.

Regarding the general use of specific letters, A is additive, B is biadditive, D is derivation, E is exponential, and L is logarithmic.

Flashback: When I went to the University of Washington, Seattle, in the early 1960's, my intention was to work in measure theory, and Professor Edwin Hewitt was on sabbatical in Australia at that time. Surprisingly after some time, I received a note from him directing me to look at Flett's [280] paper on cosine equations and see whether I could generalize the result. What I did not realize at that time was that I was hooked on functional equations, which I do not regret. My first encounter with functional equations and functional equationists came in 1966 at the Oberwolfach meeting. There were fifteen to twenty participants including Aczél, Barner, Benz, Eichhorn, Haupt, and Ostrowski, and the first speaker was Kuczma (in German). Ever since, the functional equation symposium has met regularly on an annual basis with few exceptions. I have had the opportunity to get to know personally an ever-increasing number of participants. Interest has grown many-fold, and the number of participants at the 2005 meeting was between 100 and 150. I do not see any letup in the activities of the researchers, and I have no doubt that such activity will continue in the years to come since it is a passion for many of the participants.

Professor Michiel Hazewinkel (CWI–Amsterdam), who is the managing editor of the principal mathematics series Mathematics and Its Applications of Kluwer Academic Publishers now Springer, invited me to write a book on functional equations which might be appropriate for Springer and expected this would play a part in their expansion. I accepted promptly with pleasure and full of hope. My first instinct was collaboration, and I enlisted two colleagues. I realized after some time it was not moving well. Without losing heart, I tried and got two new colleagues for the joint venture. Alas, it did not materialize either and also ended in disappointment. It took some time for me to realize that the joint work idea was not working and not feasible, and I resolved not to give up and to go it alone and honour my commitment to Professor Hazewinkel. The result was that writing this book took an unexpectedly long time. In the meantime, there was a change in the publisher to Springer.

At the time of my retirement, the chairman of my department, who was aware I was writing a book, said that continued secretarial and other assistance would be provided. But the person who succeeded him in a few months time had different ideas! He said that no secretarial assistance would be provided to typeset the book and one day unceremoniously left the manuscripts with the secretary to put in my mailbox. What can I say about this immature, thoughtless behaviour?

Usually one acknowledges support from granting agencies. Here it is different, and I present a couple of NSERC of Canada referee reports to highlight the biased considerations: "My area of expertise is not close to that of the proposal"; "I do not believe that functional equations is a desirable area for training, on the positive side it is clear that a knowledge of functional equations can be applied to a variety of areas. Caveat: I am not expert in this side of functional equations. My views of necessity are those of an interested outsider"; "In other areas of the proposal where the main objective appears to be writing a book or monograph or survey paper, it is difficult to measure the scope. While there is some merit to writing a book or survey paper, it would have been more desirable to have a greater focus put on finding solutions . . . ";

"The Committee did not note any improvement in the applicant's research record since the last decision" (sweeping statement); "Some items of the research program are quite original"; "according to my experience this type of method requires a lot of ingeniousness"; "the applicant is an expert in the proposed area"; "these results are significant contributions"; "certainly one of the leading figures in this area"; "of the five NSERC applicants I assessed this year, I would rank him in the top two"; "as also noted, Professor Kannappan's work has been well cited"; "Professor Kannappan maintains a strong rhythm of publications on various areas of functional equations"; "such a book from Professor Kannappan should be strongly encouraged"; "the referees' comments on the applicant were almost uniformly positive"; "originality and innovation"; "good". These comments had no effect or were ignored by NSERC of Canada. Is there a redemption for NSERC?

In spite of all these impediments, frustrations, and disappointments, I was not discouraged, nor did I give up or lose sight of my goal. I worked consistently to accomplish the final goal—writing this book. I take responsibility for both the credits and defects of this book.

It is my pleasure to express my deepest appreciation to the untiring Lis D'Alessio for her patience, understanding, cheerful attitude, and excellent typing. I am truly grateful for her sustained help over the years to make it possible to accomplish this task. No words can adequately describe her work throughout the entire period of writing this book.

I thank Springer for bringing out this book elegantly and for their patience.

Spread the message of functional equations. Your comments on all aspects of the book are welcome.

Pl. Kannappan
Waterloo, 2008

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1

Basic Equations: Cauchy and Pexider Equations

In this chapter, Cauchy's fundamental equations are studied, some algebraic conditions and generalizations are considered, and alternate equations, conditions on restricted domains, Jensen's equation, some special cases of Cauchy's equations, extensions of the additive equations, Pexider equations and their extensions, some applications in economics, the allocation problem, and sums of powers of first n natural numbers are treated.

There is a saying in Tamil, "paruppilla kalyanama"—marriage without lentils. (In the North American context, can one think of Thanksgiving without turkey?) Similarly, can one say "a book on functional equations without Cauchy's equations?" As Cauchy's equations permeate functional equations and are fundamental in nature, we first treat the *Cauchy equations*

(A)	$A(x+y) = A(x) + A(y)$	(additive),
(E)	$E(x+y) = E(x)E(y)$	(exponential),
(L)	$L(xy) = L(x) + L(y)$	(logarithmic),
(M)	$M(xy) = M(x)M(y)$	(multiplicative),

treated systematically in Cauchy [147], Aczél [12], and Kannappan [458]. Even though these equations occur earlier in the studies by Oresme [655, 656, 44], Euler [262, 263], Gauss [303], Stifel [777], Briggs [134], Bürgi [139], Kepler [531], Lacroix [592], Legendre [605], de Sarasa [727], etc., in the literature, these equations are known as *Cauchy's equations*. The equation (A) finds applications in almost every branch of mathematics. It appears in the problem of the parallelogram law of forces, in the problem of the measurement of areas, in the problem of simple interest, in projective geometry, in mechanics, in the theory of probability, in non-Euclidean geometry, etc. Cauchy's equations are used in the mathematics of finance, in advertising, in production theory, in economics, in taxation, in modelling, in probability theory, and in many other areas. The equation (A) is one of the equations that has been extensively studied or explored and was solved by, among numerous authors, Aczél, Alexiewicz and Orlicz, Banach, Darboux, Erdős, Fréchet, Gauss, Halperin, Hamel, Hewitt and Zuckerman, Kuczma, Kurepa, Kuwagaki, Legendre, Ostrowski,

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and Sierpinski, under various hypotheses of the function, domain, and range. We will deal with them in detail later. The existence of discontinuous solutions of (A), due to Hamel [339], is based on the axiom of choice. Where the domain and range of (A) are abstract sets (groupoids, groups, fields, etc.), these equations play an important role as the equations of homomorphism, endomorphism, isomorphism, automorphism, etc. The Cauchy equations play a prominent role in the theory of functional equations and have fascinated investigators. Researchers fell in love with these equations, and the romance will continue and will result in many more interesting results. These equations are the subject of study under various contexts.

1.1 Additive Equations

First we consider the (Cauchy) *additive equation* (A)

$$A(x+y) = A(x) + A(y). \tag{A}$$

We begin with the domain and range of A being reals. Let $A : \mathbb{R} \to \mathbb{R}$ satisfy the equation (A). Then A is said to be *additive*.

Letting $x = 0,\ y = 0$ in (A), we have

$$A(0) = 0. \tag{1.1}$$

Now, changing y into $-x$ in (A) and using (1.1), we get

$$A(-x) = -A(x) \quad \text{for all } x \in \mathbb{R}; \tag{1.2}$$

that is, A is odd. From (A) and by finite induction, it follows that

$$A(x_1 + x_2 + \cdots + x_n) = A(x_1) + A(x_2) + \cdots + A(x_n).$$

Setting $x_j = x$ $(j = 1, 2, \ldots, n)$ in the above and using (1.2), one gets

$$A(nx) = nA(x) \quad \text{for } x \in \mathbb{R},\ n \in Z. \tag{1.3}$$

Let $n(\neq 0)$ be in Z. Replace x by $\frac{x}{n}$ in (1.3) to obtain

$$A\left(\frac{x}{n}\right) = \frac{1}{n}A(x) \quad \text{for } x \in \mathbb{R},\ n \in Z^*. \tag{1.4}$$

Let $r = \frac{m}{n}$ be any rational. Then (1.3) and (1.4) yield

$$A(rx) = A\left(m \cdot \frac{x}{n}\right) = mA\left(\frac{x}{n}\right) = \frac{m}{n}A(x) = rA(x) \tag{1.5}$$

for $x \in \mathbb{R},\ r \in Q$. That is, A is *rational homogeneous.*

With $x = 1$, (1.5) becomes

$$A(r) = cr, \quad \text{for } r \in Q, \tag{1.6}$$

for some constant $c = A(1)$. So far only the condition that A is additive is used.

Now, if we assume that A is continuous or continuous at a point, we can conclude that

$$A(x) = cx \quad \text{for all } x \in \mathbb{R}. \tag{1.7}$$

Indeed, from the continuity and (1.6), (1.7) follows easily since Q is dense in $\mathbb{R}$. On the other hand, suppose A is continuous at a point $x_0 \in \mathbb{R}$ and y_0 is an arbitrary real number. Let $y_n \to y_0$, where $y_n \in \mathbb{R}$. Then $y_n - y_0 + x_0 \to x_0$ and the use of (A) and the continuity at x_0 yield $A(y_n - y_0 + x_0) \to A(x_0)$; that is, $A(y_n) \to A(y_0)$, showing that A is continuous at y_0. Hence A is continuous on $\mathbb{R}$ and (1.7) holds. Thus, we have proved the following theorem.

Theorem 1.1. *Let A be a real-valued function of a real variable satisfying the additive equation* (A)*. Then, if A is either continuous or continuous at a point, it has the form* (1.7)*.*

Remark 1.1a. If $A : \mathbb{R} \to \mathbb{R}$ is a nonzero solution of (A), then A is unbounded.

There are as many conditions known in the literature as one can imagine for the solution of (A) to be (1.7) and thus continuous. The condition of continuity can be weakened considerably in (A) to obtain the same conclusion. Several equivalent conditions for continuity and proofs are known in the literature (Aczél [12], Aczél and Dhombres [44], Kannappan [426], and Kuczma [552, 558]). In the next few theorems, we achieve the goal of continuity.

Theorem 1.2. *Suppose $A : \mathbb{R} \to \mathbb{R}$ satisfies* (A) *with $c = A(1) > 0$. Then the following conditions are equivalent:*

(i) *A is continuous at a point x_0.*
(ii) *A is monotonically increasing.*
(iii) *A is nonnegative for nonnegative x.*
(iv) *A is bounded above on a finite interval.*
(v) *A is bounded below on a finite interval.*
(vi) *A is bounded above* (*below*) *on a bounded set of positive Lebesgue measure.*
(vii) *A is bounded on a bounded set of positive measure* (*Lebesgue*).
(viii) *A is bounded on a finite interval.*
(ix) *$A(x) = cx$.*
(x) *A is locally Lebesgue integrable.*
(xi) *A is differentiable.*
(xii) *A is Lebesgue measurable.*

Proof. We will prove the implications in the same order as stated (also see Aczél [12] and Kannappan [426]).

To prove that (i) $\Longrightarrow$ (ii), let $x > y$. Let $\{r_n\}$ be a sequence of rationals such that $r_n \to x - y$ as $n \to \infty$. Then $r_n + y - x + x_0 \to x_0$ as $n \to \infty$. By (i) and the additivity of A, one gets $A(r_n + y - x + x_0) \to A(x_0)$; that is, $cr_n + A(y) - A(x) \to 0$ as $n \to \infty$. But $cr_n + A(y) - A(x) \to c(x - y) + A(y) - A(x)$; that is, $A(x) - A(y) + c(y - x) = 0$, which shows that $A(x) - A(y) = c(x - y) > 0$ (recall $c = A(1) > 0$). Hence A is monotonically increasing, which proves (ii).

To show that (ii) $\Longrightarrow$ (iii), suppose $x \geq 0$. Then, by (ii), $A(x) \geq A(0) = 0$, which is (iii).

Remark 1.3. Note that, since A is odd (it satisfies (1.2)), A is nonpositive for nonpositive x.

To prove that (iii) $\Longrightarrow$ (iv), suppose $x \in [a, b]$, a finite interval. Then $b - x > 0$. By (iii) and the additivity of A, $A(b) \geq A(x)$; that is, A is bounded above on $[a, b]$ by $A(b)$.
To show that (iv) $\Longrightarrow$(v), let $[c, d]$ be any finite interval and $x \in [c, d]$. Now A is bounded above on $[c - x, d - x]$ by (iv); that is, $A(c - x) \leq m$ for some m or $A(x) \geq A(c) - m$, which shows that A is bounded below on $[c, d]$.
To prove that (v) $\Longrightarrow$ (vi), let us prove that A is bounded above on a bounded set F of positive Lebesgue measure. There exists a finite interval $[a, b]$ such that $F \subseteq [a, b]$. Then, for any $x \in [a, b]$, consider $[a - x, b - x]$. By (v), A is bounded below so that $A(a - x) \geq m$ or $A(x) \leq A(a) - m$ for some m; that is, A is bounded above on $[a, b]$. Hence A is bounded above on F. Note that A is bounded on F.
To show that (vi)$\Longrightarrow$ (vii), suppose A is bounded below on a bounded set F of positive measure. It is well known that $F + F$ has a nonempty interior (see [566, 578, 567, 766, 533, 696, 742, 44]). Then there is an interval $[c, d] \subseteq F + F$, and A is bounded below on $[c, d]$. As in the proof of (v) $\Longrightarrow$ (vi), it is easy to show that A is bounded on F. This proves (vii).
Remark 1.3a. P. Erdős observed that A may be bounded on a set C for which $C - C$ contains an interval and that A is not continuous.
To prove that (vii) $\Longrightarrow$ (viii), let $|A(x)| \leq m$ for $x \in F$, a set of positive measure. By a theorem of Steinhaus [766], there is a positive number δ such that every real number t satisfying $|t| < \delta$ may be expressed as $x - y$ for suitable $x, y \in F$. Then $|A(t)| = |A(x - y)| \leq 2m$ for $t \in]-\delta, \delta[$. Thus A is bounded on $]-\delta, \delta[$ and hence on any finite interval. This proves (viii).

Also note that a finite interval is a bounded set of positive measure.

To show that (viii) $\Longrightarrow$ (ix) (Aczél [12], Darboux [190]), suppose A is bounded on $[a, b]$. Define

$$\phi(x) = A(x) - cx \quad \text{for } x \in \mathbb{R}. \tag{1.8}$$

Then ϕ is also additive on $\mathbb{R}$, $\phi(r) = 0$, for $r \in Q$, and ϕ is bounded on $[a, b]$. Furthermore,

$$\phi(x + r) = \phi(x) \quad \text{for } x \in \mathbb{R},\ r \in Q. \tag{1.9}$$

Since for every real x we can find a rational r such that $x + r \in [a, b]$, we can conclude from (1.9) that ϕ is bounded everywhere on $\mathbb{R}$. Now claim that $\phi(x) = 0$ for all $x \in \mathbb{R}$. If not, there is an x_0 such that $\phi(x_0) = b(\neq 0) > 0$. Then $\phi(nx_0) = nb$ holds for all $n \in Z$, so, for arbitrarily large n, ϕ can take an arbitrarily large value, contradicting the boundedness of ϕ. Thus $A(x) = cx$; that is, (ix) holds.
(ix) $\Longrightarrow$(x) is obvious.

To show that (x) $\Longrightarrow$ (xi), integrate (A) over $[a, b]$ to obtain

$$\int_a^b A(x+y)dy = \int_a^b A(x)dy + \int_a^b A(y)dy;$$

that is,

$$\int_{a+x}^{b+x} A(v)dv = (b-a)A(x) + d.$$

Since A is locally integrable, the equation above shows first that A is continuous and then indeed A is differentiable. This proves (xi).
(xi) implies (xii) is obvious.

Finally, to complete the cycle, we have to prove that (xii) $\Longrightarrow$ (i). To prove that every measurable solution of (A) is continuous (that is, of the form (1.7)), a number of proofs of this result are known (Sierpinski [740], Alexiewicz and Orlicz [69]). Here we present the proof due to Banach [96].

Proof. Let $x_0 \in \mathbb{R},\ \varepsilon > 0$, and $]a, b[$ an arbitrary interval. By a theorem of Lusin, there exists for every measurable function f, (in particular here A) and for every $\sigma > 0$ (in particular, $\sigma = \frac{b-a}{3}$) a continuous function F (for all real x) such that

$$A(x) = F(x)$$

holds for all $x \in]a, b[$, except perhaps for x's forming a set C of measure $< \sigma$. The function F being continuous, for every $\varepsilon > 0$ there exists a $\delta\ (< \sigma)$ such that, for $x \in]a, b[,\ |F(x+h)-F(x)| < \varepsilon$ whenever $|h| < \delta$. Since $A(x) = F(x)$ for $\varepsilon\ ]a, b[$ except over a set C of measure $< \sigma$, we can conclude that $A(x+h) = F(x+h)$ is true for all $x \in]a, b[$ except over a set C_1 of measure $< \sigma + |h| < \sigma + \delta$. The set of $x \in]a, b[$ for which neither $A(x) = F(x)$ nor $A(x+h) = F(x+h)$ hold is therefore of measure $\leq m(C \cup C_1) < 2\sigma + \delta < 3\sigma < b - a$, where m is the Lebesgue measure on $\mathbb{R}$. Hence there is a point $x \in]a, b[$ (dependent on h) for which $A(x) = F(x),\ A(x+h) = F(x+h)$, and $|F(x+h) - F(x)| < \varepsilon$ are valid. Hence we have $|A(x+h) - A(x)| < \varepsilon$. Since $A(x+h) - A(x) = A(x_0+h) - A(x_0)$, for any real $x_0,\ |A(x_0+h) - A(x_0)| < \varepsilon$. Hence A is continuous. This proves (i). □

Corollary 1.4. *If A satisfies* (A) *on $\mathbb{R}$ and is measurable on some subset C of positive Lebesgue measure, then A is continuous.*

Proof. Because the set of $x \in C$ for which $|A(x)| < n$ has positive measure for sufficiently large n, the proof follows by Theorem 1.2 (vii). □

Corollary 1.5. *Let $A : \mathbb{R} \to \mathbb{R}$ satisfy* (A)*. Suppose there is a subset C of positive Lebesgue measure on which A is bounded above by a Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$. Then A is continuous.*

Proof. (Dhombres [218]). Since $f : C \to \mathbb{R}$ is Lebesgue measurable and the measure of C is positive, there is an integer k such that $C_k = \{x \in C : f(x) \leq k\}$ is of positive measure. So, there exists a subset C_k of positive measure on which A is bounded above. Thus, by (vi) of Theorem 1.2, A is continuous. □

1.1.1 Discontinuous Solutions

Whereas the solution of (A) under the hypothesis of continuity is of the form (1.7), Hamel [339] bases are used successfully in constructing a function A (using the axiom of choice), which is a solution of (A) that is not of the form (1.7) and so is not continuous (see also Aczél and Daróczy [43], Aczél [12], Kuczma [566]).

If A is not continuous, how does it look? Obviously, any discontinuous solution of (A) on an interval is unbounded (see (viii) of Theorem 1.2). First we construct the most general solution of (A). Then we show the existence of discontinuous solutions of (A). As a matter of fact, we prove the following theorem.

Theorem 1.6. *Let H be a Hamel basis of $\mathbb{R}$ and f be defined arbitrarily on H. Then there exists an $A : \mathbb{R} \to \mathbb{R}$ that is additive and $A|H = f$.*

Proof. Let H be a Hamel basis of $\mathbb{R}$ over Q rationals. Let f be defined arbitrarily on H. Every real number x can be represented uniquely as a finite linear combination

$$x = \sum_{k=1}^{m} r_k h_k, \quad r_k \in Q,\ h_k \in H.$$

Define $A : \mathbb{R} \to \mathbb{R}$ by

$$A(x) = \sum_{k=1}^{m} r_k f(h_k).$$

Clearly this A is a solution of (A), and every solution of (A) can be written in this form.

Obviously, $A|H = f$. It is easy to check that such an A is continuous if and only if

$$f(h_k) = ch_k, \quad \text{for } h_k \in H,$$

for some constant c.

In order to obtain a discontinuous solution of (A) or solutions that are not of the form (1.7), choose two Hamel basis elements h_1, h_2 in H such that

$$f(h_1)/h_1 \neq f(h_2)/h_2.$$

Such solutions are nowhere continuous (refer to Theorems 1.1 and 1.2).

This proves the theorem. □

Theorem 1.6a. (Aczél [44], Eichhorn [250]). *Let the real function A satisfy* (A) *and not be continuous. Then the graph of $A = \{(x, A(x))|x \in \mathbb{R}\}$ is dense in $\mathbb{R}^2$.*

Proof. Pick $t_1 \neq 0$. Since A is not continuous, there is a $t_2 \neq 0$ such that $\frac{A(t_1)}{t_1} \neq \frac{A(t_2)}{t_2}$ (otherwise $A(t_2) = ct_2$ for all $t_2 \in \mathbb{R}$). This means $(t_1, A(t_1))$ and $(t_2, A(t_2))$ are linearly independent and form a basis for $\mathbb{R}^2$. Let $(x, y) \in \mathbb{R}^2$. Then there exist constants c_1 and c_2 in $\mathbb{R}$ such that $(x, y) = c_1(t_1, A(t_1)) + c_2(t_2, A(t_2))$. Choose rationals r_1 and r_2 as close to c_1 and c_2 (respectively) as desired. Then $(r_1t_1 + r_2t_2, A(r_1t_1 + r_2t_2)) = (r_1t_1 + r_2t_2, r_1A(t_1) + r_2A(t_2))$ is as close to

$(c_1t_1 + c_2t_2, c_1A(_1) + c_2A(t_2))$ as desired; that is, an element on the graph of A is as close to (x, y) as desired. So, the graph of A is dense in $\mathbb{R}^2$. □

Remark 1.6b. These results reveal that solutions of (A) are either very regular or extremely pathological.

Corollary 1.7. *Let A be a noncontinuous solution of* (A)*. Then the image of any open interval by A is dense in* $\mathbb{R}$.

Proof. Let $]a, b[$ be any open interval. Suppose $A(]a, b[)$ the image of $]a, b[$ by A, is not dense in $\mathbb{R}$. Then there is an open interval $]c, d[$ that has void intersection with $A(]a, b[)$. Now the open set $C =]a, b[\times]c, d[$ in $\mathbb{R}^2$ has no point of the graph of (A), contrary to Theorem 1.6a. □

1.2 Algebraic Conditions—Derivation

Theorem 1.2 shows that a solution of (A) under some regularity condition is continuous and is of the form (1.7). The question now arises as to whether one can look for supplementary algebraic conditions of A implying the continuity of A. This may lead to many interesting problems and representations of A (see Problem 1.16). First we deal with questions raised by Halperin [335] and Rothenberg [712].

Problem 1.8. (I. Halperin [335]). Suppose $A : \mathbb{R} \to \mathbb{R}$ is additive and satisfies

$$A\left(\frac{1}{x}\right) = \frac{1}{x^2}A(x), \quad x(\neq 0) \in \mathbb{R}. \tag{1.10}$$

Does it follow that A is continuous?

The answer turns out to be yes. Several proofs are known (Jurkat [406], Kurepa [584, 585, 586],Kannappan [426]). Here we present a different proof.

For $x \neq 0, 1, -1$,

$$\begin{aligned}
A\left(x - \frac{1}{x}\right) &= \left(x - \frac{1}{x}\right)^2 A\left(\frac{x}{x^2 - 1}\right) && \text{(use (1.10))}\\
&= \frac{(x^2-1)^2}{2x^2} A\left(\frac{1}{x-1} + \frac{1}{x+1}\right) && \text{(use (A))}\\
&= \frac{(x+1)^2}{2x^2} A(x-1) + \frac{(x-1)^2}{2x^2} A(x+1) && \text{(use (A) and (1.10))}\\
&= \frac{x^2+1}{x^2} A(x) - \frac{2}{x} A(1) && \text{(use (A))}\\
&= A(x) - A\left(\frac{1}{x}\right) = A(x) - \frac{1}{x^2}A(x).
\end{aligned}$$

Hence $A(x) = A(1)x$, for $x \neq 0, 1, -1$. Evidently, this relation holds for $x = 0$, $1, -1$ also. Thus A is continuous.

Derivation
A mapping $D : \mathbb{R} \to \mathbb{R}$ is called a *derivation* provided D satisfies (A) and

$$\text{(D)} \qquad D(xy) = xD(y) + yD(x), \quad \text{for all } x, y \in \mathbb{R}.$$

Proposition 1.9. *Suppose $D : \mathbb{R} \to \mathbb{R}$ is additive, that is, D satisfies* (A)*. Then the following are equivalent:*

(i) *D satisfies* (D)*.*
(ii) $D\left(\frac{1}{x}\right) = -\frac{1}{x^2}D(x)$ *for* $x \neq 0$.
(iii) $D(x^2) = 2xD(x)$ *for* $x \in \mathbb{R}$.
(iv) $D(x^2)^2 = 4x^2D(x)^2$.

Proof. We prove in the order listed above and take $D \neq 0$.
To prove that (i) $\Longrightarrow$ (ii), first of all, notice that $D(1) = 0$. The $y = \frac{1}{x}$ in (D) gives (ii).
To show that (ii) $\Longrightarrow$ (iii), again note that $D(1) = 0$. For $x \neq \pm 1$, using (A) and (ii), we have

$$D(x^2) = D(x^2-1) = -(x^2-1)^2 D\left(\frac{1}{x^2-1}\right) = -\frac{(x^2-1)^2}{2} D\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$
$$= \frac{(x+1)^2}{2} D(x-1) - \frac{(x-1)^2}{2} D(x+1) = 2xD(x) \quad \text{for } x \neq \pm 1.$$

Obviously, $D(x^2) = 2xD(x)$ holds for $x = \pm 1$ also. So, (iii) is true.
(iii) implying (iv) is obvious.
To complete the cycle, we will show that (iv) $\Longrightarrow$ (i), first note that $D(1) = 0$ and so $D(r) = 0$ for $r \in Q$. Change x to $x + r$, where r is a rational in (iv), and use D additive, $D(r) = 0$, and (iv) to get $D(x^2) = 2xD(x)$ (which is (iii)). For $x, y \in \mathbb{R}$, use (A) and (iii) to get

$$D((x+y)^2) = 2(x+y)D(x+y) = 2(x+y)(D(x) + D(y))$$
$$= D(x^2) + 2D(xy) + D(y^2);$$

that is, (i) is true. This proves the proposition. □

Remark 1.10. If D is a derivation, then $D(r) = 0$ for all $r \in Q$. If a derivation D is continuous, then $D = 0$. There exist nontrivial derivations, and derivations can be defined on abstract structures such as rings, and fields (see Zariski and Samuel [839], Kuczma [558], and Aczél and Dhombres [44]).

Problem 1.11. (Rothenberg [712]). Suppose the real function A is additive with

$$(1.11) \qquad A(x)A\left(\frac{1}{x}\right) > 0 \quad \text{for } x(\neq 0) \in \mathbb{R}.$$

Is A continuous?

The answer turns out to be no this time. First we consider a stronger condition, (1.12), than (1.11) ([445, 30]),

$$A(x)A\left(\frac{1}{x}\right) = b(>0) \quad \text{for } x \neq 0. \tag{1.12}$$

Observe that A is not constant and that $A(1)^2 = b$. Then, from (A) and (1.12), for $x \neq 1, -1$, we obtain

$$A\left(\frac{1}{x^2-1}\right) = \frac{1}{2}A\left(\frac{1}{x-1}\right) - \frac{1}{2}A\left(\frac{1}{x+1}\right);$$

that is,

$$\frac{b}{A(x^2)-A(1)} = \frac{b}{2(A(x)-A(1))} - \frac{b}{2(A(x)+A(1))},$$

which yields $A(x^2)A(1) = A(x)^2$, so that $A(x) > 0$ for $x > 0$ if $A(1) > 0$ (or $A(x) < 0$ for $x > 0$ if $A(1) < 0$). In either case, A is continuous (see (iii) of Theorem 1.2). Then $A(x) = \sqrt{b}x$ or $A(x) = -\sqrt{b}x$, according to whether $A(1) > 0$ or $A(1) < 0$.

Now we present the solution due to Benz [117]. Consider a noncontinuous automorphism $\phi : \mathbb{C} \to \mathbb{C}$, and put $A(x) = \text{Re } \phi(x)$ for all $x \in \mathbb{R}$. Then A is additive on $\mathbb{R}$ and A is discontinuous. Moreover,

$$A\left(\frac{1}{x}\right) = \frac{A(x)}{A(x)^2 + g(x)^2} \quad \text{for } x \in \mathbb{R},\ x \neq 0,$$

where $g(x) = \text{Im } \phi(x)$ (use $\phi(x)\phi(\frac{1}{x}) = 1$). Hence, $1 \geq A(x)A(\frac{1}{x}) \geq 0$ for all x ($\neq 0$) in $\mathbb{R}$. Benz ends this remark with a comment: If there exists a discontinuous automorphism ϕ of $\mathbb{C}$ such that $\phi(x) \notin i\mathbb{R}$ for all $x \in \mathbb{R}^*$, then this automorphism would lead to a counterexample $1 \geq A(x)A\left(\frac{1}{x}\right) > 0$ for all $x \in \mathbb{R}^*$.

Bergman [120] has constructed such an automorphism $\phi : \mathbb{C} \to \mathbb{C}$. □

Proposition 1.12. *Let $A : \mathbb{R} \to \mathbb{R}$ be additive and satisfy*

$$A(xy) = A(x)A(y) \quad \textit{for } x, y \in \mathbb{R}. \tag{1.13}$$

Then A is continuous and either $A = 0$ or $A(x) = x$; that is, if A is additive and multiplicative, then $A = 0$ or is an identity map.

Proof. Put $y = x$ in (1.13) to get

$$A(x^2) = A(x)^2 \quad \text{for } x \in \mathbb{R}; \tag{1.14}$$

that is, $A(x) \geq 0$ for all $x \geq 0$. Since A is also additive, by (iii) of Theorem 1.2, A is continuous and $A(x) = cx$. This A in (1.13) gives $c = c^2$; that is, $c = 0$ or 1. This proves the proposition. □

Remark 1.13. [Aczél [12], Aczél and Dhombres [44], Kannappan [426]]. All automorphisms $A : \mathbb{R} \to \mathbb{R}$ are given by $A(x) = x$.

Proposition 1.14. *Let $A : \mathbb{R} \to \mathbb{R}$ satisfy* (A) *and*

$$A(x^m) = A(x)^m \quad \text{for } x \in \mathbb{R}, \tag{1.15}$$

where m is an integer different from 0 and 1. Then A is continuous and either $A = 0$, $A(x) = x$, or $A(x) = -x$.

Proof. (See also [218].) Note that (1.14) is a special case of (1.15) and that $A = 0$ is a solution of (1.15). So, we look for a not identically zero solution A.

First let us consider the case where m is a positive, even integer. Then $A(x) \geq 0$ for $x > 0$. Thus, by (iii) of Theorem 1.2, $A(x) = cx$ with $c = c^m$. Then $c = 0$ or 1, giving the solution $A(x) = x$.

Assume now that m is a positive, odd integer. Choose any rational r $(\neq 0)$ and replace x by $x + r$ in (1.15) to obtain

$$\begin{aligned}[A(x) + A(1)r]^m &= A(x)^m + \binom{m}{1} A(x)^{m-1} A(1)r + \cdots + A(1)^m r^m \\ &= A((x + r)^m) \\ &= A(x^m) + \binom{m}{1} A(x^{m-1})r + \cdots + r^m A(1).\end{aligned}$$

Treating the right sides as a polynomial in r and equating the coefficients of r, we have $A(1)A(x)^{m-1} = A(x^{m-1})$. Since m is odd, $A(x)^{m-1} \geq 0$. Hence, if $A(1) > 0$, then $A(x) \geq 0$, and if $A(1) < 0$, then $A(x) \leq 0$ for $x \geq 0$. In either case, A is continuous by (iii) of Theorem 1.2 and $A(x) = cx$ with $c = c^m$. Since m is odd, $c = 0, 1$, or -1, yielding the solutions $A(x) = x$ and $A(x) = -x$.

Finally, let us assume that m is a negative integer different from -1. If $m = -1$, then (1.15) gives $A(x)A(\frac{1}{x}) = 1$, a special case of (1.12), and $A(x) = x$ or $A(x) = -x$ (see Problem 1.11).

Now, by (1.15), $A(x^{m^2}) = A(x)^{m^2}$, m^2 being a positive integer different from 1. By the previous case, $A(x) = x$ if m^2 (and thus m) is even or $A(x) = -x$ if m^2 (that is, m) is odd. This proves the proposition. □

Proposition 1.15. (Kannappan and Kurepa [486], Kannappan [458]). *Suppose that $A : \mathbb{R} \to \mathbb{R}$ is additive and*

$$A(x^n) = x^{n-m} A(x^m), \quad \text{for } x \in \mathbb{R}, \tag{1.16}$$

holds for all nonzero integers m and n, $m \neq n$. Then again A is continuous and $A(x) = A(1)x$.

Proof. Without loss of generality, take $n > m$. If $n < m$, we can consider $A(x^m) = x^{m-n} A(x^n)$.

Replace x by $x + r$, where r is a nonzero rational in (1.16), to have

$$A\left(x^n + \binom{n}{1}x^{n-1}r + \cdots + \binom{n}{n-1}xr^{n-1} + r^n\right)$$
$$= A\left(x^m + \binom{m}{1}x^{m-1}r + \cdots + \binom{m}{m-1}xr^{m-1} + r^m\right)$$
$$\left[x^{n-m} + \binom{n-m}{1}x^{n-m-1}r + \cdots + \binom{n-m}{n-m-1}xr^{n-m-1} + r^{n-m}\right].$$

Treating the equation above as a polynomial in r and equating the coefficient of r^{n-1} on both sides, we obtain $A(x) = A(1)x$. This proves the proposition. □

Remark 1.15a. For $n = 1$, $m = -1$, (1.16) becomes the Halperin Problem (Problem 1.8).

1.2.1 More Algebraic Conditions

Now we treat the following more general (equation) problem. Let V be the set of all additive functions $A : \mathbb{R} \to \mathbb{R}$. Then V is a vector space over $\mathbb{R}$. Let U be the subspace of V spanned by $x \to A(x) = A(1)x$, continuous functions, and by all derivations on $\mathbb{R}$.

Problem 1.16. (Kannappan and Kurepa [486], Kannappan [458]). Let u_i's be rational functions in x, the p_i's be continuous functions on $\mathbb{R}$ except at the singular points of u_i, and the A_i's be additive functions. When does a condition of the form

$$\sum_{i=1}^{n} p_i(x)A_i(u_i(x)) = 0, \quad \text{for } x \in \mathbb{R}, \tag{1.17}$$

imply that $A_i \in U$ $(i = 1, 2, \ldots, n)$; that is, that the A_i's are linearly dependent relative to U?

This problem has not been solved completely. Here we present a result from [486] that includes Halperin's Problem (Problem 1.8) and Proposition 1.15 as special cases. We will start with a lemma.

Lemma 1.17. *An additive $D : \mathbb{R} \to \mathbb{R}$ is a derivation if, and only if, D satisfies*

$$D(x^{2n}) = -nx^{n+1}D(x^{n-1}) + (n+1)x^n D(x^n), \quad \textit{for all } x \in \mathbb{R}, \tag{1.18}$$

with $D(r) = 0$, for $r \in Q$, where n is a positive integer.

Proof. When D is a derivation, it is easy to check (1.18). Suppose (1.18) holds. Note that, when $n = 1$, (1.18) is (iii) in Proposition 1.9.

Replace x by $x + r$, $(r \in Q)$ in (1.18) to obtain

$$D((x+r)^{2n}) = -n(r+x)^{n+1}D((r+x)^{n-1}) + (n+1)(r+x)^n D((x+r)^n).$$

Noting that D is additive, $D(r) = 0$, and the equation above is a polynomial in r, equate the coefficient of r^{2n-2} to get

$$\frac{2n(2n-1)}{2}D(x^2) = -n\left[\frac{(n-1)(n-2)}{2}D(x^2) + (n+1)(n-1)xD(x)\right] + (n+1)\left[\frac{n(n-1)}{2}D(x^2) + n^2xD(x)\right],$$

which is $D(x^2) = 2xD(x)$. By (iii) of Proposition 1.9, D is a derivation. □

Theorem 1.18. (Kannappan and Kurepa [486]). *Let A_1 ($\neq 0$) and A_2 be additive functions from $\mathbb{R}$ into $\mathbb{R}$ and satisfy*

$$A_1(x^n) = P(x)A_2(x^m) \tag{1.19}$$

for all $x \neq 0$, with $P : \mathbb{R}^* \to \mathbb{R}$ as a continuous function with $P(1) = 1$ and m and n nonzero integers with $m \neq n$. Then $P(x) = x^{n-m}$. Further, if $D_i(x) = A_i(x) - A_i(1)x$ for $i = 1, 2$, then D_i are derivations and $nD_1(x) = mD_2(x)$.

Conversely, if D_1 and D_2 are derivations on $\mathbb{R}$ and $A_1(x) = bx + D_1(x)$, $A_2(x) = bx + D_2(x)$, where b is any real number, $mD_2(x) = nD_1(x)$, and $P(x) = x^{n-m}$, where $m, n \in Z$, then A_1, A_2, and P satisfy (1.19) for all $x \in \mathbb{R}$.

Proof. It is easy to verify the converse part. First we shall show that $P(x) = x^{n-m}$. Replacing x by rx, r any rational in (1.19), and using (1.19), we have

$$[r^{m-n}P(rx) - P(x)]A_2(x^m) = 0.$$

Since $A_2 \neq 0$ and P is continuous with $P(1) = 1$, the equation above yields $P(x) = x^{n-m}$. Now (1.19) can be rewritten as

$$A_1(x^n) = x^{n-m}A_2(x^m), \ x \neq 0. \tag{1.20}$$

Let $D_i(x) = A_i(x) - A_i(1)x$ for $x \in \mathbb{R}$, $i = 1, 2$. Observe that $A_1(1) = A_2(1)$. First we consider the case $n \neq m$, $n, m > 0$.

We subdivide this case into three subcases: $m = 1$, $n = 2$; $m = 1$, $n > 2$; and $m \geq 2$, $n > m$.

Case $m = 1$, $n = 2$. Now (1.20) becomes

$$A_1(x^2) = xA_2(x) \quad \text{for } x \neq 0. \tag{1.21}$$

Replacing x by $x + y$ in (1.21) and utilizing (1.21), we have

$$2A_1(xy) = xA_2(y) + yA_2(x) \quad \text{for } x, y \in \mathbb{R}. \tag{1.22}$$

Set $y = 1$ in (1.22) to get

$$2A_1(x) = A_2(x) + A_2(1)x \tag{1.23}$$

and $2D_1(x) = D_2(x)$ since $A_1(1) = A_2(1)$. From (1.23) and (1.22), we have

$$A_2(xy) + A_2(1)xy = xA_2(y) + yA_2(x);$$

that is, $D_2(xy) = xD_2(y) + yD_2(x)$. Thus D_2 and hence D_1 are derivations.

Case $m = 1$ and $n > 2$. Now (1.20) becomes

$$A_1(x^n) = x^{n-1}A_2(x). \tag{1.24}$$

Changing x into $x + r$, with r any rational in (1.24), we obtain

$$A_1((x+r)^n) = (x+r)^{n-1}[A_2(x) + A_2(1)r].$$

Treating the equation above as a polynomial in r, equate the coefficients of r^{n-1} and r^{n-2} to get

$$nA_1(x) = A_2(x) + (n-1)A_2(1)x \tag{1.25}$$

and

$$\binom{n}{2}A_1(x^2) = (n-1)xA_2(x) + \binom{n-1}{2}A_2(1)x^2, \tag{1.26}$$

respectively. Equation (1.25) gives $nD_1(x) = D_2(x)$.

Use of (1.25) in (1.26) yields

$$\begin{aligned} nA_1(x^2) &= 2xA_2(x) + (n-2)A_2(1)x^2 \\ &= 2x[nA_1(x) - (n-1)A_2(1)x] + (n-2)A_2(1)x^2 \end{aligned}$$

or

$$A_1(x^2) - A_1(1)x^2 = 2x(A_1(x) - A_1(1)x);$$

that is, $D_1(x^2) = 2xD_1(x)$, showing thereby that D_1 is a derivation. Hence D_2 is also a derivation.

Case $m \geq 2$ and $n > m$. So, $n \geq 3$. As before, replace x by $x + r$, $r \in Q$, in (1.19) to have

$$A_1((x+r)^n) = (x+r)^{n-m}A_2((x+r)^m).$$

Now comparing the coefficients of r^{n-1} and r^{n-2} on both sides, we get

$$nA_1(x) = mA_2(x) + (n-m)A_2(1)x \tag{1.27}$$

and

$$\begin{aligned} \frac{n(n-1)}{2}A_1(x^2) = \frac{m(m-1)}{2}A_2(x^2) + (n-m)mxA_2(x) \\ + \frac{(n-m)(n-m-1)}{2}A_2(1)x^2. \end{aligned} \tag{1.28}$$

Now (1.27) gives the $nD_1(x) = mD_2(x)$ we seek. To prove that D_1 is a derivation, use (1.27) in (1.28) to have

$$\begin{aligned} n(n-1)A_1(x^2) = (m-1)[nA_1(x^2) - (n-m)A_1(1)x^2] \\ + 2(n-m)x[nA_1(x) - (n-m)A_1(1)x] \\ + (n-m)(n-m-1)A_1(1)x^2; \end{aligned}$$

that is,

$$A_1(x^2) - A_1(1)x^2 = 2x(A_1(x) - A_1(1)x),$$

showing thereby by (iii) of Proposition 1.9 that D_1 is a derivation. So, D_2 is also a derivation.

Case n, $m > 0$ and $n < m$ can be dealt with similarly by considering $A_2(x^m) = x^{m-n}A_1(x^n)$.

Case n, $m < 0$ follows from the case above by replacing x by $\frac{1}{x}$ in (1.20).

Finally, we consider the **case $n > 0$ and $m < 0$**. We now treat $n > 0$ and $m = -1$; that is,

$$A_1(x^n) = x^{n+1}A_2\left(\frac{1}{x}\right), \quad x \neq 0. \tag{1.29}$$

For $r \in Q$, using (1.29), we get

$$2rA_2\left(\frac{1}{x^2 - r^2}\right) = A_2\left(\frac{1}{x-r}\right) - A_2\left(\frac{1}{x+r}\right);$$

that is,

$$2rA_1((x^2 - r^2)^n) = (x+r)^{n+1}A_1((x-r)^n) - (x-r)^{n+1}A_1((x+r)^n).$$

Now, comparing the coefficient of r on both sides, we have

$$A_1(x^{2n}) = -nx^{n+1}A_1(x^{n-1}) + (n+1)x^nA_1(x^n);$$

that is, $D_1(x^{2n}) = -nx^{n+1}D_1(x^{n-1}) + (n+1)x^nD_1(x^n)$, which is (1.18). By Lemma 1.17, D_1 is a derivation. Equation (1.29) can be rewritten as

$$D_1(x^n) + A_1(1)x^n = x^{n+1}\left[D_2\left(\frac{1}{x}\right) + A_2(1)\left(\frac{1}{x}\right)\right],$$

from which, by using D_1 as a derivation, $A_2(1) = A_1(1)$, and replacing x by $\frac{1}{x}$ gives $nD_1(x) = -D_2(x)$. Hence D_2 is also a derivation.

For this case and the general case $n > 0$, $m < 0$, see also [486, 558, 327]. This proves the theorem. □

Corollary 1.19. *Halperin'sroblem* (1.8) *and* Proposition 1.15 *follow from* (1.29) *and* (1.20) *with* $A_2 = A_1$.

We will consider another special case of (1.17) and prove the following proposition.

Theorem 1.20. (Kannappan and Kurepa [486], Kuczma [558]). *If* $A_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ *satisfy* (A) *and*

$$A_1(t) = -t^2 A_2\left(\frac{1}{t}\right), \quad t\ (\neq 0) \in \mathbb{R}, \tag{1.30}$$

then $D_j(t) = A_j(t) - A_j(1)t$, $(j = 1, 2)$ *are derivations,* $D_1 = D_2$, *and* $A_1 - A_2$ *is continuous.*

Proof. For $r \in Q$, from

$$2r A_2\left(\frac{1}{x^2 - r^2}\right) = A_2\left(\frac{1}{x - r}\right) - A_2\left(\frac{1}{x + r}\right)$$

and (1.30), we have

$$2r A_1(x^2 - r^2) = (x + r)^2 A_1(x - r) - (x - r)^2 A_1(x + r).$$

By collecting the coefficient of r, we get $A_1(x^2) = 2x A_1(x) - A_1(1)x^2$; that is, D_1 is a derivation. Note that $A_1(1) = -A_2(1)$, and (1.30) can be rewritten as

$$D_1(x) + A_1(1)x = -x^2\left[D_2\left(\frac{1}{x}\right) + A_2(1)\frac{1}{x}\right]$$

or

$$D_2(x) = -x^2 D_1\left(\frac{1}{x}\right) = D_1(x).$$

Hence D_2 is also a derivation.

Now $A_1(x) - A_2(x) = D_1(x) + A_1(1)x - D_2(x) - A_2(1)x = 2A_1(1)x$. This proves the proposition. □

1.3 Additive Equation on $\mathbb{C}$, $\mathbb{R}^n$, and $\mathbb{R}_+$

1.3.1 Additive Equation on Complex Numbers

Theorem 1.21. *The general solution* $A : \mathbb{C} \to \mathbb{C}$ *of* (A) *is given by*

$$A(z) = A_1(x) + i A_2(x) + A_3(y) + i A_4(y), \tag{1.31}$$

where $z = x + iy$, $x, y \in \mathbb{R}$, *and* $A_j : \mathbb{R} \to \mathbb{R}$ *are solutions of* (A) $(j = 1, 2, 3, 4)$.

Proof. Let $z = x + iy,\ w = u + iv,\ x, y, u, v \in \mathbb{R}$. Further, let

$$A(z) = B_1(x, y) + iB_2(x, y). \tag{1.32}$$

Then, since A satisfies (A), it is easy to see from (1.32) that

$$B_j(x + u, y + v) = B_j(x, y) + B_j(u, v), \quad j = 1, 2. \tag{1.33}$$

Set $y = 0,\ v = 0$ in (1.33). Then, with

$$A_j(x) = B_j(x, 0), \quad x \in \mathbb{R}\ (j = 1, 2), \tag{1.34}$$

we see that $A_j\ (j = 1, 2)$ satisfy (A) on $\mathbb{R}$. Similarly, we obtain by defining

$$A_3(y) = B_1(0, y),\ A_4(y) = B_2(0, y), \quad \text{for } y \in \mathbb{R}, \tag{1.35}$$

that A_3 and A_4 are solutions of (A) on $\mathbb{R}$. From (1.23), (1.34), and (1.35), we have

$$B_1(x, v) = B_1(x, 0) + B_1(0, v) = A_1(x) + A_3(v)$$

and

$$B_2(u, y) = B_2(u, 0) + B_2(0, y) = A_2(u) + A_4(y).$$

Hence (1.31) holds. This completes the proof of this theorem. □

Corollary 1.22. *A complex-valued function of a real variable $A : \mathbb{R} \to \mathbb{C}$ is a solution of* (A) *if and only if*

$$A(x) = A_1(x) + iA_2(x), \quad x \in \mathbb{R}, \tag{1.36}$$

where $A_j : \mathbb{R} \to \mathbb{R}$ are solutions of (A) $(j = 1, 2)$.

Theorem 1.23. *The general continuous solution $A : \mathbb{C} \to \mathbb{C}$ of* (A) *is given by*

$$A(z) = cz + d\bar{z}, \tag{1.37}$$

where c and d are arbitrary complex constants. The same conclusion holds when A is either continuous at a point or measurable.

Proof. From Theorem 1.21, we see that $A_j\ (j = 1, 2, 3, 4)$ are continuous, continuous at a point, or measurable on $\mathbb{R}$. Hence, by Theorem 1.2, $A_j(x) = c_j x$, for $x \in \mathbb{R}$, and $c_j\ (j = 1 \text{ to } 4)$ are real constants. Hence $A(z) = c_1 x + ic_2 x + c_3 y + ic_4 y = cz + d\bar{z}$ for complex constants c and d. This proves the theorem. □

Remark 1.23a. The general differentiable solution of (A) where $A : \mathbb{C} \to \mathbb{C}$ is given by $A(z) = cz$, where c is a complex number, since $\bar{z}$ is not differentiable.

Theorem 1.24. *The most general solution $A : \mathbb{R}^n \to \mathbb{R}$ of* (A) *has the form*

$$A(x_1, x_2, \cdots, x_n) = A_1(x_1) + A_2(x_2) + \cdots + A_n(x_n), \tag{1.38}$$

where $A_k : \mathbb{R} \to \mathbb{R}$ satisfies (A) *for $k = 1$ to n. If $A : \mathbb{R}^n \to \mathbb{C}$ satisfies* (A), *then A is given by*

$$A(x_1, \cdots, x_n) = \sum_{k=1}^{n} \left(A_{k1}(x_k) + i\, A_{k2}(x_k)\right), \tag{1.39}$$

where $A_{k1}, A_{k2} : \mathbb{R} \to \mathbb{R}$ ($k = 1$ to n) are solutions of (A). *Further, the continuous solutions are given by*

$$A(x_1, x_2, \cdots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n,$$

where c_k's are reals when A is real valued or are complex constants when A is complex valued.

Proof. It is similar to the proof of Theorem 1.21 and follows from Corollary 1.22 and Theorem 1.2.

Theorem 1.25. *If a function $A : \mathbb{R}_+ \to \mathbb{R}$ satisfies* (A) *and is* (i) *continuous at a point,* (ii) *monotonically increasing,* (iii) *nonnegative for small x,* (iv) *locally integrable,* (v) *measurable,* (vi) *bounded in an interval, or* (vii) *bounded on a bounded set of positive measure, then $A(x) = cx$, for $x \in \mathbb{R}$, where c is an arbitrary constant.*

Proof. The proof is similar to proofs of Theorem 1.1 and Theorem 1.2. Later we will look at it from the point of view of extension of A.

1.3.2 More Algebraic Conditions

Theorem 1.26. (Kurepa [590], Vrbová [823]). Additive functions $A_1, A_2 : \mathbb{C} \to \mathbb{C}$ satisfying (A) and

$$\text{(a) } A_1(z) = |z|^2 A_1\left(\frac{1}{z}\right), \quad \text{(b) } A_2(z) = -|z|^2 A_2\left(\frac{1}{z}\right), \quad z \in \mathbb{C},\ z \neq 0, \tag{1.40}$$

are continuous and are given by

$$A_1(z) = A_1(1)\text{Re } z,\ A_2(z) = A_2(i)\text{Im } z. \tag{1.41}$$

Proof.

Define

$$d_1(z) = A_1(z) - A_1(1)\text{Re } z$$

and

$$d_2(z) = A_2(z) - A_2(i)\text{Im } z \quad \text{for } z \in \mathbb{C}.$$

Note that $d_1(1) = 0$ and $d_2(i) = 0$, and d_1 and d_2 satisfy (A).

From (A) and (1.40) for $z \neq -1$,

$$\begin{aligned} d_1(z) = d_1(z+1) &= A_1(z+1) - A_1(1)\text{Re}\,(z+1) \\ &= |z+1|^2 A_1\left(\frac{1}{z+1}\right) - A_1(1)\text{Re}\,(z+1) \\ &= |z+1|^2 A_1\left(1 - \frac{z}{z+1}\right) - A_1(1)\text{Re}\,(z+1) \\ &= -|z+1|^2 A_1\left(\frac{z}{z+1}\right) + (\text{Re}\,z + |z|^2)A_1(1) \\ &= -|z|^2 A_1\left(\frac{z+1}{z}\right) + (\text{Re}\,z + |z|^2)A_1(1) \\ &= -\left(|z|^2 A_1\left(\frac{1}{z}\right) - A_1(1)\text{Re}\,z\right) = -d_1(z). \end{aligned}$$

Hence $d_1(z) = 0$ for all z, including 0 and -1, and $A_1(z) = A_1(1)\text{Re}\,z$ for all $z \in \mathbb{C}$, which is the first part of (1.41).

To obtain the second part of (1.41), now, by (A) and (1.40)(b) for $z \neq 0$,

$$\begin{aligned} d_2(z) &= A_2(z) - A_2(i)\text{Im}\,z = -|z|^2 A_2\left(\frac{1}{z}\right) - A_2(i)\text{Im}\,z \\ &= -|z|^2\left(A_2\left(\frac{1}{z}\right) - A_2(i)\text{Im}\,\left(\frac{1}{z}\right)\right) = -|z|^2 d_2\left(\frac{1}{z}\right),\ z \neq 0. \end{aligned}$$

From (A) and (1.40), since $d_2(i) = 0$, we have

$$\begin{aligned} d_2(z) = d_2(z-i) &= -|z-i|^2 d_2\left(\frac{1}{z-i}\right) = -|z-i|^2 d_2\left(i - \frac{zi}{z-i}\right) \\ &= |z-i|^2 d_2\left(\frac{zi}{z-i}\right) = -|zi|^2 d_2\left(\frac{z-i}{zi}\right) = |z|^2 d_2\left(\frac{1}{z}\right) \\ &= -d_2(z), \quad \text{implying that } d_2(z) = 0 \end{aligned}$$

for all, z including $z = 0$; that is, $A_2(z) = A_2(i)\ \text{Im}\,z$.

Equation (1.40)(b) was used in [823] in connection with the characterization of complex inner product spaces. Now we present another proof of the same.

With $z = iy,\ y \in \mathbb{R}$, (1.40)(b) yields

$$A_2(iy) = y^2 A_2\left(\frac{i}{y}\right), \quad y\ (\neq 0) \in \mathbb{R}.$$

For $r \in Q,\ y \in \mathbb{R}$,

$$A_2(i(y+r)) = (y+r)^2 A_2\left(\frac{i}{y+r}\right);$$

that is,

$$A_2(iy) + A_2(ir) = (y^2 + 2yr + r^2)A_2\left(\frac{i}{y+r}\right)$$

or

$$y^2 A_2\left(\frac{i}{y}-\frac{i}{y+r}\right)+A_2\left(ir-\frac{ir^2}{y+r}\right)=2yrA_2\left(\frac{i}{y+r}\right)$$

or

$$y^2 A_2\left(\frac{i}{y(y+r)}\right)+A_2\left(\frac{iy}{y+r}\right)=2yA_2\left(\frac{i}{y+r}\right)\quad(\text{cancelling } r)$$

or

$$A_2(i(y+r)y)+y^2A_2\left(\frac{i(y+r)}{y}\right)=2yA_2(i(y+r))\quad\left(\text{cancelling }\frac{1}{(y+r)^2}\right)$$

or

$$A_2(iy^2)+rA_2(iy)+y^2A_2(i)+ry^2A_2\left(\frac{i}{y}\right)=2y[A_2(iy)+rA_2(i)].$$

Equating the coefficient of r, we get

$$A_2(iy)=A_2(i)y\quad\text{for all } y\in\mathbb{R}.$$

Now we will prove that $A_2(x)=0$ for $x\in\mathbb{R}$.

$$A_2(z)=A_2(x)+A_2(iy)=-(x^2+y^2)A_2\left(\frac{x-iy}{x^2+y^2}\right)\quad(\text{using (1.40)(b)});$$

that is,

$$A_2(x)=-(x^2+y^2)A_2\left(\frac{x}{x^2+y^2}\right)=x^2A_2\left(\frac{x^2+y^2}{x}\right)\quad(\text{using (1.40)(b)}).$$

Now $y=0$ gives $A_2(x)=x^2A_2(x)$. Hence $A_2(x)=0$ for $x\neq\pm1$. But $A_2(\pm1)=0$. So, $A_2(x)=0$ for $x\in\mathbb{R}$. This proves the second part of (1.41) and the proposition. □

We now prove the following proposition, which is a generalization of Theorem 1.26.

Theorem 1.27. (Kurepa [590], Kannappan [458]). *The general solutions* $A_1, A_2 : \mathbb{C}\to\mathbb{C}$ *satisfying* (A) *and*

$$A_1(z)=-|z|^2A_2\left(\frac{1}{z}\right),\quad z\neq0, \tag{1.42}$$

are continuous and have the forms

$$A_1(z)=a\,\mathrm{Re}\,z+b\,\mathrm{Im}\,z,\ A_2(z)=-a\,\mathrm{Re}\,z+\mathrm{Im}\,z,\quad \textit{for } z\in\mathbb{C}, \tag{1.43}$$

where $a=A_1(1)=-A_2(1)$ *and* $b=A_1(i)=A_2(i)$ *are complex constants.*

Proof. Defining $d_j(z) = A_j(z) - A_j(1)\mathrm{Re}\ z,\ (j = 1, 2),\ z \in \mathbb{C}$, and using (A), (1.42), and $A_1(1) = -A_2(1)$, we obtain

$$d_1(z) = A_1(z) - A_1(1)\mathrm{Re}\ z = -|z|^2 A_2\left(\frac{1}{z}\right) - A_1(1)\mathrm{Re}\ z$$

$$= -|z|^2(A_2\left(\frac{1}{z}\right) - A_2(1)\mathrm{Re}\ \left(\frac{1}{z}\right) = -|z|^2 d_2\left(\frac{1}{z}\right), \quad z \neq 0. \tag{1.44}$$

Observing that d_1 and d_2 are additive and $d_1(1) = 0 = d_2(1)$, we get

$$d_1(z) = d_1(z-1) = -|z-1|^2 d_2\left(\frac{1}{z-1}\right) = -|z-1|^2 d_2\left(\frac{z}{z-1} - 1\right)$$

$$= -|z-1|^2 d_2\left(\frac{z}{z-1}\right) = |z|^2 d_1\left(\frac{z-1}{z}\right) = -|z|^2 d_1\left(\frac{1}{z}\right), \tag{1.45}$$

which is (1.40). Hence, by (1.41), $d_1(z) = d_1(i)\mathrm{Im}\ z$; that is, $A_1(z) = d_1(z) + A_1(1)\mathrm{Re}\ z = A_1(i)\mathrm{Im}\ z + A_1(1)\mathrm{Re}\ z$.

Further, (1.44) and (1.45) yield $D_1 = D_2$ and $A_2(z) = D_1(i)\mathrm{Im}\ z + A_2(1)\mathrm{Re}\ z = A_2(i)\mathrm{Im}\ z - A_1(1)\mathrm{Re}\ z$. Thus (1.43) holds. This completes the proof of this proposition. □

1.4 Alternative Equations, Restricted Domains, and Conditional Cauchy Equations

Authors working in these areas are numerous, and the literature is rather voluminous (refer to Fischer [272, 278], Forti [284], Ger [304, 305], Dhombres [218, 221], Kannappan and Kuczma [485], Swiatak and Hosszú [786], and Vincze [819], which is only a partial list). When studying functional equations, one clearly notices that their solution depends not only on the functional equation and the class to which it belongs (like measurable or continuous functions) but also the domain of definition of the function (for example, the equation $f(xy) = f(x) + f(y)$ has solution $f(x) = c\log x$ on positive reals for measurable f, whereas on nonnegative reals or on $\mathbb{R}$ the only solution is $f(x) = 0$). Traditionally, authors used to suppose that the equation holds for all values of the variables on a certain set that appears normal for the equation. The classical example cited is the equation (A) holding on all of $\mathbb{R}$. But it is found that in many situations the equation is either postulated on a certain subset (for example, (A) on nonnegative reals in economics) or is satisfied on some subset of the domain of definition of the function, leading to the study of functional equations on *restricted domains*. We will later study in some detail the extensions of functions. Some of the equations come from non-Euclidean relativity theory. Here we will deal with some of the problems above.

1.4.1 Alternative Equations

A function $A : \mathbb{R} \to \mathbb{R}$ that is additive also obviously satisfies

$$[A(x+y)]^2 = [A(x) + A(y)]^2 \tag{1.46}$$

for $x, y \in \mathbb{R}$. The converse implication is less obvious. Several authors have studied the problem of the equivalence of (A) and (1.46). The equation (1.46) yields either $A(x+y) = A(x) + A(y)$ or $A(x+y) = -A(x) - A(y)$, earning it the name *alternative functional equation.* We will prove in the next theorem the equivalence of (A) and (1.46) (see [819, 272]). Equations (1.46) and (1.50) are special cases of the equation

$$[f(x+y)]^n = [f(x) + f(y)]^n,$$

which is equivalent to (A) ([819]).

Theorem 1.28. *A nonzero function $A : \mathbb{R} \to \mathbb{R}$ that satisfies* (1.46) *also satisfies* (A)*; that is,* (A) *and* (1.46) *are equivalent.*

Remark. The equation (1.46) is equivalent to the conditional Cauchy (additive) equation (A) for all $x, y \in \mathbb{R}$ such that the following holds: $A(x+y) + A(x) + A(y) \neq 0$.

Proof. (Vincze [819]). For x, y, z in $\mathbb{R}$, utilizing (1.46), we have

$$\begin{aligned}
[A(x+y+z)]^2 &= [A(x+y)+A(z)]^2 = [A(x+y)]^2+[A(z)]^2+2A(x+y)A(z) \\
&= [A(x)]^2 + [A(y)]^2 + [A(z)]^2 + 2A(x)A(y) + 2A(x+y)A(z) \\
&= [A(x) + A(y+z)]^2 \\
&= [A(x)]^2 + [A(y)]^2 + [A(z)]^2 + 2A(y)A(z) + 2A(y+z)A(x);
\end{aligned}$$

that is, $A(x)[A(y+z) - A(y)] = A(z)[A(x+y) - A(y)]$. Since $A \neq 0$, there is a $z_0 \in \mathbb{R}$ such that $A(z_0) \neq 0$. With $z = z_0$, the equation above can be rewritten as

$$A(x+y) = A(y) + A(x)k(y) \quad \text{for } x, y \in \mathbb{R}, \tag{1.47}$$

where $k(y) = \frac{A(y+z_0)-A(y)}{A(z_0)}$.

Interchanging x and y in (1.47), we get

$$A(x+y) = A(x) + A(y)k(x) \quad \text{for } x, y \in \mathbb{R}. \tag{1.48}$$

We will now show that $k(y) = 1$ for all y in $\mathbb{R}$, so that (1.47) becomes (A). Indeed, by considering $A(x+y+z)$ as $A[x+(y+z)]$ and $A[(x+y)+z]$ (using associativity) and using (1.47), we obtain $A(z)k(y+x) = A(z)k(y)k(x)$. Since $A \not\equiv 0$, k satisfies the functional equation

$$k(y+x) = k(y)k(x) \quad \text{for } x, y \in \mathbb{R}. \tag{1.49}$$

If $k = 0$, then (1.47) gives that $A =$ constant, and then by (1.46) we conclude that $A = 0$, which is not the case. Then $k(x) > 0$ for all $x \in \mathbb{R}$. Letting $y = x$ in (1.46) and (1.47), we have

$$[A(2x)]^2 = 4[A(x)]^2 \quad \text{and} \quad A(2x) = A(x)(1 + k(x)).$$

Squaring the latter and noting that $k(x) > 0$, we get $k(x) = 1$ when $A(x) \neq 0$. Suppose $A(y) = 0$. Use (1.47) and (1.48) to get $A(x + y) = A(x)k(y) = A(x)$ (that is, $k(y) = 1$) since $A \not\equiv 0$. Hence $k \equiv 1$ and A satisfies (A).

The same conclusion can also be obtained by the following argument. First $x = 0,\ y = 0$ in (1.46) yields $A(0) = 0$, and then $y = -x$ again in (1.46) gives $A(-x) = -A(x)$; that is, A is odd. Equating the right sides of (1.47) and (1.48) results in $A(y)[k(x) - 1] = A(x)[k(y) - 1]$.

If $k(y) = 1$ for all y in $\mathbb{R}$, there is nothing to prove. If not, $k(x) - 1 = k_1 A(x)$ for k_1 a nonzero constant. Then (1.46) becomes

$$A(x + y) = A(x) + A(y) + k_1 A(x)A(y).$$

Setting $y = -x$, the equation above yields $k[A(x)]^2 = 0$ (that is, $A(x) = 0$) since $k_1 \neq 0$, which is again a contradiction. This proves the theorem. □

Theorem 1.29. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation*

$$|f(x + y)| = |f(x) + f(y)| \quad \textit{for } x, y \in \mathbb{R}. \tag{1.50}$$

Then f satisfies (A)*; that is,* (A) *and* (1.50) *are equivalent.*

Proof. First set $x = 0,\ y = 0$ in (1.50) to get $f(0) = 0$ and then $y = -x$ in (1.50) to obtain $f(-x) = -f(x)$ for $x \in \mathbb{R}$; that is, f is odd.

Further, for $x, y \in \mathbb{R}$, by using (1.50), we have

$$|f(x + y) - f(x)| = |f(x + y) + f(-x)| = |f(y)|. \tag{1.51}$$

For $a, b \in \mathbb{R}$, it is true that $|a + b|^2 + |a - b|^2 = 2a^2 + 2b^2$. First take $a = f(x + y),\ b = f(x) - f(y)$ and then $a = f(x + y) - f(x),\ b = f(y)$, and use (1.50) and (1.51) to have

$$\begin{aligned} &|f(x + y) + f(x) - f(y)|^2 + |f(x + y) - f(x) + f(y)|^2 \\ &\quad = 2|f(x + y)|^2 + 2|f(x) - f(y)|^2 \\ &\quad = 2|f(x) + f(y)|^2 + 2|f(x) - f(y)|^2 \\ &\quad = 4|f(x)|^2 + 4|f(y)|^2 \end{aligned} \tag{1.52}$$

and

$$\begin{aligned} &|f(x + y) - f(x) + f(y)|^2 + |f(x + y) - f(x) - f(y)|^2 \\ &\quad = 2|f(x + y) - f(x)|^2 + 2|f(y)|^2 \\ &\quad = 2|f(y)|^2 + 2|f(y)|^2 \\ &\quad = 4|f(y)|^2. \end{aligned}$$

Interchange x and y in the equation above, add the resultant to it, and use (1.52) to obtain $|f(x+y)-f(x)-f(y)|^2 = 0$, so that f is additive. This proves the theorem. □

Remark 1.29a. Observe that no regularity assumption is made either in Theorem 1.28 or Theorem 1.29. As a matter of fact, the results above hold in a more general setting.

Result 1.30. (Kuczma [558], Fischer [272], Fischer and Muszély [278]). *Let* $(\mathbf{X}, +)$ *be a semigroup and let* $(\mathbf{Y}, +, \cdot)$ *be a commutative ring. The functional equations* (A) *and* (1.46) *are equivalent in the class* $\mathbf{Y}^{\mathbf{X}}$ *if and only if* $\mathbf{Y}$ *contains no genuine nilpotent elements nor elements of order* 3 *in the group* $(\mathbf{Y}, +)$.

Result 1.31. (Fischer and Muszély [278], Dhombres [218]). *In the class of functions* $f : S \to H$, *where* S *is a commutative semigroup and* H *is a Hilbert space* (*either real or complex*), *the equation* (A) *is equivalent to*

$$||f(x+y)|| = ||f(x) + f(y)|| \quad \textit{for all } x, y \in S.$$

Here is a result along the same lines from complex variables due to H. Haruki [350]. Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function satisfying

$$|f(x+iy)| = |f(x) + af(iy)|, \quad x, y \in \mathbb{R},\ a \text{ in } \mathbb{C}.$$

Then the solutions are given by

$$f(z) = \alpha \exp(bz) \quad \text{when } a = 0,\ z \in \mathbb{C},$$

or

$$f(z) = \alpha z,\ \alpha \sin bz,\ \alpha \sin h(bz)$$

when $a = 1$ for $z \in \mathbb{C}$ (see also [706, 349, 348]), or

$$f(z) = \alpha z + \beta z^2,\ \alpha \sin z + \beta \cos bz - \beta,\ \alpha \sin h\ bz + \beta \cos h\ bz - \beta \quad \text{when } a = -1,$$

or

$$f(z) = 0 \quad \text{when } |1+a| \neq 1,\ f(z) = \text{constant},$$

when $|1 + a| = 1$ in all other cases of $a \neq 0, 1, -1$ where b is real and α, β are complex constants.

1.4.2 Restricted Domains and Conditional Cauchy Equations

Frequently, in many situations, one encounters functional equations with a restricted domain when compared with what could be thought of as the natural domain. In the literature, "functional equations on restricted domains" (Kuczma [558]) and "conditional functional equations" (Dhombres and Ger [221]) have been used to describe this situation. We will return to this later in Sections 1.6.4, 1.7, etc.

In the study of the measure of nonlocalized oriented angles, Dubikajtis and Kuczma [232] came across the functional equation $f(x_1 + x_2) = f(x_1) + f(x_2)$ for $|x_1| < \frac{\pi}{2},\ |x_2| < \frac{\pi}{2}$, the Cauchy equation (A) on a restricted domain.

1.4.3 Mikusiński's Functional Equation

J. Mikusiński asked for one-to-one, onto mappings (bijections) $T : \mathbb{R}^2 \to \mathbb{R}^2$ that map straight lines onto straight lines. The problem could be solved without the use of functional equations, and this is what was done by Mikusiński and S. Karnik. However, here we present the earlier version of the argument leading to what is now called *Mikusiński's functional equation* (also refer to [558, 218, 44]).

As T is one-to-one, it maps parallel lines into parallel lines and so transforms parallelograms into parallelograms. First of all, write $T : \mathbb{R}^2 \to \mathbb{R}^2$ as $T(x, y) = (F(x, y), G(x, y))$ for $x, y \in \mathbb{R}^2$. We will determine F and G, where $F, G : \mathbb{R}^2 \to \mathbb{R}$. We may assume without loss of generality that $F(0, 0) = 0 = G(0, 0)$.

The parallelogram property applied to the points $(0, 0),\ (x, 0),\ (x, y)$, and $(0, y)$ yields the system of functional equations

$$F(x, y) = F(x, 0) + F(0, y),\ G(x, y) = G(x, 0) + G(0, y). \tag{1.53}$$

The collinearity of the points $(0, 0),\ (1, 0)$, and $(x, 0),\ x \in \mathbb{R},\ x \neq 0$, results in the collinearity of $T(0, 0),\ T(1, 0)$, and $T(x, 0)$, whence

$$aF(x, 0) = bG(x, 0) \quad \text{for } x \in \mathbb{R}, \tag{1.54}$$

where the constants $a = G(1, 0)$ and $b = F(1, 0)$ are not zero simultaneously; we may assume $b \neq 0$.

In the same way, the collinearity of $(0, 0)$, $(0, 1)$, and $(0, y)$ leads to

$$cF(0, y) = dG(0, y), \quad \text{for } y \in \mathbb{R}, \text{ with } d \neq 0. \tag{1.55}$$

From (1.53), (1.54), and (1.55), we obtain

$$d[bG(x, y) - aF(x, y)] = (bc - ad)F(0, y) \quad \text{for } x, y \in \mathbb{R}. \tag{1.56}$$

In view of (1.56), $bc - ad \neq 0$, for otherwise we would have $bG(x, y) = aF(x, y)$, which implies that the image $T(\mathbb{R}^2)$ is contained in the line $y = \frac{a}{b}x$, contradicting the assumption that T is onto.

As x varies in $\mathbb{R}$, the line (x, x) maps to the line $(F(x, x), G(x, x))$; that is,

$$\alpha F(x, x) + \beta G(x, x) = 0, \quad \text{with } |\alpha| + |\beta| \neq 0. \tag{1.57}$$

Combining (1.53), (1.54), (1.55), and (1.57), we get

$$(\alpha b + \beta a)F(x, 0) + (\alpha d + \beta c)F(0, x) = 0,$$

where $|\alpha b + \beta a| + |\alpha d + \beta c| \neq 0$ (otherwise $bc - ad = 0$). If $\alpha b + \beta a = 0$, then $F(0, x) = 0$, which by (1.56) contradicts the bijectivity of T. Similarly, $\alpha d + \beta c = 0$ would imply $F(x, 0) = 0$, which due to (1.54) gives $G(x, 0) = 0$, again contradicting that T is one-to-one. Hence, we have

$$F(0, x) = kF(x, 0), \quad \text{for } x \in \mathbb{R}, \text{ with } k \neq 0. \tag{1.58}$$

Define $f(x) = kF(x, 0),\ x \in \mathbb{R}$. The problem boils down to finding f.

Finally, from the collinearity of $(x+y,0)$, (x,y), and $(0,x+y)$, and by (1.53), (1.54), (1.55), and (1.58), we obtain

$$\begin{vmatrix} F(x+y,0) & G(x+y,0) & 1 \\ F(x,y) & G(x,y) & 1 \\ F(0,x+y) & G(0,x+y) & 1 \end{vmatrix} = 0;$$

that is,

$$\begin{vmatrix} f(x+y) & \frac{a}{b}f(x+y) & 1 \\ f(x)+kf(y) & \frac{a}{b}f(x)+\frac{kc}{d}f(y) & 1 \\ kf(x+y) & \frac{kc}{d}f(x+y) & 1 \end{vmatrix} = 0,$$

the equation

$$f(x+y)[f(x+y)-f(x)-f(y)]=0 \quad \text{for } x,y\in\mathbb{R}. \tag{1.59}$$

This equation is a conditional Cauchy equation known as *Mikusiński's equation*. It is obvious that (A) implies (1.59). But it is less obvious that in fact (1.59) implies (A) without any regularity condition. It is still more interesting that this conclusion fails to hold if we restrict the domain of f to the set of integers Z. The equation (1.59) is equivalent to the condition

$$\text{if } f(x+y)\neq 0, \text{ then } f(x+y)=f(x)+f(y). \tag{1.59a}$$

We now proceed to determine the solution of (1.59) in a general setup.

Result 1.32. (Dhombres [218], Kuczma [558], Ger [304, 305], Dhombres and Ger [221]). *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be groups* (*not necessarily Abelian*). *If the group* $\mathbf{X}$ *has no* (*normal*) *subgroup of index* 2, *then the conditional equation* (1.59) *for functions* $f:\mathbf{X}\to\mathbf{Y}$ *is equivalent to* (A). *If* $\mathbf{X}$ *has a normal subgroup* K *of index* 2, *then the family of all solutions* $f:\mathbf{X}\to\mathbf{Y}$ *of* (1.59) *consists of all solutions of* (A) *and all solutions*

$$f(x)=\begin{cases} 0, & x\in K \\ c, & x\notin K \end{cases}, \tag{1.59b}$$

where c *is an arbitrary constant.*

Remark. It is evident that if $\mathbf{X}=\mathbb{R}=\mathbf{Y}$, then (1.59) or (1.59a) and (A) are equivalent, and that if $\mathbf{X}=Z=\mathbf{Y}$, the class of solutions of (1.59a) consists of the linear functions $f(x)=cx$ (c an integer) and all functions of the form $f(x)=0$ for $x\in$ even and $f(x)=c$ for $x\in$ odd, where $c\neq 0$ is an integer.

We now return to the Mikusiński problem.

Theorem 1.33. *Let* $T:\mathbb{R}^2\to\mathbb{R}^2$ *be a bijection mapping straight lines into straight lines and continuous at a point. Then* T *is linear; that is,* $T(x,y)=(\alpha x+\beta y,\gamma x+\delta y)$, *and* $\alpha,\beta,\gamma,\delta$ *are constants.*

Proof. From the arguments above, Result 1.32, and Theorem 1.2, we can conclude that $f(x) = F(x,0)$ is additive and $f(x) = \alpha x$ for some constant α. Now, from (1.53), (1.54), (1.55), and (1.58), we have $F(x,y) = \alpha x + k\alpha y$, $G(x,y) = \frac{c}{d}k\alpha x + \frac{a}{b}\alpha y$, and the conclusion holds. □

The conditional functional equation

$$f(x) + f(y) \neq 0 \quad \text{implies } f(x+y) = f(x) + f(y) \tag{1.59c}$$

symmetric to (1.59a) was treated by Dhombres and Ger [218, 221].

Result 1.34. *Let* $\mathbf{X}$, $\mathbf{Y}$ *be groups* (*not necessarily Abelian*), *and suppose that* $\mathbf{Y}$ *does not contain elements of order* 2. *Then a function* $f : \mathbf{X} \to \mathbf{Y}$ *satisfies* (1.59b) *if, and only if, it satisfies* (A).

The functional equation

$$[f(x+y) - af(x) - bf(y)]\,[f(x+y) - f(x) - f(y)] = 0, \tag{1.60}$$

where a, b are constants, is a joint generalization of (1.46) and (1.59). The following result was proved by Kannappan and Kuczma [485].

Result 1.35. *Let* $G(+)$ *be a commutative group and* $H(+,\cdot)$ *be an integral domain of characteristic zero. Let* $f : G \to H$ *be a solution of* (1.60). *Then* $K = \{x \in G \mid f(x) = 0\}$ (*if not empty*) *is a subgroup of* G *and only the following four possibilities can occur:*

(i) $a + b = 1$, $K = \phi$, *and* f *is constant.*
(ii) $a + b = 0$, K *is of index* 2, *and* f *has the form given by* (1.59b).
(iii) $a+b = -1$, K *is of index* 3, *and* f *has the form* $f(x) = 0$ *for* $x \in K$, $f(x) = c$ *for* $x \in x_0 + K$, *and* $f(x) = -c$ *for* $x \in -x_0 + K$, *where* $x_0 \notin K$.
(iv) a, b *are arbitrary constants,* f *is additive (that is,* f *satisfies* (A)), *and the index of* K *is either* 1 (*in which case* $f = 0$) *or infinite.*

1.5 The Other Cauchy Equations

Let us treat the functional equations

$$E(x+y) = E(x)E(y) \qquad \text{(exponential)}, \tag{E}$$
$$L(xy) = L(x) + L(y) \qquad \text{(logarithmic)}, \tag{L}$$
$$M(xy) = M(x)M(y) \qquad \text{(multiplicative)}, \tag{M}$$

also known as Cauchy's equations. In order to distinguish one from the other, we will identify them as additive, exponential, logarithmic, or multiplicative equations. All of them can also be thought of as homomorphisms between semigroups with appropriate algebraic structures.

For example, let S and H be semigroups. Then (A) is a homomorphism between $S(+)$ and $H(+)$, (E) is between $S(+)$ and $H(\cdot)$, (L) is between $S(\cdot)$ and $H(+)$, and (M) is between $S(\cdot)$ and $H(\cdot)$.

One can find solutions of these equations either by a method similar to that employed for solving (A) or by other means. But one can also find the solutions more promptly by putting them in a form analogous to that of (A). First let us consider (E).

1.5.1 Exponential Equation

Suppose E satisfies (E) for $x, y \in T$, where either $T = \mathbb{R}$, $T = \mathbb{R}_+^*$, or $T = \mathbb{R}_+$. In the first two cases (that is, *for all real or all positive* x, y), we will prove that *a solution of* (E) *is either identically zero or nowhere zero.*

Indeed, suppose $E(x_0) = 0$ for some x_0. Then (E) yields

$$E(x) = E(x - x_0) \cdot E(x_0). \tag{1.61}$$

If $T = \mathbb{R}$, (1.61) shows that $E(x) = 0$ for all real $x \in \mathbb{R}$; that is, everywhere. If case (E) holds only for positive x, y, (1.61) shows that $E(x) = 0$ for $x \geq x_0$. Let $x \in]0, x_0[$. Then there is an integer n such that $nx \geq x_0$. Now, from (E) and (1.61), we have $0 = E(nx) = E(x)^n$, and thus again $E(x) = 0$ for all $x \in \mathbb{R}_+^*$.

The argument above implies that $E(x) = 0$ for all $x > 0$ but not for $x = 0$. As a matter of fact, E given by

$$E(x) = 0, \quad \text{for } x > 0, \ E(0) = 1, \tag{1.62}$$

is a solution of (E) on the domain $T = \mathbb{R}_+$.

As $E(x) \equiv 0$ is a solution of (E) in all these cases, without loss of generality we can assume that $E(x) \neq 0$ for all $x \in T$; that is, E is *nowhere zero.*

Replacing x and y in (E) by $\frac{x}{2}$, we obtain

$$E(x) = E\left(\frac{x}{2}\right)^2 > 0,$$

from which it follows that any nontrivial solution of (E) is always positive. Now taking the logarithm on both sides of (E), (E) goes over to (A), where

$$A(x) = \log E(x) \quad \text{and} \quad E(x) = e^{A(x)}. \tag{1.63}$$

Thus we have proved the following theorem.

Theorem 1.36. *Let* $E : T \to \mathbb{R}$ *satisfy* (E). *Then the most general solutions of* (E) *are given by* $E(x) = 0$, *and by* (1.63), $E(x) = e^{A(x)}$, *where* A *is an arbitrary solution of* (A) *on the same domains. If* $T = \mathbb{R}_+$, *we have the additional solution* (1.62), *but no others.*

Corollary 1.36a. *The most general continuous* (*continuous at a point, measurable, measurable on a set of positive measure, strictly monotonic on* $x,\ y \geq a \geq 0$, *etc.*) *solutions of* (E) *are given by*

$$E(x) = 0 \quad \textit{and} \quad E(x) = e^{cx}, \tag{1.64}$$

where c *is a constant on the domain* $T = \mathbb{R}$ *or* $\mathbb{R}_+^*$. *If* (E) *holds on* $T = \mathbb{R}_+$, *continuous solutions are still given by* (1.64), *but in the case of measurability by the solution* (1.62) *also.*

The proof follows from Theorem 1.36 and Theorem 1.2.

Theorem 1.37. *Suppose* $\nu : \mathbb{R}_+ \to \mathbb{C}$ *is a nontrivial complex-valued function and satisfies* (E) *with* $\nu(0) = 1$ *and* $|\nu(x)|$ *is bounded in some interval* $[a, b]$. *Then* $|\nu(x)| = \exp(cx)$ *for some real constant* c.

Proof. (Kannappan [426]). Define $A(x) = \log|\nu(x)|$ for $x \in \mathbb{R}_+$. Then A is well defined and satisfies (A) with $A(0) = 0$ and A is bounded from above on $[a, b]$. Then, by Theorem 1.2, $A(x) = cx$ for $x \geq 0$. This proves the theorem. □

Definition: Character. Set $\chi(x) = \frac{\nu(x)}{|\nu(x)|}$ for $x \in \mathbb{R}_+$, where ν satisfies (E) with $\nu(0) = 1$. Then it is clear that χ also satisfies (E) for $x, y \geq 0$ and $|\chi(x)| = 1$. For negative x, we may set $\chi(x) = [\chi(-x)]^{-1}$. Then χ satisfies (E) for all real x. Such a function is called a *character* ([369, 376]) of the real line.

Theorem 1.38. (Hille and Phillips [376]). *A mapping* $\chi : \mathbb{R} \to \mathbb{C}$ *is a measurable character of the real line if and only if* $\chi(x) = e^{ibx}$ *for some real constant* b.

Proof. For $|\delta| > 0$ and $a > 0$, we have

$$[\chi(x+\delta) - \chi(x)]a = \int_0^a [\chi(x+\delta-y) - \chi(x-y)]\chi(y)dy.$$

Hence

$$|\chi(x+\delta) - \chi(x)| \leq \frac{1}{a}\int_0^a |\chi(x+\delta-y) - \chi(x-y)|dy,$$

which tends to zero with $|\delta|$. Thus χ is continuous. Further,

$$\begin{aligned}\frac{\chi(\delta)-1}{\delta}\int_0^a \chi(x)dx &= \frac{1}{\delta}\int_0^a [\chi(x+\delta) - \chi(x)]dx \\ &= \frac{1}{\delta}\int_\delta^{a+\delta} \chi(x)dx - \frac{1}{\delta}\int_0^a \chi(x)dx.\end{aligned}$$

Choose a such that $\int_0^a \chi(x)dx \neq 0$. Since the limit as $|\delta| \to 0$ exists for terms on the right-hand side of this equation, it follows that the derivative of χ at $x = 0$ exists, and let $\frac{d\chi(x)}{dx}\,|_{x=0} = \beta$. Since χ satisfies (E), we can conclude that the derivative of χ exists for all $x \in \mathbb{R}$ and that $\frac{d\chi(x)}{dx} = \beta\chi(x)$. In fact, since

$$\frac{\chi(x+\delta) - \chi(x)}{\delta} = \chi(x)\frac{\chi(\delta)-1}{\delta},$$

taking the limit as $|\delta| \to 0$, the result follows. Hence $\chi(x) = ce^{\beta x}$. Since $\chi(0) = 1$, $c = 1$. As $|\chi(x)| = 1$, we see that β is purely imaginary; that is, $\beta = ib$ for some real b. □

Corollary 1.39. *If $\nu : \mathbb{R}_+ \to \mathbb{C}$ is measurable and satisfies* (E) *nontrivially with $\nu(0) = 1$, then*

$$\nu(x) = e^{(\alpha+i\beta)x} \quad \textit{for some reals } \alpha, \beta.$$

Proof. Define $A(x) = \log|\nu(x)|$. In the first place, A is measurable and additive on $\mathbb{R}_+$. Hence A is continuous, and $A(x) = \alpha x$ for some real α. Thus $|\nu(x)| = e^{\alpha x}$. Further, $\chi(x) = \frac{\nu(x)}{|\nu(x)|}$ is measurable. So, by Theorem 1.38, $\chi(x) = e^{i\beta x}$ for some real β. The result is now immediate. □

Result 1.40. (Aczél and Dhombres [44]). *The general solutions $E : \mathbb{R} \to \mathbb{C}$ of* (E) *are given by*

$$E(x) = 0 \quad \textit{and} \quad E(x) = e^{A(x)+iA_2(x)},$$

where $A : \mathbb{R} \to \mathbb{R}$ is additive and A_2 is an arbitrary solution of

$$A_2(x+y) \equiv A_2(x) + A_2(y) \pmod{2\pi} \quad \textit{for all } x, y \in \mathbb{R}. \tag{1.65}$$

Finally, we have the following result.

Result 1.41. [44]. *The general solutions $E : \mathbb{C} \to \mathbb{C}$ of* (E) *are given by $E(z) = 0$ and $E(z) = e^{A_1(x)+A_2(y)+iA_3(x)+iA_4(y)}$, where A_1 and A_2 satisfy* (A) *on $\mathbb{R}$ and A_3 and A_4 satisfy* (1.65) *on $\mathbb{R}$. Further, if E is measurable, then $E(z) = 0$ or $E(z) = e^{\alpha z+\beta\bar{z}}$, where α and β are complex numbers.*

1.5.2 Logarithmic Equation

Theorem 1.42. *If $L : \mathbb{R}^* \text{ or } \mathbb{R}_+^* \to \mathbb{R}$ is a solution of* (L), *then the most general form of L is given by*

$$L(x) = A(\log|x|), \quad \textit{for } x \in \mathbb{R}^* \textit{ or } L(x) = A(\log x),\ x \in \mathbb{R}_+^*, \tag{1.66}$$

where A satisfies (A) *on $\mathbb{R}^*$ or $\mathbb{R}_+^*$.*

Proof. First notice that L is even when (L) holds on $\mathbb{R}^*$ because, with $y = x$, (L) gives $L(x^2) = 2L(x)$, which when x is replaced by $-x$ yields $L(x^2) = 2L(-x)$, showing thereby that L is even; that is, $L(-x) = L(x)$.

Now let x and y be positive. We will show that (L) can be reduced to (A). There exists $u, v \in \mathbb{R}$ such that $x = e^u$ and $y = e^v$. By defining

$$A(u) = L(e^u) \quad \text{for } u \in \mathbb{R},$$

the equation (L) becomes $A(u+v) = A(u) + A(v)$ for $u, v \in \mathbb{R}$; that is, $L(x) = A(\log x)$ for $x \in \mathbb{R}_+^*$.

The result on $\mathbb{R}^*$ follows from the fact that L is even. □

Remark 1.42a. The only solution of (L) on $\mathbb{R}$ or $\mathbb{R}_+$ is $L(x) = 0$.

Corollary 1.43. *The general solution of* (L) *that is continuous, continuous at a point, measurable, or strictly monotonic on* $\mathbb{R}_+^*$ *is given by* $L(x) = c\log x$, *and on* $\mathbb{R}^*$, *the solution of* (L) *that is continuous, continuous at a point, measurable, or bounded from one side on a set of positive measure is given by* $L(x) = c\log|x|$ *for* $x \neq 0$.

The proof follows from Theorem 1.42 and Theorem 1.2.

The domain of the logarithmic equation (L) can be restricted and can be seen from the following results.

Suppose (L) holds for $x, y \in]0, 1]$. Setting $x = e^{-u},\ y = e^{-v}$ $(u, v \geq 0$; that is, $u, v \in \mathbb{R}_+)$, (L) goes over to (A) on $\mathbb{R}_+$ by the substitution $A(u) = L(e^{-u})$. Thus we have the following result.

Result 1.44. *If* $L :]0, 1] \to \mathbb{R}$ *satisfies* (L), *then* L *is given by* $L(x) = A(-\log x)$ *for* $x \in]0, 1]$, *where* A *is additive on* $\mathbb{R}_+$. *If* L *of* (L) *is continuous or measurable, then* $L(x) = c\log x$ *for a real* c, *and if* L *of* (L) *is nonnegative on* $]0, 1]$, *then* $L(x) = -c\log x$ *for a nonnegative constant* c.

Result 1.45. [44]. *Let* $L : Z_+^* \to \mathbb{R}$ *satisfy* (L); *that is,*

$$L(mn) = L(m) + L(n) \tag{1.67}$$

holds for all positive integers m *and* n *with*

$$n \xrightarrow{\liminf} \infty[L(n+1) - L(n)] \geq 0.$$

Then there is a constant c *such that* $L(n) = c\log n,\ n \in Z_+^*$.

Result 1.45a. [43]. *Suppose* $L : Z_+^* \to \mathbb{R}$ *satisfies* (1.67) *with*

$$nL(n) \leq (n+1)L(n+1) \quad \text{for } n = 1, 2, \ldots.$$

Then $L(n) = c\log n$ *for some constant* $c \geq 0$.

The same conclusion holds when L *satisfies* (1.67) *and* $\lim_{n\to\infty}[L(n+1) - L(n)] = 0$.

Result 1.46. [218, 43]. *The general monotonic solution of*

$$L(xn) = L(x) + L(n), \quad \text{for } x \in [1, \infty[,\ n \in Z_+^*,$$

is given by $L(x) = c\log x$, *for* $x \in [1, \infty[$, *where* c *is an arbitrary real number.*

1.5.3 Characterization of Exponential and Logarithmic Functions

The functions e^x and $\log x$ can be characterized by means of the equations (E) and (L), respectively (see the theorems above Theorem 1.42 and Result 1.44). But these functions can also be characterized with the help of the equations

$$E(2x) = E(x)^2, \tag{1.68}$$

(1.68a) $$L(x^2) = 2L(x),$$

in a single variable and some additional conditions. The following result will be of interest in that direction (see [550, 558]).

Result 1.47. (Kuczma [550]). *The function $E(x) = e^x$ is the only function that is either* (i) *differentiable in $\mathbb{R}_+$ and satisfies* (1.68) *with $E(0) = E'(0) = 1$ or* (ii) *logarithmically convex in $\mathbb{R}_+^*$ and satisfies* (1.68) *and the condition $E(1) = e$.*

The function $L(x) = \log x$ is the only function that is either (i) *differentiable in* $[1, \infty[$ *and satisfies* (1.68a) *with $L'(1) = 1$ or* (ii) *concave in* $]1, \infty[$ *and satisfies* (1.68a) *and the condition $L(e) = 1$.*

There are other characterizations of the logarithmic function (see also Chapter 17).

Theorem 1.48. (Heuvers [367]). *Let $f : \mathbb{R}_+^* \to \mathbb{R}$ satisfy the functional equation*

(1.69) $$f(x+y) - f(x) - f(y) = f\left(\frac{1}{x} + \frac{1}{y}\right) \quad \text{for } x, y \in \mathbb{R}_+^*.$$

Then f satisfies (L) *and conversely.*

Here we present a modified proof from [367].

Proof. If f satisfies (L), it is easy to check that f also satisfies (1.69). Now let f be a solution of (1.69). Define

$$F(x, y) = f(x+y) - f(x) - f(y) \quad \text{for } x, y \in \mathbb{R}_+^*.$$

F is called a Cauchy difference and satisfies

$$F(x+y, z) + F(x, y) = F(y+z, x) + F(y, z)$$

(see Chapter 13). Now (1.69) yields

(1.70) $$f\left(\frac{1}{x+y} + \frac{1}{z}\right) + f\left(\frac{1}{x} + \frac{1}{y}\right) = f\left(\frac{1}{y+z} + \frac{1}{x}\right) + f\left(\frac{1}{y} + \frac{1}{z}\right),$$

for $x, y, z \in \mathbb{R}_+^*$. Note that $f(1) = 0$ and $f\left(\frac{1}{x}\right) = -f(x)$.

Set $\frac{1}{y} + \frac{1}{z} = 1$ in (1.69). Clearly y and $z > 1$ and $z = \frac{y}{y-1}$. Now

$$\frac{1}{x+y} + \frac{1}{z} = \frac{1}{x+y} + \frac{y-1}{y} = \frac{y^2 + xy - x}{y(x+y)}$$

and

$$\frac{1}{y+z} + \frac{1}{x} = \frac{y^2 + xy - x}{xy^2}.$$

Now (1.70) becomes

$$f\left(\frac{y^2 + xy - x}{y(x+y)}\right) + f\left(\frac{1}{x} + \frac{1}{y}\right) = f\left(\frac{y^2 + xy - x}{xy^2}\right),$$

for $x > 0$ and $y > 1$, which by the substitutions

$$u = \frac{1}{x} + \frac{1}{y},\ v = \frac{y^2 + xy - x}{y(x+y)},$$

can be written as

$$f(uv) = f(u) + f(v). \tag{1.71}$$

Since $x > 0$ and $y > 1$, u and v are positive and there may be further restrictions on u and v. We have

$$u = \frac{1}{x} + \frac{1}{y},\ v = 1 - \frac{x}{y(x+y)}.$$

Then evidently $0 < v < 1$ since $x > 0,\ y > 1$, and

$$v = 1 - \frac{1}{uy^2};$$

that is,

$$uv = u - \frac{1}{y^2}$$

or

$$\frac{1}{y} = \sqrt{u(1-v)}$$

and

$$\frac{1}{x} = u - \sqrt{u(1-v)} = \frac{u^2 - (u(1-v))}{u + \sqrt{u(1-v)}} = \frac{u(u+v-1)}{u + \sqrt{u(1-v)}}.$$

Since $0 < v < 1$ and $x > 0,\ u + v - 1 > 0$, which will be satisfied if $u > 1$.

Thus (1.71) holds for $u > 1,\ 0 < v < 1$. Also, since

$$f(u \cdot 1) = f(u) = f(u) + f(1),$$
$$f(v \cdot 1) = f(v) = f(v) + f(1),$$

(1.71) holds for $u \geq 1$ and $0 < v \leq 1$.

Now we will show that (1.71) holds for all $u, v > 0$. For $u, v > 0$, choose n sufficiently large that $nu > 1$ and $\frac{v}{n} < 1$. Now

$$f(u) = f\left(nu \cdot \frac{1}{n}\right) = f(nu) + f\left(\frac{1}{n}\right), \quad \text{by (1.71)},$$

and

$$f(v) = f\left(\frac{v}{n} \cdot n\right) = f\left(\frac{v}{n}\right) + f(n), \quad \text{again by (1.71)},$$

so that

$$\begin{aligned} f(u)+f(v) &= f(nu)+f\left(\frac{1}{n}\right)+f\left(\frac{v}{n}\right)+f(n) \\ &= f(nu)+f\left(\frac{v}{n}\right) \quad \left(\text{since } f\left(\frac{1}{x}\right)=-f(x)\right) \\ &= f\left(nu\cdot\frac{v}{n}\right), \quad \text{by (1.71)}, \\ &= f(uv). \end{aligned}$$

Hence f satisfies (L) on $\mathbb{R}_+^*$. This proves the theorem. □

Theorem 1.48a. *Suppose $f:\mathbb{R}_+^*\to\mathbb{R}$ satisfies either*

$$f(xy)-f(x)=-f\left(\frac{1}{y}\right), \quad \textit{for } x,y>0, \tag{1.69a}$$

or

$$f(xy)=-f\left(\frac{1}{x}\right)-f\left(\frac{1}{y}\right), \quad \textit{for } x,y>0. \tag{1.69b}$$

Then f satisfies (L) *on $\mathbb{R}_+^*$.*

Proof. It is easy to see that $f(1)=0$ in either case. Suppose f satisfies (1.69a). Then interchanging of x and y shows that $f(x)+f\left(\frac{1}{x}\right)=c$ constant. Set $x=1$ to get $c=0$ and $f\left(\frac{1}{x}\right)=-f(x)$. Then (1.69a) is (L). As for (1.69b), let $y=\frac{1}{x}$ to get $f(x)+f\left(\frac{1}{x}\right)=0$. Thus f satisfies (L). This completes the proof. □

1.5.4 Multiplicative Equation

The equation (M) is the most complicated of all the three equations. We obtain the solution of (M) in the following theorem. Let T be either $\mathbb{R}_+$, $\mathbb{R}_+^*$, $\mathbb{R}$, or $\mathbb{R}^*$.

Theorem 1.49. *The general solutions $M:T\to\mathbb{R}$ of* (M) *are given by*

$$M(x)=0 \quad \textit{or} \quad M(x)=e^{A(\log x)}, \quad \textit{for } x\in\mathbb{R}_+^* \ (\textit{if } T=\mathbb{R}_+^*), \tag{1.72}$$

$$M(x)=0,\ M(x)=1,\ M(x)=\begin{cases} e^{A(\log x)}, & x\in\mathbb{R}_+^* \\ 0, & x=0 \end{cases} \ (\textit{if } T=\mathbb{R}_+), \tag{1.72a}$$

(1.72b)

$$M(x)=0,\ M(x)=1,\ M(x)=\operatorname{sign} x,\ M(x)=|\operatorname{sign} x| \textit{ for } T=\mathbb{R} \textit{ or } \mathbb{R}^*,$$

$$M(x)=\begin{cases} e^{A|\log|x|)}, & x\in\mathbb{R}^* \\ 0, & x=0 \end{cases} \quad \textit{or } M(x)=\begin{cases} \operatorname{sign} x\, e^{A(\log|x|)}, & x\in\mathbb{R}^* \\ 0, & x=0 \end{cases},$$

where A is a solution of (A) *on $\mathbb{R}$.*

Further, $M : T \to \mathbb{R}$ is a continuous (continuous at a point) solution of (M) if, and only if, either $M = 0$ or $M = 1$ or M has one of the forms

$$M(x) = |x|^c, \ M(x) = (\operatorname{sign} x)|x|^c, \quad x \in T,$$

for some constant c. If $0 \in T$, then $c > 0$.

Proof. (Aczél [12], Kuczma [558], Kannappan [426], Aczél and Dhombres [44]). Let $x, y \in \mathbb{R}_+^*$. There exist $u, v \in \mathbb{R}$ such that $x = e^u$ and $y = e^v$. Setting $E(u) = M(e^u)$, $u \in \mathbb{R}$, the equation (M) goes over to (E). Then, by Theorem 1.36, (1.72) follows. Thus the continuous solutions of (M) in this case are $M(x) = 0$ or $M(x) = x^c$.

Now put $x = 0$ in (M) to get $M(0)(1 - M(y)) = 0$; that is, either $M(0) = 0$ or $M(y) = 1$ for all $y \in \mathbb{R}$ or $\mathbb{R}_+$. This proves (1.72a).

Letting $y = 1$ be put into (M), we obtain either $M(1) = 1$ or $M(x) = 0$ for all $x \in T$. Now we will consider $T = \mathbb{R}$ or $\mathbb{R}^*$ with $M(1) = 1$. Setting $x = -1 = y$ in (M), we get $1 = M(1) = M(-1)^2$; that is, either $M(-1) = 1$ or $M(-1) = -1$. Finally, set $y = \operatorname{sign} x$ $(x \neq 0)$ in (M) to get

$$M(|x|) = M(x)M(\operatorname{sign} x), \quad \text{for } x \neq 0,$$

which is the same as

$$M(x) = M(|x|), \qquad \text{for } x \in \mathbb{R}^*,$$

or

$$M(x) = (\operatorname{sign} x)M(|x|), \qquad \text{for } x \in \mathbb{R}^*.$$

Now, (1.72b) follows from (1.72) and (1.72a). Further, continuous solutions can be obtained from (1.72), (1.72a), (1.72b), and Theorem 1.2. □

1.6 Some Generalizations of the Cauchy Equations

First we start with the Jensen and the Pexider equations. We study them in a general setting, too. One of the interesting and surprising aspects of functional equations is that one equation can determine many unknown functions. We will notice this in this section when dealing with the Pexider equations and several times in later sections, too.

1.6.1 Jensen's Equation

The equation

$$\text{(J)} \qquad f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$$

is known as *Jensen's equation* (Jensen [394], Aczél [12], Aczél, Chang, and Ng [40]). Whereas (J) makes sense in algebraic systems that are 2-divisible (that is, division by 2 is permissible), by replacing x by $x + y$ and y by $x - y$, (J) goes over to

$$f(x + y) + f(x - y) = 2f(x), \tag{J_1}$$

which in a way eliminates this problem and makes sense in algebraic systems that need not be 2-divisible. The equations (J) and (J_1) are equivalent in 2-divisible systems. Both equations can be solved by relating them to the additive equation (A). The equation (J) is meaningful in an interval (a, b) without any restriction, whereas (J_1) holds on an interval with restrictions. The equation (J) is "geometrical" in nature, which preserves midpoints. We will use either of them, depending on the situation.

Let $G(+)$ be a group not necessarily Abelian and $H(+)$ be a 2-divisible Abelian group. Suppose $f : G \to H$ satisfies (J_1) with

$$f(x + y + z) = f(x + z + y) \quad \text{for } x, y, z \in G. \tag{K}$$

Remark. We will encounter this (K) condition several times later on, too.

As a consequence of (K), we obtain

$$f(y + z) = f(z + y) \quad \text{for } z, y \in G. \tag{1.73}$$

Note that in general (1.73) does not imply (K).

Let $x = 0$ in (J_1) to get $f(y) + f(-y) = 2f(0)$ for $y \in G$. Interchanging x and y in (J_1), adding the resultant to (J_1), and using (1.73), we have

$$2f(x + y) + 2f(0) = 2f(x) + 2f(y), \quad x, y \in G.$$

Defining

$$A(x) = f(x) - f(0),$$

we see from the equation above that A satisfies (A) since H is 2-divisible. Thus we have proved the following theorem.

Theorem 1.50. (Aczél, Chung, and Ng [40]). *The general solution $f : G \to H$ of* (J_1) *satisfying* (K) *is given by*

$$f(x) = A(x) + c, \quad x \in G, \tag{1.74}$$

where A satisfies (A) *and c is an arbitrary constant.*

Corollary 1.51. *Let $f : T \to \mathbb{C}$, where $T = \mathbb{R}$ or $\mathbb{C}$ is a solution of either* (J_1) *or* (J). *Then the general solution is given by* (1.74) *for $x \in T$, where $A : T \to \mathbb{C}$ satisfies* (A) *and c is a complex constant and the solution continuous or continuous at a point is given by*

$$f(x) = ax + c \ (\text{when } T = \mathbb{R}) \quad \text{and} \quad f(z) = az + b\bar{z} + c \text{ when } T = \mathbb{C},$$

where a, b, c are complex constants.

The equation (J) is more meaningful on an "interval". The following results will be of interest in this direction.

Theorem 1.52. *Let f be a real-valued function defined on an arbitrary interval and satisfy* (J)*. If f is continuous, then f is affine; that is,*

$$f(x) = cx + b \tag{1.75}$$

for all x in the interval, where c and b are real constants.

Proof. Notice that f maps midpoints to midpoints.

Without loss of generality, let us assume the interval is $[0, 1]$. Let $f(0) = b$ and $f(1) = a$. For every x, y in $[0, 1]$, it is evident that $\frac{x+y}{2} \in [0, 1]$. The proof is based on induction.

First, let us show that (1.75) holds for x, a dyadic number in $[0, 1]$. Then, since the dyadic numbers are dense in $[0, 1]$ and f is continuous, we will have the required result (1.75) for all $x \in [0, 1]$. From (J), we have

$$f\left(\frac{1}{2}\right) = f\left(\frac{0+1}{2}\right) = \frac{f(0) + f(1)}{2} = \frac{b+a}{2} = b + \frac{1}{2}c,$$

where $c = a - b$. Suppose (1.75) holds for $\frac{1}{2^n}$. Then

$$f\left(\frac{1}{2^{n+1}}\right) = f\left(\frac{0 + 1/2^n}{2}\right) = \frac{b + b + \frac{1}{2^n}c}{2} = b + \frac{1}{2^{n+1}}c.$$

Now we will prove by induction that (1.75) holds for $x = \frac{t}{2^n}$, where t is an integer such that $0 \le t \le 2^n$.

Suppose (1.75) holds for $\frac{k}{2^n}$ for $k < m$ and $\frac{k}{2^n} \in [0, 1]$. Now, to prove (1.75) for $\frac{m}{2^n}$, if $m = 2^l$, the result follows from the induction hypothesis. Otherwise, we can write $m = 2^k + r$ such that $2^{k+1} > m$. Notice that $n - k \ge 1$. Then

$$\begin{aligned} f\left(\frac{m}{2^n}\right) &= f\left(\frac{\frac{1}{2^{n-k-1}} + \frac{r}{2^{n-1}}}{2}\right) = \frac{f(\frac{1}{2^{n-k-1}}) + f(\frac{r}{2^{n-1}})}{2} \\ &= b + \left(\frac{1}{2^{n-k-1}} + \frac{r}{2^{n-1}}\right)\frac{1}{2}c = b + \frac{m}{2^n}c, \end{aligned}$$

which completes the proof. □

Remark 1.52a. Associativity is a useful concept. We will use it to solve Jensen and Pexider equations and at many other places in later chapters, too.

Now we will consider $f : G \to \mathbb{R}$ satisfying (J) with (K), where G is a 2-divisible group, and show that f has the form (1.74).

Replacing y by $y + z$ in (J), we have

$$\begin{aligned} f\left(\frac{x+y+z}{2}\right) &= \frac{f(x) + f(y+z)}{2}, \quad \text{for } x, y, z \in G, \\ &= \frac{f(x+y) + f(z)}{2} \quad \text{(use associativity);} \end{aligned}$$

that is,

$$f(x+y) - f(x) = f(y+z) - f(z) \quad \text{for } x, y, z \in G.$$

So, for fixed $y \in G$, we define $h : G \to \mathbb{R}$ by

$$h(x) = f(x+y) - f(x).$$

Then h is a constant, say a function of y, $g(y)$. Then

$$\begin{aligned} f(x+y) &= f(x) + g(y), \quad \text{for } x, y \in \mathbb{R}, \\ &= f(y) + g(x) \quad \text{(by (K))}, \end{aligned}$$

interchanging x and y, or

$$f(x) - g(x) = d, \quad \text{a constant},$$

and

$$f(x+y) = f(x) + f(y) - d.$$

Define $A : \mathbb{R} \to \mathbb{R}$ by

$$A(x) = f(x) - d,$$

so that A is additive (A) and

$$f(x) = A(x) + d \quad \text{for } x \in \mathbb{R},$$

which is (1.74). □

A Generalization

Use associativity again.

Theorem 1.53. *Suppose $f, g, h : G \to \mathbb{R}$ satisfy*

$$f\left(\frac{x+y}{2}\right) = \frac{g(x)+h(y)}{2} \quad \textit{for } x, y \in G, \tag{1.76}$$

with f satisfying (K)*, where G is a 2-divisible group. Then there exists an additive A on G such that*

$$f(x) = A(x) + \frac{a+b}{2},\ g(x) = A(x) + a,\ h(x) = A(x) + b, \tag{1.76a}$$

where a, b are real constants.

Using only associativity twice, we can obtain from (1.76)

$$\begin{aligned} f\left(\frac{x+y+z}{2}\right) &= \frac{g(x+y)+h(z)}{2} \\ &= \frac{g(x)+h(y+z)}{2}, \quad \text{for } x, y, z \in G; \end{aligned}$$

that is,

$$\begin{aligned} g(x+y)-g(x) &= h(y+z)-h(z) \\ &= \text{a function of } y \\ &= A(y) \text{ (say)}, \quad \text{for } x, y, z \in G, \end{aligned}$$

$$g(x+y) = g(x) + A(y), \tag{1.76b}$$

$$h(y+z) = h(z) + A(y), \quad \text{for } x, y, z \in G. \tag{1.76c}$$

Put $x = 0$ (identity of G) in (1.76b) to get

$$g(y) = a + A(y).$$

Hence (1.76b) yields that A is additive.

Set $z = 0$ in (1.76c) to have

$$h(y) = A(y) + b,$$

and these values of h, g in (1.76) yield f in (1.76a). (f and g can also be obtained by using associativity.) This proves the theorem. □

Remark. Notice that the argument above is valid when the domain is a non-commutative semigroup and the range is a commutative group.

Remark. Note that one equation (1.76) determines three unknown functions f, g, h.

Corollary 1.53a. *Suppose* $f, g, h : \mathbb{R} \to \mathbb{R}$ *satisfy* (1.76)*. The continuous solutions of* (1.76) *are given by*

$$f(x) = cx + \frac{a+b}{2},\ g(x) = cx + a,\ h(x) = cx + b, \tag{1.76d}$$

for $x \in \mathbb{R}$.

Result. (Kuczma [558]). *Let* $C \subseteq \mathbb{R}^n$ *be a convex set such that the interior of* $C \neq \emptyset$*, and let* $f : C \to \mathbb{R}$ *be a solution of* (J)*. Then* f *is given by*

$$f(x) = A(x) + b \quad \textit{for } x \in C,$$

where $A : \mathbb{R}^n \to \mathbb{R}$ *satisfies* (A) *and* b*, a constant.*

The proof of this result involves extensions of functions, which is the topic of discussion in the next section.

Isometry

Suppose E and F are two normed linear spaces over $\mathbb{R}$. A mapping $f : E \to F$ satisfying the equation

$$||f(x) - f(y)|| = ||x - y||$$

is called an *isometry*.

Evidently f is one-to-one and continuous. We know from Theorem 1.50 that if $f : E \to F$ satisfies (J_1) or (J), then f has the form (1.74).

Result 1.54. [44]. *The general solution* $f : E \to F$ *of* (1.76)*, which is onto and satisfies* (J)*; that is, every isometry between two real normed linear spaces that is onto is an affine transformation* (*a mapping composed of a linear transformation and a translation*).

1.6.2 Pexider's Equations

The functional equations

(PA) $$f(x+y) = g(x) + h(y),$$

(PE) $$f(x+y) = g(x)h(y),$$

(PL) $$f(xy) = g(x) + h(y),$$

(PM) $$f(xy) = g(x)h(y),$$

treated by Pexider [666] with three unknown functions f, g, h are known as *Pexider's equations*, which are generalizations of the Cauchy equations (A), (E), (L), and (M), respectively. The solutions of these equations can be obtained by reducing them to the corresponding Cauchy equations and can be thought of as homomorphisms between two algebraic structures; the Pexider equations can be thought of as homotopies between two algebraic systems. Before we treat (PA) to (PM) individually on various structures, we will unite them in the following result as a homotopism.

Definition. Let $N_1(*)$ and $N_2(0)$ be *groupoids* (a set endowed with a binary operation is called a *groupoid*). A triplet $f, g, h : N_1 \to N_2$ is called a *homotopy* provided it satisfies the functional equation

$$f(x * y) = g(x)0h(y) \quad \text{for } x, y \in N_1. \tag{1.77}$$

Theorem 1.55. *Let* $G(*)$ *be a groupoid with an identity* e *and* $H(0)$ *be a group not necessarily commutative. The general system of solutions* $f, g, h : G \to H$ *of* (1.77) *is given by*

$$f(x) = a0k(x)0b,\ g(x) = a0k(x),\ h(x) = k(x)0b, \quad \textit{for } x \in G, \tag{1.78}$$

where k *is a homomorphism from* G *to* H*; that is,*

$$k(x * y) = k(x)0k(y), \quad \textit{for } x, y \in N_1, \tag{1.79}$$

and a *and* b *are arbitrary constants in* H.

Proof. Letting $x = e$ and $y = e$ separately in (1.77), we have

$$f(y) = a0h(y),\ \ f(x) = g(x)0b, \quad x, y \in G, \tag{1.80}$$

respectively, where $a = g(e)$ and $b = h(e)$ are in H. From (1.77) and (1.80) there results

$$f(x * y) = f(x)0b^{-1}0a^{-1}0f(y) \quad \text{for } x, y \in N_1.$$

By defining

$$k(x) = a^{-1}0\,f(x)0b^{-1} \quad \text{for } x \in G, \tag{1.81}$$

the equation (1.81) reduces to (1.79). Thus (1.81) and (1.80) give the result (1.78) we seek. □

From Theorem 1.55 there results the following theorem regarding the Pexider equations (PA) to (PE).

Result 1.56. (a) *The general system of solutions* $f, g, h : G(+) \to H(+)$ *of* (PA), *where* $G(+)$ *is a groupoid with identity and* $H(+)$ *is a group, is given by*

$$f(x) = a + A(x) + b,\ g(x) = a + A(x),\ h(x) = A(x) + b,$$

where A *is additive; that is,* A *satisfies* (A) *and* a, b *are arbitrary constants.*

When $H = G = \mathbb{R}$ *and any one of* f, g, h, *say* f, *is continuous at a point* (*or continuous or measurable, etc.*), *the solutions of* (PA) *are given by*

$$f(x) = cx + a + b,\ g(x) = cx + a,\ h(x) = cx + b,$$

where a, b, c *are real constants, and when* $H = \mathbb{R}^m$, $G = \mathbb{R}^n$, *the solutions of* (PA) *with continuous* f *are given by*

$$f(x) = c \cdot x + a + b,\ g(x) = c \cdot x + a,\ h(x) = c \cdot x + b, \quad x \in \mathbb{R}^n,$$

where a, b *are constants in* $\mathbb{R}^m$ *and* c *is an* $m \times n$ *matrix over* $\mathbb{R}$.

(b) *The most general system of solutions* $f, g, h : G(+) \to H(\cdot)$ *of* (PE), *where* G *is a groupoid with a neutral element and* H *is a group, is of the form*

$$f(x) = aE(x)b,\ g(x) = aE(x),\ h(x) = E(x)b,$$

where E *satisfies* (E) *and* a, b *are constants, together with the trivial solutions*

$$f(x) = 0, g(x) \text{ is arbitrary},\ h(x) = 0 \text{ and } f(x) = 0 = g(x),\ h(x) \text{ is arbitrary.}$$

When $G = \mathbb{R}(+)$ *and* $H = \mathbb{R}(\cdot)$, *the solutions of* (PE) *with a continuous* f *are given by*

$$f(x) = abe^{cx},\ g(x) = ae^{cx},\ h(x) = be^{cx},$$

where a, b, c *are arbitrary nonzero constants, together with the two trivial solutions above.*

(c) *The most general system of solutions* $f, g, h : G(\cdot) \to H(+)$ *of* (PL), *where* G *is a groupoid with identity and* H *is a group, is given by*

$$f(x) = a + L(x) + b,\ g(x) = a + L(x),\ h(x) = L(x) + b,$$

where L *is logarithmic; that is,* L *satisfies* (L) *on* G *and* a, b *are arbitrary constants.*

When $G = \mathbb{R}_+^*(\cdot)$ *or* $\mathbb{R}^*(\cdot)$ *and* $H = \mathbb{R}(+)$, *the general solutions of* (PL) *with* f *continuous at a point are given by*

$$f(x) = c \log|x| + a + b,\ g(x) = c \log|x| + a,\ h(x) = c \log|x| + b,$$

where a, b, c *are arbitrary constants.*

(d) *Functions* $f, g, h : T \to \mathbb{R}$, *where* $T = \mathbb{R}$ *or* $\mathbb{R}^*$ *or* $\mathbb{R}^*_+$, *satisfy* (PM) *if and only if either they have the* (*trivial*) *form* $f = 0 = g$, h *arbitrary or* $f = 0 = h$, g *arbitrary, or there exists a function* $M : T \to \mathbb{R}$ *satisfying the multiplicative equation* (M) *such that*

$$f(x) = abM(x),\ g(x) = aM(x),\ h(x) = bM(x)$$

hold, where a, b *are nonzero reals.*

If f, g, h *are continuous solutions of* (PM), *then they have the trivial forms and*

$$f(x) = ab,\ g(x) = a,\ h(x) = b,$$

or

$$f(x) = ab|x|^c,\ g(x) = a|x|^c,\ h(x) = b|x|^c,$$

or

$$f(x) = ab|x|^c \operatorname{sgn} x,\ g(x) = a|x|^c \operatorname{sgn} x,\ h(x) = b|x|^c \operatorname{sgn} x,$$

where $a, b \in \mathbb{R}^*$ *and* $c \in \mathbb{R}$ (*with* $c > 0$ *if* $0 \in T$).

In Result 1.56, identity in G plays a crucial role. Similar results hold even if G does not possess an identity element, but we supplement it by other requirements. For example, we prove the following result (refer to [558, 12, 40]) using associativity to solve (1.77) and then deduce the solutions of (PA) to (PM).

Theorem 1.57. *Let* $G(*)$ *be a commutative semigroup and* $H(0)$ *be a group* (*not necessarily Abelian*). *Functions* $f, g, h : G \to H$ *satisfy* (1.77) *if and only if*

$$\begin{aligned} f(x * y) &= a \circ b \circ k(x * y) \circ b \circ d, \\ g(x) &= a \circ k(x) \circ b, \\ h(x) &= b \circ k(x) \circ d, \quad \text{for } x, y \in G, \end{aligned} \tag{1.82}$$

where k *satisfies*

$$k(x * y) = k(x) \circ k(y) \quad \text{for } x, y \in G \tag{1.83}$$

(*that is,* k *is a homomorphism*).

Proof. Using associativity of $(*)$ and (1.77), we have

$$\begin{aligned} f(x * y * z) &= g(x * y) \circ h(z) \\ &= g(x) \circ h(y * z), \quad \text{for } x, y, z \in G; \end{aligned}$$

that is,

$$g(x)^{-1} \circ g(x * y) = h(y * z) \circ h(z)^{-1} = k(y) \quad \text{(say)}. \tag{1.84}$$

From

$$g(x * y) = g(x) \circ k(y), \quad \text{for } x, y \in G,$$

using associativity again, we get

$$\begin{aligned} g(x * y * z) &= g(x * y) \circ k(z) = g(x) \circ k(y) \circ k(z) \\ &= g(x) \circ k(y * z). \end{aligned}$$

Thus k satisfies (1.83). Using $(*)$ commutative and (1.84), we get

$$g(x) \circ k(y) = g(y) \circ k(x),\ k(y) \circ h(z) = k(z) \circ h(y)$$

or

$$g(x) = a \circ k(x) \circ b,\ h(z) = b \circ k(z) \circ d, \tag{1.85}$$

where $a = g(y_0),\ b = k(y_0)^{-1},\ d = h(y_0)$.

Note that b commutes with $k(x)$ (use (1.83)). Now (1.77) gives, using (1.85),

$$f(x * y) = a \circ k(x) \circ b \circ b \circ k(y) \circ d = a \circ b \circ k(x * y) \circ b \circ d. \tag{1.86}$$

Equations (1.85) and (1.86) yield (1.82). This proves the theorem.

Corollary 1.57a.

(i) *Functions* $f, g, h : G(+) \to H(+)$ (*where* G *is a commutative semigroup and* H *a group not necessarily Abelian*) *satisfy* (PA) *if and only if*

$$f(x+y) = a+b+A(x+y)+b+d,\ g(x) = a+A(x)+b,\ h(x) = b+A(x)d+a,$$

where A *satisfies* (A).

(ii) *Let* $G(+)$ *be a commutative semigroup and* $H(\cdot)$ *be a group. The general system of solutions* $f, g, h : G \to H$ *of* (PE) *is given by*

$$f(x+y) = abE(x+y)bd,\ g(x) = aE(x)b,\ h(x) = bE(x)d,$$

where E *is a solution of* (E).

(iii) *Suppose* $f, g, h : G(\cdot) \to H(+)$ (*where* G*is a commutative semigroup and* H *a group*) *are solutions of* (PL). *Then there exists an* L *satisfying* (L) *such that*

$$f(xy) = a + b + L(xy) + b + d,\ g(x) = a + L(x) + b,\ h(x) = b + L(x) + d.$$

(iv) *The general system of solutions* $f, g, h : G(\cdot) \to H(\cdot)$ (*where* G *is a commutative semigroup and* H *a group*) *satisfying* (PM) *is of the form*

$$f(xy) = abM(xy)bd,\ g(x) = aM(x)b,\ h(x) = bM(x)d,$$

where M *satisfies* (M).

1.6.3 Some Generalizations

Proposition 1.58. *Let us first consider the simple functional equation*(*s*)

$$f(x+y) = f(x) + f(y) \pm f(x)f(y), \tag{1.87}$$

where $f : S \to F$, $S(+)$ *is a groupoid, and* F *is a field. Then*

$$f(x) = E(x) - 1 \quad \text{or} \quad f(x) = 1 - E(x)$$

according to whether a $+$ *or* $-$ *sign is taken in* (1.87)*, where* E *satisfies* (E)*.*

Proof. When a $+$ sign occurs on the right side of (1.87), define $E(x) = 1 + f(x)$ and add 1 to both sides; when a $-$ sign occurs in (1.87), add -1 to both sides and define $E(x) = 1 - f(x)$. Then (1.87) reduces to (E).

For $S = \mathbb{R} = F$ with f continuous at a point, the solution of (1.87) is given by

$$f(x) = e^{cx} - 1 \quad \text{and} \quad f(x) = -1$$

or

$$f(x) = 1 - e^{cx} \quad \text{and} \quad f(x) = 1$$

according to whether a $+$ or $-$ sign occurs in (1.87).

The continuous solutions of

$$f(x_1 + x_2, y_1 + y_2) = f_1(x_1, y_1) + f_2(x_1, y_2) + f_3(x_2, y_1) + f_4(x_2, y_2) \tag{1.88}$$

are obtained in Kuwagaki [591]. In Theorem 1.59, we obtain the general solution of (1.88) in a general setting related to the Pexider equation and a biadditive function. Note that this equation has five unknown functions.

Theorem 1.59. *The general solution* f, $f_i : H \times H \to G$ $(i = 1, 2, 3, 4)$ *of* (1.88)*, where* H *is a groupoid with neutral element* 0 *and* G *is a commutative group, is given by*

$$\begin{cases} f(x,y) = B(x,y) + A_1(x) + A_2(Y) + c, \\ f_1(x,y) = B(x,y) + u_1(x) + v_1(y) - c_1, \\ f_2(x,y) = B(x,y) + A_1(x) - u_1(x) + v_2(y) + c_1, \\ f_3(x,y) = B(x,y) + u_2(x) + A_2(y) - v_1(y) + c_1, \\ f_4(x,y) = B(x,y) + A_1(x) - u_2(x) + A_2(y) - v_2(y) + c - c_1, \end{cases} \tag{1.89}$$

where B *is* biadditive (*that is,* $B : H \times H \to G$ *is additive in each variable*)*,* A_1, A_2 *are additive* (*satisfy* (A) *on* H)*,* u_1, u_2, v_1, v_2 *are arbitrary functions, and* c, c_1 *are arbitrary constants.*

Proof. Setting $x_1 = 0 = x_2 = y_1 = y_2$ in (1.88), we have

$$f(0,0) = \sum_{i=1}^{4} f_i(0,0).$$

With $y_1 = 0 = y_2$, (1.88) results in the Pexider equation (PA)

$$f(x_1 + x_2, 0) = (f_1 + f_2)(x_1, 0) + (f_3 + f_4)(x_2, 0),$$

and by Result 1.56 (a), there exists an additive $A_1 : H \to G$ and constants A_1 and b_1 such that

$$\begin{cases} f(x,0) = A_1(x) + a_1 + b_1, \\ (f_1 + f_2)(x,0) = A_1(x) + a_1, \\ (f_3 + f_4)(x,0) = A_1(x) + b_1. \end{cases} \tag{1.90}$$

Similarly, setting $x_1 = 0 = x_2$, (1.88) results in a Pexider equation (PA) such that

$$\begin{cases} f(0,y) = A_2(y) + a_2 + b_2, \\ (f_1 + f_3)(0,y) = A_2(y) + a_2, \\ (f_2 + f_4)(0,y) = A_2(y) + b_2, \end{cases} \tag{1.91}$$

where $A_2 : H \to G$ satisfies (A) and $a_2,\ b_2$ are constants.

Putting $s_2 = 0 = y_2,\ x_2 = 0 = y_1,\ x_1 = 0 = y_1$, and $x_1 = 0 = y_2$ separately in (1.88), we have

$$\begin{cases} f_1(x_1, y_1) = f(x_1, y_1) - f_2(x_1, 0) - f_3(0, y_1) - f_4(0,0), \\ f_2(x_1, y_2) = f(x_1, y_2) - f_1(x_1, 0) - f_4(0, y_2) - f_3(0,0), \\ f_4(x_2, y_2) = f(x_2, y_2) - f_3(x_2, 0) - f_2(0, y_2) - f_1(0,0), \\ f_3(x_2, y_1) = f(x_2, y_1) - f_4(x_2, 0) - f_1(0, y_1) - f_2(0,0). \end{cases} \tag{1.92}$$

Substituting these values of $f_1,\ f_2,\ f_3,\ f_4$ into (1.88) and using (1.90) and (1.91) yields

$$\begin{aligned} f(x_1 + x_2, y_1 + y_2) = {} & f(x_1, y_1) + f(x_1, y_2) + f(x_2, y_1) + f(x_2, y_2) \\ & - f(0,0) - A_1(x_1) - a_1 - A_1(x_2) \\ & - b_1 - A_2(y_1) - a_2 - A_2(y_2) - b_2. \end{aligned}$$

Noting that A_1 and A_2 are additive and $f(0,0) = a_1 + b_1 = a_2 + b_2 = c$ (say), the equation above can be rewritten as

$$B(x_1 + x_2, y_1 + y_2) = B(x_1, y_1) + B(x_2, y_1) + B(x_1, x_2) + B(x_2, y_2),$$

where

$$B(x,y) = f(x,y) - A_1(x) - A_2(y) - c. \tag{1.93}$$

It is easy to check that B is biadditive (that is, additive in each variable).

From (1.93), (1.92), (1.90), and (1.91), we obtain the solution of (1.88) given by (1.89), where $u_1(x) = f_1(x, 0),\ v_1(y) = f_1(0, y),\ v_2(y) = f_2(0, y),\ u_2(x) = f_3(x, 0)$.

Remark. Suppose $H = \mathbb{R} = G$ in Theorem 1.59. If any one of the functions, say f, is continuous at a point, then the solution of (1.88) is given by

$$\begin{aligned}
f(x, y) &= axy + a_1x + a_2y + c,\\
f_1(x, y) &= axy + u_1(x) + v_1(y) - c,\\
f_2(x, y) &= axy + a_1x - u_1(x) + v_2(y) + c_1,\\
f_3(x, y) &= axy + u_2(x) + a_2y - v_1(y) + c_1,\\
f_4(x, y) &= axy + a_1x - u_2(x) + a_2y - v_2(y) + c - c_1.
\end{aligned}$$

1.6.4 Some Special Cases

Let us now consider the functional equation

$$f(x + y) = g(x)h(y) + k(y). \tag{1.94a}$$

This equation contains (A) (when $h = 1,\ g = k = f$), (E) (when $k = 0,\ g = h = f$), (PA) ($h = 1$), and (PE) ($k = 0$). The solution of (1.94a) can be obtained in relation to the equations

$$F(x + y) = F(y) + G(x)h(y), \tag{1.94a$_1$}$$

$$G(x + y) = G(x)\phi(y) + G(y) \tag{1.94a$_2$}$$

(refer to Aczél [12], Aczél and Daróczy [43], and Castillo and Ruiz-Cobo [144]).

Suppose $f, g, h, k : S \to H$ satisfy (1.94a), where $S(+)$ is a commutative semigroup with identity 0 and H is a field.

(1.95) Either $g \equiv 0$ or $h \equiv 0$ in (1.94a) leads to $f = \text{constant} = k(0)$

(put $y = 0$). Assume $g \not\equiv 0$ and $h \not\equiv 0$.

Now $f(x) = c_1$ gives $g(x)h(y) + k(y) = c_1$; that is,

$$g(x) = c_2,\ k(y) = c_1 - c_2h(y), \quad h \text{ is arbitrary.} \tag{1.96}$$

Let us now assume that $f \not\equiv$ constant.

Substitute $x = 0$ in (1.94a) to obtain

$$f(y) = g(0)h(y) + k(y) \quad \text{for } y \in S. \tag{1.97}$$

Subtracting (1.97) from (1.94a), we have (1.94a$_1$), where

$$F(x) = f(x) - f(0),\ G(x) = g(x) - g(0). \tag{1.98}$$

First we determine F and G from (1.94a$_1$) and (1.94a$_2$) using associativity.

Considering $F(x+y+z)$ as $F((x+y)+z)$ and $F(x+(y+z))$ (that is, using associativity) and using (1.94a$_1$), we get

$$(G(x+y)-G(y))h(z)=G(x)h(y+z) \quad \text{for } x,y,z\in S. \tag{1.99}$$

Fixing z_0 in (1.99) such that $h(z_0)\neq 0$, (1.99) becomes (1.94a$_2$), where

$$\phi(y)=h(y+z_0)h^{-1}(z_0).$$

Note that $\phi(0)=1$. From (1.94a$_2$), we have

$$\begin{aligned} G(x+y+z)&=G(x+y)\phi(z)+G(z)=[G(x)\phi(y)+G(y)]\phi(z)+G(z)\\ &=G(x)\phi(y+z)+G(y)\phi(z)+G(z); \end{aligned}$$

that is, $G(x)[\phi(y+z)-\phi(y)\phi(z)]=0$.

Now $G(x)=0$ implies $F(x)=$ constant (from (1.94a$_1$)); that is, $f=$ constant, which is not the case.

Thus ϕ is exponential and satisfies (E), say $\phi=E$. Interchanging of x and y in (1.94a$_2$) yields (note that $\phi=E$)

$$G(x)(E(y)-1)=G(y)(E(x)-1). \tag{1.100}$$

Consider the case $E(x)=1$.

Equation (1.94a$_2$) shows that G is additive, say $G=A$, $g(x)=A(x)+c_2$ (use (1.98)), and that (1.99) gives $h=$ constant $=c_3$. Again (1.94a$_1$) with $y=0$ gives $F(x)=c_3G(x)=c_3A(x)$ (note $F(0)=0$) or

$$f(x)=A(x)c_3+c_1. \tag{1.101}$$

Substituting this f in (1.97) yields

$$k(y)=c_3A(y)+c_1-c_2c_3. \quad \text{Note that } h=c_3,\ g(x)=A(x)+c_2. \tag{1.101a}$$

Finally, consider the case $E(x)\not\equiv 1$. Use (1.100) to get G (and thus g); that is,

$$g(x)=c(E(x)-1)+c_2,\ c\neq 0. \tag{1.102}$$

($c=0$ gives $G=0$ and that F in (1.94a$_1$), so f is constant.)

Now $y=0$ in (1.94a$_1$) gives $F(x)=c_3G(x)=c_3c(E(x)-1)$; that is,

$$f(x)=cc_3(E(x)-1)+c_1. \tag{1.102a}$$

Using (1.102) and (1.102a) in (1.94a$_1$), we get (specializing x)

$$h(x)=c_3E(x). \tag{1.102b}$$

From (1.102a), (1.102b), and (1.97), we get

$$k(y)=(cc_3-c_2c_3)(E(y)-1)+c_1-c_2c_3. \tag{1.102c}$$

Theorem 1.60. *The most general systems of solutions of* (1.94a) *on S are given by* (1.101), (1.101a), (1.102), (1.102a), (1.102b), (1.102c), (1.96), *and* (1.95), *where E satisfies* (E) *and A is a solution of* (A).

Remark 1.60a. As for ($1.94a_1$), in addition to the trivial solutions $G = 0,\ h$ arbitrary, $F =$ constant and G arbitrary, $h = 0,\ F =$ constant, the solutions of ($1.94a_1$) are given by

$$G(x) = A(x),\ F(x) = c_3A(x) + c_1,\ h(x) = c_3,$$

or

$$G(x) = c(E(x) - 1),\ F(x) = cc_3(E(x) - 1) + c_1,\ h(x) = c_3E(x),$$

where E and A satisfy (E) and (A), respectively, on S.

Remark 1.60b. Barring the trivial solutions $G = 0,\ \phi$ arbitrary or $\phi = 0,\ F =$ constant of ($1.94a_1$), the most general solutions of ($1.94a_2$) are given by

$$\phi = 1, G(x) = A(x) \quad \text{or} \quad \phi(x) = E(x),\ G(x) = c(E(x) - 1),$$

where A and E are solutions of (A) and (E), respectively.

The solutions above follow from the proof of Theorem 1.60. We also provide a direct proof of ($1.94a_1$) starting from

$$[G(x + y) - G(y)]h(z) = G(x)h(y + z) \quad \text{for } x, y, z \in S. \tag{1.99}$$

Since $h \not\equiv 0$, we obtain

$$G(x + y) = G(y) + G(x) \cdot \beta(y), \tag{1.99i}$$

where

$$\begin{aligned} \beta(y) &= h(y + z_0)h^{-1}(z_0) \quad (\text{put } z = z_0) \\ &= G(x) + G(y)\beta(x), \quad \text{for } x, y \in S; \end{aligned}$$

that is,

$$G(x)[\beta(y) - 1] = G(y)[\beta(x) - 1]. \tag{1.99ii}$$

Case 1. Suppose $\beta(x) \equiv 1$. Then (1.99i) shows that G is additive, say $G = A$. This $G = A$ in (1.99) gives $h =$ constant $= c_3$. Use ($1.94a_1$) to get

$$F(x + y) = F(y) + c_3A(x);$$

that is,

$$F(x) = c_1 + c_3A(x) \quad (\text{put } y = 0).$$

Case 2. Assume $\beta(x) \not\equiv 1$. Equation (1.99ii) yields

$$G(x) = c(\beta(x) - 1),$$

which in turn in (1.99i) shows that β is exponential, say $\beta = E$. These values of G and β in (1.99) give

$$h(y + z) = h(z)E(y);$$

that is,

$$h(y) = c_3 E(y) \quad (\text{take } z = 0).$$

Substituting these values of G, h in (1.94a$_1$) yields $F(x) = zc_3(E(x) - 1)$.

Remark 1.60c. Note that (1.94a) determines five unknown functions and no regularity condition is used (see also [12, 44, 818, 144]).

Remark 1.60d. The commutativity of S is used only once to obtain (1.100). The results could be achieved by assuming that f satisfies the (K) condition.

Corollary. The general nontrivial system of continuous solutions of (1.94a) for $S = H = \mathbb{R}$ is given by

$$(1.94a') \qquad \begin{cases} & f(x) = bc_3x + c_1,\ g(x) = bx + c_2, \\ & h(x) = c_3,\ k(x) = bc_3x + c_1 - c_2c_3, \\ \text{and} & \\ & f(x) = cc_3(e^{bx} - 1) + c_1,\ g(x) = c(e^{bx} - 3) + c_2, \\ & h(x) = c_3e^{bx},\ k(x) = (cc_3 - c_2c_3)(e^{cx} - 1) + c_1 - c_2c_3, \end{cases}$$

that of (1.94a$_2$) is given by

$$(1.94a'_2) \qquad \phi(x) = 1,\ G(x) = bx \quad \text{or} \quad \phi(x) = e^{bx},\ G(x) = c(e^{bx} - 1),$$

and that of (1.94a$_1$) is given by

$$(1.94a'_1) \qquad \begin{cases} & G(x) = bx,\ F(x) = c_3bx + e_1,\ h(x) = c_3 \\ \text{or} & \\ & G(x) = c(e^{bx} - 1),\ F(x) = cc_3(e^{bx} - 1) + c_1,\ h(x) = c_3e^{bx}. \end{cases}$$

Sometimes one encounters the dual of (1.94a), namely

$$(1.94b) \qquad f(xy) = g(x)h(y) + k(y),$$

where $f, g, h, k : S \to H$, $S(\cdot)$ is a commutative semigroup with identity e and H is a field. This equation contains (L), (M), (PL), and (PM). From Theorem 1.60, we deduce the following result.

Result 1.61. *The general system of solutions of* (1.94b) *is given by* (1.95), (1.96), *and* (1.94b′)

$$(1.94b') \qquad \begin{cases} & f(x) = c_3L(x) + c_1,\ g(x) = L(x) + c_3, \\ & h(x) = c_3,\ k(x) = c_3L(x) + c_1 - c_2c_3, \\ \text{and} & \\ & f(x) = cc_3(M(x) - 1) + c_1,\ g(x) = c(M(x) - 1) + c_2, \\ & h(x) = c_3M(x),\ k(x) = (cc_3 - c_2c_3)(M(x) - 1) + c_1 - c_2c_3, \end{cases}$$

where L (*logarithmic*) *is a solution of* (L) *and M multiplicative is a solution of* (M) *on S*.

Note that Result 1.61 also gives the solution of

$$f(xy) = g(y)h(x) + k(x), \quad \text{for } x, y \in S, \tag{1.94b$_1$}$$

as (1.94b′).

For $S = \mathbb{R}$, $H = \mathbb{R}$, the system of solutions of (1.94b) for which f is strictly monotonic is given by

$$\text{(1.94b}'')\qquad \begin{cases} & f(x) = c_3 bk \log gx + c_1,\ g(x) = bk \log gx + c_3, \\ & h(x) = c_3,\ k(x) = bk \log x + c_1 - c_2 c_3, \\ \text{and} & \\ & f(x) = cc_3(x-1) + c_1,\ g(x) = c(x^b - 1) + c_2, \\ & h(x) = c_3 x^b,\ k(x) = (cc_3 - c_2 c_3)(x^b - 1) + c_1 - c_2 c_3. \end{cases}$$

Remark 1.62. The general system of solutions of

$$\psi(x+y) = \psi(x)\alpha(y) + \beta(y) \tag{1.94a$_3$}$$

where $\psi, \alpha, \beta : S \to H$ is given by

$$\text{(1.94a}_3')\qquad \begin{cases} & \alpha = 0,\ \psi = c = \beta, \\ & \psi(x) = c,\ \alpha \text{ arbitrary},\ \beta(x) = c(1 - \alpha(x)), \\ & \psi(x) = A(x) + b_1,\ \alpha(x) = 1,\ \beta(x) = A(x), \\ & \psi(x) = c(E(x) - 1) + c_1,\ \alpha(x) = E(x), \\ \text{or} & \\ & \beta(y) = (c - c_1)(E(y) - 1), \end{cases}$$

respectively, where E satisfies (E) and A satisfies (A) on S.

The general nontrivial system of continuous solutions of (1.94a$_3$) for $S = \mathbb{R} = H$ is given by

$$\text{(1.94a}_3'')\qquad \begin{cases} \text{or} & \psi(x) = bx + b_1,\ \alpha(x) = 1,\ \beta(x) = bx, \\ & \psi(x) = c(e^{bx} - 1) + c_1,\ \alpha(x) = e^{bx},\ \beta(x) = (c - c_1)(e^{cx} - 1), \end{cases}$$

respectively.

Let us give an application of (1.94a$_3$) in the theory of mean values and operations (see Aczél [12], p. 152). We start with the functional equation

$$f(ax + by + c) = g(x) + h(y) + c', \quad \text{with } ab \neq 0, \tag{1.103a}$$

and the general linear equation

$$f(ax + by + c) = a' f(x) + b' f(y) + c', \quad \text{with } aa'bb' \neq 0 \tag{1.103b}$$

(refer to Aczél [12], Daróczy [191, 192], Kuczma [558], Losonczi [613], and Castillo and Ruiz-Cobo [144]).

Let $f, h, g : F_1 \to F_2$, where F_1 and F_2 are fields. If in (1.103a) either f, g, or h is a constant, so are the others. So, we will consider only nonconstant solutions of (1.103a).

Putting $y = 0$ and $x = 0$ separately in (1.103a), we get

$$f(ax + c) = g(x) + h(0) + c', \quad a \neq 0, \tag{1.104a}$$

$$f(by + c) = g(0) + h(y) + c', \quad b \neq 0. \tag{1.104b}$$

Note that $f(c) = g(0) + h(0) + c'$.

Substituting (1.104a) and (1.104b) into (1.103a), we obtain

$$f(ax + by + c) = f(ax + c) + f(by + c) - f(c), \quad ab \neq 0;$$

that is,

$$f(x + y + c) = f(x + c) + f(y + c) - f(c).$$

Hence $A(x) = f(x + c) - f(c)$ satisfies (A), so that

$$f(x) = A(x) + f(c) - A(c) = A(x) + f(0). \tag{1.105}$$

From (1.105), (1.104a), and (1.104b), we have

$$g(x) - g(0) = A(ax),\ h(y) - h(0) = A(by), \quad a, b \neq 0. \tag{1.106a}$$

If we consider (1.103b) (take $g(x) = a' f(x),\ h(y) = b' f(y)$) as before, we obtain (1.105) and

$$a' A(x) = A(ax),\ b' A(y) = A(by), \quad aba'b' \neq 0. \tag{1.106b}$$

Thus we have proved the following theorem.

Theorem 1.63. *Let $f, g, h : F_1 \to F_2$. Then the general nonconstant solutions of* (1.103a) *and* (1.103b) *are given by* (1.105) *and* (1.106a) *and* (1.105) *with* (1.106b)*, respectively, where A is a solution of* (A)*. Furthermore, if $F_1 = \mathbb{R} = F_2$ and any one of the functions is continuous or continuous at a point or monotonic, etc., then the solution of* (1.103a) *is given by*

$$f(x) = dx + \alpha_1,\ g(x) = dax + \alpha_2,\ h(y) = dby + \alpha_3, \tag{1.103a$'$}$$

with $dc + \alpha_1 = \alpha_2 + \alpha_3 + c$, and that of (1.103b) *is given by*

$$f(x) = dx + \alpha \text{ with } a = a',\ b = b',\ dc - c' = (a + b - 1)\alpha, \tag{1.103b$'$}$$

with $d \neq 0$. From (1.106b) *follows $a = a'$ and $b = b'$. As a matter of fact, we also have the following from* (1.103b)*: $c = c'$, $a = 1 = b = a' = b'$ gives* (A)*; $a + b = 1$, $c = 0 = c'$ gives $f(ax + (1 - a)y) = af(x) + (1 - a) f(y)$ ($a = \frac{1}{2}$ gives Jensen's equation*)*; and $a + b = 1$, $c \neq 0$ gives $f(ax + (1 - a)y + c) = af(x) + (1 - a) f(y) + c'$.*

We will now consider the following functional equations arising in the theory of mean values using (1.94a$_3$) and (1.103b):

$$\text{(1.107a)} \qquad \begin{cases} g\left(\frac{af(x+t)+bf(y+t)}{a+b}\right) = g\left(\frac{af(x)+bf(y)}{a+b}\right) + t, \\ f(g(x)) = x, \end{cases}$$

$$\text{(1.107b)} \qquad g\left(\frac{af(xt)+bf(yt)}{a+b}\right) = g\left(\frac{af(x)+bf(y)}{a+b}\right) t, \;\; f(g(x)) = x.$$

Suppose f is a strictly monotonic surjective continuous mapping on $\mathbb{R}_+$ satisfying (1.107a) for $x \geq 0$. Note that $g = f^{-1}$ and the codomain of f is a nondegenerate interval, say J.

With $f_t(x) = f(x+t)$, $g_t(x) = g(x) - t$, for fixed t in $\mathbb{R}_+^*$, (1.107a) goes over to

$$g_t\left(\frac{af_t(x)+bf_t(y)}{a+b}\right) = g\left(\frac{af(x)+bf(y)}{a+b}\right), \;\; f_t(g_t(x)) = x.$$

By hypothesis on f for $u, v \in J$, there exist unique $x, y \in \mathbb{R}_+$ such that $f(x) = u$, $g(y) = v$, $x = g(u)$, $y = g(v)$, are satisfied so that the equation above becomes

$$\text{(1.107a}_1\text{)} \qquad f_t\left(g\left(\frac{au+bv}{a+b}\right)\right) = \frac{af_t(g(u))+bf_t(g(v))}{a+b},$$

which is of the form (1.103b), where f is replaced by $f_t g$.

Remark. The continuous solution of the Jensen equation (J) is given by $f(x) = cx + b$ (see Theorem 1.52). The same holds for (1.107a$_1$) $(f_t g)\left(\frac{au+bv}{a+b}\right) = \frac{a(f_t g)(u)+b(f_t g)|v|}{a+b}$, $a, b > 0$. Hence

$$(f_t g)(u) = c(t)u + \alpha(t);$$

that is,

$$f(x+t) = c(t)\, f(x) + \alpha(t),$$

which is (1.94a$_3$). Thus follows the result.

Result 1.64. *The general strictly monotonic continuous solutions of* (1.107a) *are given by*

$$\text{(1.107a}'\text{)} \qquad f(x) = dx + \alpha \quad \textit{and} \quad f(x) = de^{cx} + \alpha.$$

Similarly, for (1.107b) *the dual solutions are*

$$\text{(1.107b}'\text{)} \qquad f(x) = d\log x + \alpha \quad \textit{and} \quad f(x) = dx^c + \alpha \quad (c, d \neq 0).$$

1.6.5 More Generalizations

The *Mikusiński-Jensen functional equation*

$$\text{(M-J)} \qquad f\left(\frac{x+y}{2}\right)\left[f(x)+f(y)-2f\left(\frac{x+y}{2}\right)\right]=0,$$

where $f:\mathbb{R}\to\mathbb{R}$ or $f:(a,b)\to\mathbb{R}$, is a special case of the functional equation

$$\text{(A-G)} \qquad \begin{aligned} &f(x)f(px+(1-p)y)+f(y)f((1-p)x+py)\\ &\quad =f(px+(1-p)y)^2+f((1-p)x+py)^2 \end{aligned}$$

(for $p=\frac{1}{2}$), which was introduced by Alsina and Garcia-Roig [71] in connection with the classical problem of duplicating a cube. We will deal with (A-G) later in the application. We now treat (M-J) and relate it to (1.59), the Mikusiński functional equation, and produce the following result due to Lajko [599].

Theorem 1.65. *If the mapping $f:\mathbb{R}\to\mathbb{R}$ satisfies* (M-J)*, then there exist an additive A and a constant b such that*

$$\text{(SM-J)} \qquad f(x)=A(x)+b \quad \textit{for } x\in\mathbb{R}.$$

Proof. As remarked by Lajko, a proof can be presented based on a result of Baron and Ger or Ger in connection with a generalization of (1.59).

Case $f(0)\neq 0$. With $y=-x$, (M-J) goes over to

$$f(x)+f(-x)-2f(0)=0 \quad \text{for } x\in\mathbb{R}.$$

The function $h:\mathbb{R}\to\mathbb{R}$ defined by

$$h(x)=f(x)-f(0)$$

is odd, $h(0)=0$, and (M-J) becomes

$$\left[h\left(\frac{x+y}{2}\right)+f(0)\right]\left[h(x)+h(y)-2h\left(\frac{x+y}{2}\right)\right]=0 \quad \text{for } x,y\in\mathbb{R}.$$

Replacing x and y by $-x$ and $-y$ in the equation above and adding the resultant equation to the equation above, using h odd and $f(0)\neq 0$, we get

$$h(x)+h(y)=2h\left(\frac{x+y}{2}\right) \quad \text{for } x,y\in\mathbb{R},$$

which is from Jensen [394]. Thus we get (SM-J) with $b=f(0)\neq 0$.

Case $f(0)=0$. Equation (M-J) can be rewritten as

$$f\left(\frac{x+y}{2}\right)\left[f(x+y)-2f\left(\frac{x+y}{2}\right)\right]=0$$

by replacing x and y by $x + y$ and 0, respectively. Thus, if $f\left(\frac{x+y}{2}\right) \neq 0$, then $f(x + y) = 2f\left(\frac{x+y}{2}\right) \neq 0$, and it follows from (M-J) and $f(x + y) \neq 0$ that f satisfies (A); that is, (1.59a), which is equivalent to (1.59). Then, by [218, 221, 232], there exists an additive $A : \mathbb{R} \to \mathbb{R}$ such that $f(x) = A(x)$; that is, (SM-J) holds with $b = 0$. □

The following result also holds (Lajko [599]).

Result 1.66. *If* $f : \mathbb{R} \to \mathbb{R}^*$ (*or* $f : (a, c) \subseteq \mathbb{R} \to \mathbb{R}^*$) *satisfies* (M-J), *then* (SM-J) *holds where* $b \neq 0$.

1.7 Extensions

Extension problems arise in many branches of mathematics; for example, extensions of homomorphisms and isomorphisms are known in algebra. These can be considered extensions of functional equations and arise in many applications in the study of functional equations. The problem is that one is interested in either extending functions satisfying a functional equation on a "smaller" or "restricted" set to functions that satisfy the same equation on a "larger" or "maximal" set or proving that the given functions satisfying a functional equation on a "smaller" or "restricted" set indeed satisfy the same equation on a "larger" set. The next question is, when the extension exists, is it unique? One can cite the following well-known example. Suppose $f : T \to \mathbb{R}$ ($T = [0, \infty]$ or $[0, 1]$) satisfies (A) (in the case of $[0, 1]$, (A) holds when $x + y$ also is in $[0, 1]$). Is it true that there is an additive $A : \mathbb{R} \to \mathbb{R}$ that is an extension of f? The answer to this question is affirmative, and the extension is also unique (as we will show later). But, in general, there may not be an extension, and when it exists, it need not be unique (see Section 1.6.5). We illustrate these extensions in the results to follow. We start off with the following theorem related to the example above.

Theorem 1.67. (Aczél, Baker, Djokovic, Kannappan, and Rado [36]). *Let* $G(\cdot)$ *and* $H(\circ)$ *be groups* (*that need not be Abelian*), S *be a nonempty subsemigroup of* G *such that* $G = SS^{-1} = \{st^{-1} : s, t \in S\}$, *and* $f : S \to H$ *be a homomorphism; that is,*

$$f(st) = f(s) \circ f(t) \quad \text{for } s, t \in S.$$

Then there exists a homomorphism $g : G \to H$ *that is a unique extension of* f.

Proof. Define $g : G \to H$ by

$$g(st^{-1}) = f(s) \circ f(t)^{-1} \quad \text{for } s, t \in S.$$

This g is well defined, for if $st^{-1} = uv^{-1}$, for $u, v, s, t \in S$, then $s = uv^{-1}t$ and there are $r, w \in S$ such that $v^{-1}t = rw^{-1}$ and $tw = vr$, $sw = ur$, so that $f(t) \circ f(w) = f(v) \circ f(r)$ and $f(s) \circ f(w) = f(u) \circ f(r)$. Hence $f(s) \circ f(w) \circ f(r)^{-1} = f(u)$, and $f(w) \circ f(r)^{-1} = f(t)^{-1} \circ f(v)$; that is, $f(s) \circ f(t)^{-1} = f(u) \circ f(v)^{-1}$. That is, g is well defined.

We will now prove that g indeed is a homomorphism. Let $x, y \in G$. Then there are $s, t, u, v, r, w \in S$ such that

$$x = uv^{-1},\ y = rs^{-1},\ v^{-1}r = tw^{-1} \quad \text{and} \quad f(r) \circ f(w) = f(v) \circ f(t).$$

Now,

$$\begin{aligned} g(xy) &= g(uv^{-1}rs^{-1}) = g(ut \cdot (sw)^{-1}) = f(ut) \circ f(sw)^{-1} \\ &= f(u) \circ f(t) \circ f(w)^{-1} \circ f(s)^{-1} = f(u) \circ f(v)^{-1} \circ f(r) \circ f(s)^{-1} \\ &= g(x) \circ g(y);\ \text{that is, } g \text{ is a homomorphism.} \end{aligned}$$

Further, for $s, t \in S$, $g(s) = g(st \cdot t^{-1}) = f(st) \circ f(t)^{-1} = f(s)$ shows that g is an extension of f. Finally, there is the uniqueness of g. Suppose h is also an extension of f. Then

$$h(st^{-1}) = h(s)_0 h(t)^{-1} = f(s) \circ f(t)^{-1} = g(st^{-1}).$$

This proves the theorem. □

Remark 1.68. If G and H are Abelian groups, the proof is even simpler (see [53]). Also when $G = H = \mathbb{R}(+)$, $S = [0, \infty[$ (the example cited before Aczél and Erdős [45]), every function $f : [0, \infty[\to \mathbb{R}$ satisfying (A) can be extended to a solution A of the same equation on the whole plane, $A(x + y) = A(x) + A(y)$, for $(x, y) \in \mathbb{R}^2$, so that $A(x) = g(x)$, for $x \geq 0$.

The same result can also be obtained from the following result.

Result 1.69. (Aczél, Kannappan, Ng, and Wagner [53]). *Let $G(\cdot)$ and $H(\circ)$ be groups and S be a subsemigroup of G such that every element x of G different from the neutral element of G belongs either to S or $x^{-1} \in S$ (or both). Then every homomorphism $f : S \to H$ has a unique extension to a homomorphism $g : G \to H$.*

Note that S generates G and $G = SS^{-1}$.

Now we will see an example of a homomorphism on a group finitely generated by a subsemigroup.

Theorem 1.70. (Mokanski [637]). *Let $G(\cdot)$ and $H(\circ)$ be groups and S a subsemigroup of G such that $G = SS^{-1}S$. If a mapping $f : G \to H$ satisfies the equation*

$$f(ab^{-1} \cdot c) = f(ab^{-1}) \circ f(c), \quad \text{for } ab^{-1} \in SS^{-1} \text{ and } c \in S \text{ or } a, b, c \in S,$$

then

$$f(xy) = f(x) \circ f(y) \quad \text{for all } x, y \in G.$$

Proof. First we will show that

$$f(ab) = f(a) \circ f(b) \quad \text{for } a, b \in S;$$

that is, f is a homomorphism of S into H.

For $a, b \in S$, by the definition of f,

$$f(ab) = f(abb^{-1} \cdot b) = f(abb^{-1}) \circ f(b) = f(a) \circ f(b).$$

Also, $f(ab) = f(ab^{-1} \cdot b^2) = f(ab^{-1}) \circ f(b^2)$, so that

$$f(ab^{-1}) = f(a) \circ f(b)^{-1} \quad \text{for } a, b \in S.$$

Note that $f(ab^{-1}c) = f(a) \circ f(b)^{-1} \circ f(c)$ for $a, b, c \in S$.

Let $x, y \in G$. Then there exist $a, b, c, u, v, w, r, s, t$ in S such that $x = ab^{-1}c$, $y = uv^{-1}w$, $b^{-1}cuv^{-1} = rs^{-1}t$, so that $cu = brs^{-1} \cdot tv$ and

$$f(cu) = f(br) \circ f(s)^{-1} \circ f(tv) = f(b) \circ f(r) \circ f(s)^{-1} \circ f(t) \circ f(r).$$

Now,

$$\begin{aligned} f(xy) &= f(ab^{-1}c \cdot uv^{-1}w) = f(ar \cdot s^{-1} \cdot tw) = f(ar) \circ f(s)^{-1} \circ f(tw) \\ &= f(a) \circ \underline{f(r) \circ f(s)^{-1} \circ f(t)} \circ f(w) \\ &= f(a) \circ \underline{f(b)^{-1} \circ f(c) \circ f(u) \circ f(v)^{-1}} \circ f(w) \\ &= f(x) \circ f(y). \end{aligned}$$

This proves the theorem. □

Before discussing more results on extensions, we will provide examples to show that an extension may not be possible, and when it exists (is possible), it need not be unique.

Examples

(1) Let $f : \mathbb{R}_+ \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 0, & \text{for } x > 0, \\ 1, & \text{for } x = 0. \end{cases}$$

Obviously, $f(x + y) = f(x) f(y)$ for $x, y \in \mathbb{R}_+$. This f has no extension E to $\mathbb{R}$ satisfying (E) on $\mathbb{R}$ (whenever E is zero at a point, then E is identically 0 in $\mathbb{R}$).

(2) Let $f : \mathbb{R}_+^* \to \mathbb{R}$ be a nonzero solution of

$$f(xy) = f(x) + f(y).$$

This f has no extension to $\mathbb{R}_+$ or $\mathbb{R}$ (such an extension is identically zero on $\mathbb{R}_+$ or $\mathbb{R}$). But this f has an extension to $\mathbb{R}^*$,

$$g(x) = \begin{cases} f(x), & \text{for } x > 0, \\ f(-x), & \text{for } x < 0. \end{cases}$$

(3) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^c$ for $x > 0$. Then $g(xy) = g(x)g(y)$ for $x, y \in \mathbb{R}_+$, and this g has two extensions h_1, h_2 to $\mathbb{R}$, namely $h_1(x) = |x|^c$ for $x \in \mathbb{R}$; $h_2(x) = (\operatorname{sgn} x)\ |x|^c\ , x \in \mathbb{R}$ ($h_1|\mathbb{R}_+ = g$ and $h_2|\mathbb{R}_+ = g$).

Now we will consider more results on extensions.

Result 1.71. [229, 558, 95]. *Suppose G and H are Abelian groups with H divisible and S is a subsemigroup of G. Let $f : S \to H$ be a homomorphism. Then there exists a homomorphism $h : G \to H$ such that $h|S = f$.*

Remark 1.71a. This result fails to hold if H is not divisible. For example, let $G = H = Z$ and $S = 2Z$. Then Z is not divisible and $f : 2Z \to Z$ defined by $f(2n) = n$ for $n \in Z$ is a homomorphism with no extension $h : Z \to Z$ (such an extension has to satisfy $h(3) + h(3) = h(6) = f(6) = 3$, and $2h(3) = 3$ has no solution in Z).

Further, the Aczél and Erdős [45] problem has a solution. Also, if S does not generate G, then the extension h need not be unique [229].

1.7.1 Extension of the Additive Function on [0, 1]

Problem. $f : I \to \mathbb{R}$ such that

$$f(x + y) = f(x) + f(y), \quad \text{for } x, y, x + y \in I, \tag{1.108}$$

holds. Does there exist an additive $A : \mathbb{R} \to \mathbb{R}$ satisfying (A) such that $A|I = f$? If it exists, is A unique?

These types of problems arise in information theory, statistics, and other fields. The answer to the problem above is affirmative and was first proved by Daróczy and Losonczi [203]; also see [43, 55]. Here we present a slightly different proof, first by extending f to $[-1, 1]$ and then to $\mathbb{R}$.

Define $g : [-1, 1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x), & \text{for } x \in [0, 1], \\ -f(-x), & \text{for } x \in [-1, 0]. \end{cases}$$

First note that $g(-x) = -g(x)$ for $x \in [-1, 1]$ and $g(0) = 0$. We will show that

$$g(x + y) = g(x) + g(y), \quad \text{for } x, y, x + y \in [-1, 1]. \tag{1.108a}$$

Obviously (1.108a) holds for $x, y > 0$ with $x + y \in I$.

Let $x < 0,\ y < 0$ with $-1 < x + y < 0$. Then

$$g(x + y) = -f(-x - y) = -(f(-x) + f(-y)) = g(x) + g(y).$$

Let $x < 0,\ y > 0$. Then two cases arise according to whether $x+y < 0$ or $x+y > 0$. First take the case $x + y < 0$. Then

$$g(x) = g(x + y - y) = g(x + y) + g(-y) = g(x + y) - g(y).$$

Now consider the case $x + y > 0$. Then

$$g(y) = g(x + y - x) = g(x + y) + g(-x) = g(x + y) - g(x).$$

The cases where $x > 0$ and $y < 0$ with $x + y \in [-1, 1]$ can be treated similarly. Thus (1.108a) holds in a "hexagonal" or "diamond" neighbourhood of $(0, 0)$. This g is unique.

Now we extend g on $[-1, 1]$ to A on $\mathbb{R}$ as in Aczél and Losonczi [55].

Define $A : \mathbb{R} \to \mathbb{R}$ by

$$A(x) = ng\left(\frac{x}{n}\right), \quad \text{for } x \in \mathbb{R},\ n \in Z_+,\ \frac{x}{n} \in [-1, 1].$$

This A is well defined since, for $\frac{x}{m} \in [-1, 1]$, $n \cdot \frac{x}{mn}$, $m \cdot \frac{x}{mn}$ are in $[-1, 1]$ and $g(nt) = ng(t)$ for $t, nt \in [-1, 1]$. Now

$$mg\left(\frac{x}{m}\right) = mng\left(\frac{x}{mn}\right) = nmg\left(\frac{x}{mn}\right) = ng\left(\frac{x}{n}\right).$$

Also $A(x) = g(x)$ for $x \in [-1, 1]$ (take $n = 1$).

Finally, for $x, y \in \mathbb{R}$, choose $n \in Z_+$ such that $\frac{x}{n}, \frac{y}{n}, \frac{x+y}{n}$ are in $[-1, 1]$. Hence

$$A(x + y) = ng\left(\frac{x + y}{n}\right) = n\left[g\left(\frac{x}{n}\right) + g\left(\frac{y}{n}\right)\right] = A(x) + A(y).$$

Further, A is unique, for if A_1 is another extension on $\mathbb{R}$ of g on $[-1, 1]$, then for arbitrary $x \in \mathbb{R}$, $n \in Z_+$, with $\frac{x}{n} \in [-1, 1]$, it follows that

$$A_1(x) = nA_1\left(\frac{x}{n}\right) = ng\left(\frac{x}{n}\right) = A(x) \quad \text{for } x \in \mathbb{R}.$$

This confirms that the extension is unique. □

Remark 1.72. The result above holds on any interval $[0, s]$; that is, if $f : [0, s] \to \mathbb{R}$ satisfies (1.108) for $x, y, x + y \in [0, s]$, then f has an additive extension on $\mathbb{R}$.

Remark 1.73. Unique (additive) extensions exist in the following cases, too. If $f :]-r, r[\to \mathbb{R}$ $(r > 0)$ is additive either on $H_r = \{(x, y) : x, y, x + y \in]-r, r[\}$ (hexagonal neighbourhood of $(0, 0)$) or on $K_r = \{(x, y) : x^2 + y^2 < r^2\}$ (circular neighbourhood of $(0, 0)$), then there exists a unique $A : \mathbb{R} \to \mathbb{R}$ satisfying (A) such that A is an extension of f (see [203]).

We will see the use of these in applications later. Now we use then to prove the following result arising in information theory.

First we prove the following extension result similar to the problem discussed earlier, but this time the domain is the open unit interval as follows.

Theorem 1.74. *Let* $f :]0, 1[\to \mathbb{R}$ *satisfy*

$$f(x + y) = f(x) + f(y) \quad \textit{for } x, y, x + y \in]0, 1[.$$

Then there exists a $g : [0, 1] \to \mathbb{R}$ *satisfying*

$$g(x + y) = g(x) + g(y) \quad \text{for } x, y, x + y \in [0, 1]$$

such that $g|]0, 1[= f$. *Thus there exists an additive* $A : \mathbb{R} \to \mathbb{R}$ *that is an extension of* f. *Clearly this extension is unique.*

Proof. First note that $kf(t) = f(kt)$ for t, kt in $]0, 1[$, where k is a positive integer. For integers $n, m > 1$, claim $nf\left(\frac{1}{n}\right) = mf\left(\frac{1}{m}\right)$.

Indeed, since $\frac{1}{mn}, \frac{1}{n} = m \cdot \frac{1}{mn}, \frac{1}{m} = n \cdot \frac{1}{mn}$ are all in $]0, 1[$,

$$nf\left(\frac{1}{n}\right) = n \cdot mf\left(\frac{1}{mn}\right) = m \cdot nf\left(\frac{1}{mn}\right) = mf\left(\frac{1}{m}\right).$$

Now define g on $[0, 1]$ by

$$g(x) = \begin{cases} f(x), & \text{for } x \in]0, 1[, \\ 0, & \text{for } x = 0, \\ nf\left(\frac{1}{n}\right), & \text{for } x = 1. \end{cases}$$

Claim $g(x + y) = g(x) + g(y)$ for $x, y, x + y$ in I.

For $x, y, x + y$ in $]0, 1[$, it is true.

For $x = 0,\ y \in]0, 1[,\ x + y \in]0, 1[$, and

$$g(x + y) = g(y) = g(0) + g(y) = g(x) + g(y).$$

For $x = 0,\ y = 1,\ g(x + y) = g(1) = g(0) + g(1) = g(x) + g(y)$.

Finally, for x, y in $]0, 1[$ with $x + y = 1$, choose a positive integer n such that $\frac{x}{n}, \frac{y}{n}$, and $\frac{x+y}{n}$ are in $]0, 1[$. Since $f\left(\frac{x+y}{n}\right) = f\left(\frac{x}{n}\right) + f\left(\frac{y}{n}\right)$, $nf\left(\frac{x+y}{n}\right) = nf\left(\frac{x}{n}\right) + nf\left(\frac{y}{n}\right)$ and $g(x+y) = g(1) = nf\left(\frac{1}{n}\right) = nf\left(\frac{x}{n}\right) + nf\left(\frac{y}{n}\right) = g(x) + g(y)$. Thus g extends f on $]0, 1[$ to $[0, 1]$. So, by the previous result, there is an additive A on $\mathbb{R}$ that extends g on $[0, 1]$ and hence extends f on $]0, 1[$. □

Remark 1.75. The result above holds on any interval $]0, s[$; that is, if $f :]0, s[\to \mathbb{R}$ satisfies $f(x + y) = f(x) + f(y)$ for $x, y, x + y \in]0, s[$, then f has an additive extension on $\mathbb{R}$.

Theorem 1.76. *The functions* $f_i :]0, 1[\to \mathbb{R}$ *satisfy the functional equation*

$$\sum_{i=1}^{n} f_i(p_i) = 0 \quad \left(0 < p_i < 1, \sum_{i=1}^{n} p_i = 1\right) \tag{1.109}$$

for arbitrary (*but fixed*) $n \geq 3$ *if, and only if, there exists an additive* $A : \mathbb{R} \to \mathbb{R}$ *such that*

$$f_i(x) = A(x) + b_i \quad \text{for } x \in]0, 1[, \tag{1.110}$$

where b_i $(i = 1, 2, \ldots n)$ *are constants with* $A(1) + \sum_{i=1}^{n} b_i = 0$.

Proof. We choose $a \in]0,1[$ such that $(n-3)a < 1,\ b = 1-(n-3)a$. Putting $p_1 = x,\ p_2 = y,\ p_3 = b-x-y,\ p_4 = \cdots = p_n = a$ in (1.109), we obtain

$$f_1(x) + f_2(y) + f_3(b-x-y) = k \quad \text{for } x, y, x+y \in]0,b[,$$

where k is a constant. Interchanging x and y, we have $f_1(x) - f_2(x) = f_1(y) - f_2(y) = c$ (say), so that the equation above can be written as

$$f_1(x) + f_1(y) - c + f_3(b-x-y) = k, \tag{1.111}$$

(1.90)

with k a constant, for $x, y, x+y$ in $]0,b[$.

For $x, y, z, x+y+z \in]0,b[$, (1.111) yields

$$\begin{aligned} f_3(b-x-y-z) &= k + c - f_1(x+y) - f_1(z) \\ &= k + c - f_1(x) - f_1(y+z); \end{aligned}$$

that is, $f_1(x+y) - f_1(x) = f_1(z+y) - f_1(z) = d(y)$ (say).

Hence,

$$f_1(x+y) = f_1(x) + d(y) = f_1(y) + d(x)$$

so that $f_1(x) - d(x) = c_1$ and

$$f_1(x+y) = f_1(x) + f_1(y) - c \quad \text{for } x, y, x+y \in]0,b[.$$

Thus, by Remark 1.75, there exists an additive $A : \mathbb{R} \to \mathbb{R}$ such that

$$f_1(x) = A(x) + b_1.$$

Since f_1 is arbitrary, the form above holds for all f_i $(i = 1, 2, \ldots n)$ for $x \in [0,b]$. Thus $f_i(x) = A(x) + b_i$ $(i = 1, 2, \ldots n)$. Now, given $x \in]0,1[$, we can always find a "b" satisfying the condition above (A and b_i are independent of b because of the additivity of A). Hence the solution of (1.109) is given by (1.110) for $x \in]0,1[$ with $A(1) + \sum_{i=1}^{n} b_i = 0$.

This proves the theorem. □

1.7.2 Quasiextensions

As pointed out in the examples, there an additive extension may not exist even in some very simple cases. Suppose

$$f(x) = \begin{cases} 5x+2, & \text{for } x \in]1,4[, \\ 5x+3, & \text{if } 5 < x < 7, \\ 5x+5, & \text{if } 6 < x < 11. \end{cases} \tag{1.112}$$

Then the function $f : D_f =]1, 4[\cup]5, 7[\cup]6, 11[\to \mathbb{R}$ is additive for $(x, y) \in D =]1, 4[\times]5, 7[$, but there is no additive $A : \mathbb{R} \to \mathbb{R}$ that extends f (that is, $A|_{D_f} = f$) for such an A would be continuous on D_f and $A(x) = cx$, for $x \in \mathbb{R}$, which contradicts $A|_{D_f} = f$.

Nevertheless, $A(x) = 5x$ is a quasi extension of f; that is, A differs from f on $]1, 4[$, $]5, 7[$, and $]6, 11[$ by a constant and each is additive on $\mathbb{R}$.

However, if we let

$$\text{(1.112a)} \qquad \begin{cases} f(z) = 5z + 5 & \text{for } z \in]6, 11[, \\ g(x) = 5x + 2 & \text{when } 1 < x < 4, \text{ and} \\ h(y) = 5x + 3 & \text{if } y \in]5, 7[, \end{cases}$$

then

$$f(x + y) = g(x) + h(y) \quad \text{for } (x, y) \in]1, 4[\times]5, 7[,$$

and there exist unique functions $F, G, H : \mathbb{R} \to \mathbb{R}$ such that

$$F|_{]6,11[} = f, \ G|]1, 4[= g, \ H|]5, 7[= h,$$

and

$$F(x + y) = G(x) + H(y) \text{ for all } x, y \in \mathbb{R}.$$

These lead to the notion of *quasiextensions*. Let $\mathbf{X}(+)$, $K(*)$ be groupoids and $\mathbf{Y}(+)$ be a commutative group. Given a nonempty subset $D \subseteq \mathbf{X} \times \mathbf{X}$, we denote $D_x = \{x : (x, y) \in D \text{ for some } y \in \mathbf{X}\}$, $D_y = \{y : (x, y) \in D \text{ for some } x \in \mathbf{X}\}$, $D_0 = \{x + y : (x, y) \in D\}$, and $D_1 = D_0 \cup D_x \cup D_y$.

We say that a function f is additive on the set D if $f : D_1 \to \mathbf{Y}$ satisfies (A) for $(x, y) \in D$. A function $A : \mathbf{X} \to \mathbf{Y}$ is called an (*additive*) *extension* of a function f on D if (A) holds for A on $\mathbf{X}$ and $A|_{D_1} = f$.

If there is an additive $F : \mathbf{X} \to \mathbf{Y}$ and a point $(a, b) \in D$ such that

$$\begin{aligned} f(x + y) - f(a) - f(b) &= F(x + y) - F(a) - F(b) \quad \text{for } x + y \in D_0, \\ f(x) - f(a) &= F(x) - F(a) \quad \text{for } x \in D_x, \\ f(y) - f(b) &= F(y) - F(b) \quad \text{for } y \in D_y, \end{aligned}$$

hold, we say that F is a *quasiextension* of f.

It is easy to check that this definition is independent of the choice of (a, b) (for $(c, d) \in D$, take $x = c$ and $y = d$ in the equations above).

1.7.3 Extension of the Pexider Equation

For $f : D_0 \to K$, $g : D_x \to K$, and $h : D_y \to K$ (where $K(*)$ is a groupoid) such that

$$f(x + y) = g(x) * h(y) \quad \text{for all } (x, y) \in D \subseteq \mathbf{X} \times \mathbf{X}$$

(that is, f, g, h satisfy a Pexider equation), if there are functions $F, G, H : \mathbf{X} \to K$ with $F|D_0 = f,\ G|D_x = g,\ H|D_y = h$, and $F(x+y) = G(x) * H(y)$ for all $x, y \in \mathbf{X}$, (F, G, H) is said to be a *Pexider extension* to $\mathbf{X}$ of (f, g, h) on D.

The function f in (1.112) has quasiextension to $\mathbb{R}$, and (f, g, h) in (1.112a) has Pexider extension (not only quasiextension) to $\mathbb{R}^2$. As for quasiextension, the basic results are from Daróczy and Losonczi [203], Rimán [703], Aczél [30], and Radó and Baker [676]: If f is additive on the circular disk $D = \{(x, y) : (x-a)^2 + (y-b)^2 < r^2\}$, then there exists a quasiextension A of f to $\mathbb{R}^2$; from any open connected set D in $\mathbb{R}^2$ there exists a unique quasiextension of the Cauchy equation on D to $\mathbb{R}^2$ [203]. Let $D \subseteq \mathbb{R}^{2n}$ be an open, connected set, $D_x = \{x \in \mathbb{R}^n : (x, y)\}$ in D for some $y \in \mathbb{R}^n$, $D_y = \{y \in \mathbb{R}^n : \text{for some } x, (x, y) \in D\}$, $D_0 = \{z \in \mathbb{R}^n,\ z = x+y, (x, y) \in D\}$, $D' \subseteq \mathbb{R}^n$ such that $D_x \cup D_y \cup D_0 \subseteq D'$. If $f : D' \to \mathbb{R}$ satisfies the additive equation (A) on D, then there exists a unique additive $A : \mathbb{R}^n \to \mathbb{R}$ and constants $b, c \in \mathbb{R}$ such that

$$f(x) = A(x) + b \quad \text{for } x \in D_x, \qquad f(y) = A(y) + c \text{ for } y \in D_y,$$

and $f(x+y) = A(x+y) + b + c$ for $x + y \in D_0$ hold.

As regards Pexider extension, in general the extension may not exist. For example, if $f, g : \mathbb{R} \to \mathbb{R}$ and $h : 2Z \to \mathbb{R}$ are defined by $f(x) = k + \sin \pi x$, $g(x) = \sin \pi x$ for $x \in \mathbb{R}$, $h(2n) = k$ for $n \in Z$, and k is constant, then $f(x+2n) = g(x) + h(2n)$ for all $(x, 2n) \in \mathbb{R} \times 2Z$, but there is no H from $\mathbb{R}$ to $\mathbb{R}$ such that $H|2Z = h$ and $f(x+y) = g(x) + H(y)$ for all $x, y \in \mathbb{R}$ (see [676, 30]).

The following result is due to Baker and Radó [676].

Result 1.77. *Let U be a nonempty open, connected subset of $\mathbb{R}^n \times \mathbb{R}^n$ and K be an additive Abelian group. Suppose $f : U_0 \to K$, $g : U_x \to K$, and $h : U_y \to K$ satisfy $f(x+y) = g(x) + h(y)$ for all $(x, y) \in U$. Then there exist unique functions $F, G, H : \mathbb{R}^n \to K$ that extend f, g, h, respectively, and $F(x+y) = G(x) + H(y)$, for all $x, y \in \mathbb{R}^n$. Further, there exists a unique additive $A : \mathbb{R}^n \to K$ and constants $a, b \in K$ such that $F(x) = A(x) + a + b$, $G(x) = A(x) + a$, and $H(x) = A(x) + b$ for all $x \in \mathbb{R}^n$.*

Recall that f, g, h satisfying a Pexider equation is called a homotopism (f, g, h). Regarding a homotopism (f, g, h) of a semigroup S into a group K, we prove the following result due to Radó [675] which does not involve topology.

Theorem 1.78. *Let $G_1(\cdot)$ and $G_2(\cdot)$ be groups and S a subsemigroup of G_1 such that $G_1 = SS^{-1} = S^{-1}S$. Let (f, g, h) be a homotopism of S into G_2. Then there exists a homotopism (F, G, H) from G_1 into G_2 and a (multiplicative) $M : G_1 \to G_2$ satisfying $M(xy) = M(x)M(y)$ for $x, y \in G_1$ and constants $b, c \in G_2$ such that*

$$F(x) = bM(x)c,\ G(x) = bM(x),\ H(x) = M(x)c,$$

$$F|SS = f|SS,\ G|S = g, \quad \text{and} \quad H|S = h.$$

Proof. Suppose $f, g, h : S \to G_2$ satisfy $f(st) = g(s)h(t)$ for $s, t \in S$.

Define $M : G_1 \to K$ by

$$M(x) = M(st^{-1}) := h(s)h(t)^{-1},$$

where $x = st^{-1}$ for some $s, t \in S$. First, we will show that M is defined unambiguously and then that $M(xy) = M(x)M(y)$ holds for $x, y \in G_1$.

Suppose $st^{-1} = u\upsilon^{-1}$ and $st^{-1} = w^{-1}r$, for $u, \upsilon, w, r \in S$. Then $ws = rt$, $wu = r\upsilon$, and

$$\begin{aligned} f(ws) = g(w)h(s) = f(rt) = g(r)h(t), \\ f(wu) = g(w)h(u) = f(r\upsilon) = g(r)h(\upsilon). \end{aligned}$$

Eliminating $g(w)$ and $g(r)$, we have $h(\upsilon)h(u)^{-1}h(s) = h(t)$; that is, $h(s)h(t)^{-1} = h(u) \cdot h(\upsilon)^{-1}$. So, M is well defined.

Now, we prove that $M(st) = M(s)M(t)$ and $M(st^{-1}) = M(s)M(t)^{-1}$ for $s, t \in S$.

For $s, t \in S$,

$$M(s) = M(st \cdot t^{-1}) = h(st) \cdot h(t)^{-1};$$

that is,

$$h(st) = M(s)h(t) \quad \text{for } s, t \in S.$$

For $r, s, t \in S$,

$$\begin{aligned} h(str) &= M(st) \cdot h(r) \\ &= M(s) \cdot h(tr) = M(s)M(t)h(r); \end{aligned}$$

that is, $M(st) = M(s)M(t)$, for $s, t \in S$ (M is multiplicative on S).

Let $u\upsilon^{-1} = t^{-1}r$ for $u, \upsilon, t, r \in S$. Then $st^{-1} = st^{-1}rr^{-1} = s \cdot u\upsilon^{-1} \cdot r^{-1}$ for $s, t, r, u, \upsilon \in S$. Now $tu = r\upsilon$, $h(tu) = h(r\upsilon)$; that is, $M(t)h(u) = M(r)h(\upsilon)$ or $M(r)^{-1}M(t) = h(\upsilon)h(u)^{-1}$.

$$\begin{aligned} M(st^{-1}) &= M(su \cdot (r\upsilon)^{-1}) = h(su) \cdot h(r\upsilon)^{-1} \\ &= M(s)h(u)h(\upsilon)^{-1}M(r)^{-1} \\ &= M(s) \cdot M(t)^{-1}M(r) \cdot M(r)^{-1} = M(s) \cdot M(t)^{-1}. \end{aligned}$$

Let $x, y \in G_1$. Then there are $r, s, t, u, \upsilon, w \in S$ such that $x = rs^{-1}$, $y = tu^{-1}$, $s^{-1}t = \upsilon \cdot w^{-1}$, and $tw = s\upsilon$, with $M(t)M(w) = M(s)M(\upsilon)$.

Finally,

$$\begin{aligned} M(xy) &= M(rs^{-1} \cdot tu^{-1}) = M(r \cdot \upsilon w^{-1}u^{-1}) \\ &= M(r\upsilon) \cdot M(uw)^{-1} \\ &= M(r)\underline{M(\upsilon) \cdot M(w)^{-1}}M(u)^{-1} \\ &= M(r)\underline{M(s)^{-1} \cdot M(t)} \cdot M(u)^{-1} = M(x)M(y). \end{aligned}$$

If we define

$$F(x) = bM(x)c,\ G(x) = bM(x),\ H(x) = M(x)c, \quad \text{for } x \in G_1,$$

then

$$F(xy) = G(x)H(y) \quad \text{holds for} \quad x, y \in G_1.$$

Fix t_0, r_0 in S and take $c = M(t_0)^{-1}h(t_0)$ and $b = g(r_0)M(r_0)^{-1}$. By the definition of M, we have, for $s \in S$, $h(s) = M(st_0^{-1}) \cdot h(t_0) = M(s)c$ and $H|S = h$. Suppose $st^{-1} = r^{-1}u$ for $r, s, t, u \in S$. Then $rs = ut$, $f(rs) = g(r)h(s) = f(ut) = g(u)h(t)$, and $h(s)h(t)^{-1} = g(r)^{-1}g(u)$, so that $M(r^{-1}u) = M(st^{-1}) = h(s)h(t)^{-1} = g(r)^{-1}g(u)$. As before, $M(r^{-1}u) = M(r)^{-1}M(u)$. Now $g(u) = g(r_0)M(r_0)^{-1} \cdot M(u) = bM(u)$ for $u \in S$ and $G|S = g$. Finally, for $s, t \in S$,

$$f(st) = g(s)h(t) = bM(s) \cdot M(t)c = bM(st)c = F(st) \quad \text{and} \quad F|SS = f|SS.$$

This completes the proof of this result. □

We now extend functions satisfying the Cauchy and Pexider equations where one of the variables is contained in an arbitrary group G or a subsemigroup S of G and the other is confined to certain nonempty subsets C of G and the values of the functions are elements of a group H.

Denote $\langle C\rangle$ = the subgroup generated by C.

Result 1.79. (Mokanski [637]). *Let $G(\cdot)$ and $H(0)$ be groups and C a nonempty subset of G. Suppose $f : G \to H$ satisfies $f(x \cdot c) = f(x) \circ f(c)$, for $x \in G,\ c \in C$. Then $f(x \cdot c') = f(x) \circ f(c')$ holds for $x \in G$ and $c' \in \langle C\rangle$.*

Note that $\langle C\rangle$ is the maximal set for which $f(xc') = f(x) \circ f(c')$ is valid, as can be seen from the following example.

Take $G = H = Z(+),\ C = \{4\}$. Then $\langle C\rangle = 4Z$. Set

$$f(x) = \begin{cases} x, & \text{if } x \text{ is even,} \\ x + 1, & \text{if } x \text{ is odd.} \end{cases}$$

It is clear that $f(x + 4) = f(x) + f(4)$ for $x \in Z,\ 4 \in C$. Further, $f(x + 4n) = f(x) + f(4n)$ for $x \in Z,\ 4n \in \langle C\rangle$. But $8 = f(8) = f(5+3) \neq f(5) + f(3) = 10$.

If $\langle C\rangle = G$ in the result above, then $f(xy) = f(x) \circ f(y)$ for all $x, y \in G$.

Result 1.80. [637]. *Let $G(\cdot)$ and $H(0)$ be groups, S a subsemigroup of G, and $C \subseteq S$. If $g : S \to H$ satisfies $g(s \cdot c) = g(s) \circ g(c)$, for $s \in S,\ c \in C$, then $g(sc') = g(s) \circ g(c')$ is valid for $s \in S$ and $c' \in C'$, the subsemigroup generated by C. Further, if $G = SS^{-1}$ and $C' = S$, then there exists a unique homomorphism ϕ from G into H such that $\phi|S = g$.*

The Aczél and Erdős result [45] can be deduced easily from this result also.

Result 1.81. [637]. *Let $G(\cdot)$ and $H(0)$ be groups and S a subsemigroup of G such that $\langle S\rangle = G$. Suppose $f, g : G \to H,\ h : S \to H$ are such that $f(xs) = g(x) \circ h(s)$ holds for $x \in G,\ s \in S$. Then there exists a homomorphism $\phi : G \to H$ such that $f(x) = \phi(x) \circ b,\ g(x) = \phi(x) \circ a,\ h(s) = a_0^{-1}\phi(s) \circ b$ for $x \in G,\ s \in S$, and a, b are constants in H.*

Again the Aczél-Erdős result [45] can be deduced from this result.

1.7.4 Extension of the Logarithmic Equation

Theorem 1.82. *Let* $f : T \to \mathbb{R}$, *where* $T = \mathbb{R}_+^*$ *or* $]0, 1]$ *or* $[1, \infty[$ *or* $]\alpha, \infty[$ $(\alpha > 0)$ *satisfy* $f(xy) = f(x) + f(y)$ *for* $x, y \in T$. *Then* f *has an extension* $L : \mathbb{R}^* \to \mathbb{R}$ *satisfying* (L).

Proof. In the case $T = \mathbb{R}_+^*$, $L(x) = f(|x|)$ for $x \in \mathbb{R}^*$ serves the purpose.

In the remaining cases, T is a subsemigroup of $\mathbb{R}_+^*(\cdot)$ such that $\mathbb{R}_+^* = TT^{-1} = T^{-1}T$. Then, by Theorem 1.67, f has an extension (logarithmic) to $\mathbb{R}_+^*$ and thus to $\mathbb{R}^*$.

Without using this theorem, we give the following simple, direct proof for the case $T = [1, \infty[$.

Let $f : [1, \infty[\to \mathbb{R}$ satisfy the logarithmic equation (L). Define $L : \mathbb{R}_+^* \to \mathbb{R}$ by

$$L(x) = \begin{cases} f(x), & \text{for } x \geq 1, \\ -f\left(\frac{1}{x}\right), & \text{for } x \in]0, 1]. \end{cases}$$

Clearly $L\left(\frac{1}{x}\right) = -L(x)$ for $x \in \mathbb{R}_+^*$ and $L(xy) = L(x) + L(y)$ for $x, y \geq 1$ and $x, y \in]0, 1]$.

Let $x > 1$ and $y < 1$. Two cases arise according to whether $xy \geq 1$ or $xy \leq 1$. When $xy = 1$, $L(xy) = L(1) = f(1) = 0 = L(x) + L\left(\frac{1}{x}\right) = L(x) + L(y)$. Let $xy > 1$. Then $L(x) = L\left(xy \cdot \frac{1}{y}\right) = L(xy) + L\left(\frac{1}{y}\right) = L(xy) - L(y)$. When $xy < 1$, $L(y) = L\left(xy \cdot \frac{1}{x}\right) = L(xy) + L\left(\frac{1}{x}\right) = L(xy) - L(x)$. The cases where $x < 1$ and $y > 1$ can be argued similarly. Thus L is logarithmic on $\mathbb{R}_+^*$ and $L \mid [1, \infty[= f$.

□

As a consequence of Theorem 1.82, we obtain the following results.

Theorem 1.83. *A function* $f : [1, \infty[\to \mathbb{R}$ *satisfying*

$$f(tn) = f(t) + f(n), \quad \text{for } t \in [1, \infty[,\ n \in Z_+^*, \tag{1.113}$$

satisfies the equation $f\left(t\frac{m}{n}\right) = f(t) + f\left(\frac{m}{n}\right)$ *for* $t, \frac{m}{n} \in [1, \infty[$, $m, n \in Z_+^*$. *Further, if* f *is continuous, then* $f(ts) = f(t) + f(s)$ *for all* $s, t \in [1, \infty[$, *and there exists a continuous logarithmic extension* $L :]0, \infty[\to \mathbb{R}$ *such that* $L(t) = f(t) = c \log t$ *for* $t \in [1, \infty[$, *where* c *is an arbitrary constant.*

Proof. Take any rational $\frac{m}{n}$ in $]1, \infty[$. Then $m, n \in Z_+^*$ and $m > n$. Let $t \in]1, \infty[$. First, take $t > n$. Then $f(t) = f\left(\frac{t}{n} \cdot n\right) = f\left(\frac{t}{n}\right) + f(n)$, using (1.113), so that

$$f\left(\frac{t}{n}\right) = f(t) - f(n);\ f\left(\frac{m}{n}\right) = f(m) - f(n).$$

Further,

$$f\left(t\cdot\frac{m}{n}\right)=f\left(\frac{t}{n}\cdot m\right)=f\left(\frac{t}{n}\right)+f(m)\qquad\text{(using (1.113))}$$
$$=f(t)-f(n)+f(m)=f(t)+f\left(\frac{m}{n}\right).$$

For $t<n$, there exists an integer k such that $kt>n$, and

$$f\left(kt\cdot\frac{m}{n}\right)=f(kt)+f\left(\frac{m}{n}\right)=f(t)+f(k)+f\left(\frac{m}{n}\right)$$
$$=f(k)+f\left(t\frac{m}{n}\right).$$

Thus $f\left(t\cdot\frac{m}{n}\right)=f(t)+f\left(\frac{m}{n}\right)$ for $t,\frac{m}{n}\in[1,\infty[$.

If f is continuous, then $f(ts)=f(t)+f(s)$, $t,s\in[1,\infty]$. Further, there is a unique continuous $L:\mathbb{R}_+^*\to\mathbb{R}$ satisfying (L) such that $L|[1,\infty[=f$ and $f(t)=c\log t$, for $t\in[1,\infty[$. □

Remark. [40]. If f is a strictly monotonic function satisfying (1.113), then also $f(t)=c\log t$ for $t\in[1,\infty[$.

Result 1.84. *The general solutions $f,g,h:]0,1]\to\mathbb{R}$ of the Pexider equation*

$$f(xy)=g(x)+h(y),\quad \textit{for } x,y\in\,]0,1], \tag{1.114}$$

are given by $f(x)=L(x)+a+b$, $g(x)=L(x)+a$, $h(x)=L(x)+b$, for $x\in\,]0,1]$, where $L:\mathbb{R}_+^*\to\mathbb{R}$ is logarithmic and a and b are arbitrary constants.

Proof. Setting $y=1$ and $x=1$ separately in (1.114), we get $f(x)=g(x)+a$ and $f(y)=b+h(y)$, so that (1.114) becomes $f(xy)=f(x)+f(y)-a-b$, which is logarithmic on $]0,1]$. The use of Theorem 1.82 yields the result sought.

1.7.5 Exponential Extension

Result 1.85. *Let $f:T\to\mathbb{R}^*$, where $T=]0,\infty[=\mathbb{R}_+^*$ or $]\alpha,\infty[$ $(\alpha>0)$ satisfy $f(x+y)=f(x)f(y)$. Then there is an $E:\mathbb{R}\to\mathbb{R}^*$ satisfying* (E) *such that $E|T=f$.*

Noting that $T(+)$ is a semigroup of $\mathbb{R}(+)$ with $\mathbb{R}=T-T$, use Theorem 1.67.

Theorem 1.86. *Suppose $f:[0,1]\to\mathbb{R}^*$ is such that*

$$f(x+y)=f(x)f(y) \tag{1.115}$$

holds for $x,y,x+y\in I$. Then there is an $E:\mathbb{R}\to\mathbb{R}^*$ satisfying (E) *such that $E|I=f$.*

Proof. We do the extension by two stages, first to $[-1,1]$ and then to $\mathbb{R}$.

Define $g : [-1, 1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x), & \text{for } x \in [0, 1], \\ \frac{1}{f(-x)}, & \text{for } x \in [-1, 0]. \end{cases}$$

Note that $g(-x) = \frac{1}{g(x)}$, for $x \in [-1, 1]$, and $g(x + y) = g(x)g(y)$, for $x, y \geq 0$, with $x + y \in I$ and $x, y \leq 0$ with $-1 \leq x + y \leq 0$. Consider the case where x and y are of opposite signs; that is, say $x < 0$ and $y > 0$. Then either $x + y > 0$ or $x + y < 0$. In the former case,

$$g(y) = g(x + y - x) = g(x + y) \cdot g(-x).$$

In the latter case, $g(x) = g(x + y - y) = g(x + y)g(-y)$. Thus g satisfies (1.115) on $[-1, 1]$. Now we extend to $\mathbb{R}$.

For $t \in \mathbb{R}$, define

$$E(t) = \left[g\left(\frac{t}{n}\right) \right]^n,$$

where $\frac{t}{n} \in [-1, 1]$, for some integer n. This definition is independent of n.

Note first that $g(nx) = [g(x)]^n$ for $x, nx \in [-1, 1]$. Suppose $\frac{t}{m} \in [-1, 1]$. Then $n \cdot \frac{t}{mn}$ and $m \cdot \frac{t}{mn} \in [-1, 1]$ and

$$\left[g\left(\frac{t}{m}\right) \right]^m = \left[g\left(n \cdot \frac{t}{mn}\right) \right]^m = \left[g\left(\frac{t}{mn}\right) \right]^{mn} = \left[g\left(\frac{t}{n}\right) \right]^n.$$

Let $t, u \in \mathbb{R}$. Choose n sufficiently large that $\frac{t}{n}, \frac{u}{n}, \frac{t+u}{n} \in [-1, 1]$. Then

$$E(t + u) = \left[g\left(\frac{t+u}{n}\right) \right]^n = \left[g\left(\frac{t}{n}\right) \right]^n \cdot \left[g\left(\frac{u}{n}\right) \right]^n = E(t) \cdot E(u).$$

This proves the theorem. □

1.7.6 Multiplicative Extension

Suppose h satisfies the functional equation

$$h(pq) = h(p)h(q) \quad \text{for } p, q \in]0, 1]. \tag{1.116}$$

If $h(1) = 0$, then $h(p) = 0$ for all p (take $q = 1$). So, let us consider h to be not identically zero. If $h(p_0) = 0$ for $p_0 \in]0, 1[$, then $h(tp_0) = 0$ for $t \in]0, 1]$ and $h(q) = 0$ for $q > p_0$ (choose n sufficiently large that $q^n < tp_0$). So, we can assume that h is nowhere zero and consider $h :]0, 1] \to \mathbb{R}^*$ satisfying (1.116).

Result 1.87. *A function $f : T \to \mathbb{R}^*$, where $T =]0, 1]$ or $[1, \infty[$ satisfying $f(ts) = f(t)f(s)$, for $s, t \in T$, has an extension $M : \mathbb{R}^*_+ \to \mathbb{R}^*$ satisfying* (M). *If f is strictly monotonic, then $f(t) = t^C$* [43].

Observe that $T(\cdot)$ is a semigroup and $\mathbb{R}^*_+ = TT^{-1}$, and apply Theorem 1.67.

Theorem 1.88. *Suppose* $h : [1, \infty[\to \mathbb{R}^*$ *satisfies*

$$h(tn) = h(t)h(n) \quad \text{for } t \in [1, \infty[,\ n = 1, 2, \ldots. \tag{1.117}$$

Then $h\left(t\frac{m}{n}\right) = h(t)h\left(\frac{m}{n}\right)$ *holds for* $t, \frac{m}{n} \in [1, \infty[$. *Further, if* h *is continuous, then there exists an* $M : \mathbb{R}_+^* \to \mathbb{R}^*$ *satisfying* (M) *such that* $M|[1, \infty[= f$ *and* $M(t) = f(t) = t^C$, *for* $t \in [1, \infty[$.

Proof. Let $\frac{m}{n} \in]1, \infty[$. Then $m > n$. Choose $t \in [1, \infty[$ with $t > n$. Then $h(t) = h\left(\frac{t}{n} \cdot n\right) = h\left(\frac{t}{n}\right) \cdot h(n)$ using (1.117), so that $h\left(\frac{t}{n}\right) = \frac{h(t)}{h(n)}$, $h\left(\frac{m}{n}\right) = \frac{h(m)}{h(n)}$.

Now

$$\begin{aligned} h\left(t \cdot \frac{m}{n}\right) = h\left(\frac{t}{n} \cdot m\right) &= h\left(\frac{t}{n}\right) \cdot h(m) \qquad \text{(by (1.117))} \\ &= h(t) \cdot \frac{h(m)}{h(n)} = h(t)h\left(\frac{m}{n}\right). \end{aligned}$$

As in Theorem 1.82, the identity above is valid for all $t \in [1, \infty[$.

If h is continuous, clearly $h(st) = h(s)h(t)$ for $s, t \in [1, \infty[$. The rest follows from Theorem 1.82, Theorem 1.67, and Theorem 1.49. □

1.7.7 Extension of Derivations

Result 1.89. [558]. *Let* P *be an integral domain,* F *be its field of quotients, and* K *be a field extension of* F. *If* $f : P \to K$ *is a derivation satisfying* (D), *then there exists a unique derivation* $d : F \to K$ *such that* $d|P = f$.

Result 1.90. [558]. *Let* K *be a field of characteristic zero,* F *be a subfield of* K, B *be an algebraic base of* K *over* F, *if it exists, and otherwise let* $B = \phi$. *Let* $f : F \to K$ *be a derivation. Then, for every function* $h : B \to K$, *there exists a unique derivation* $d : K \to K$ *such that* $d|F = f$, $d|B = h$.

This result proves the existence of nontrivial derivations of $\mathbb{R}$ [558].

1.7.8 Almost Everywhere Extension

P. Erdős in 1960 [260, 558] raised the following question. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ satisfies (A) for almost all $(x, y) \in \mathbb{R}^2$ (in the sense of Lebesgue measure); that is, $f(x+y) = f(x)+f(y)$ is valid for all $(x, y) \in D$ with the Lebesgue measure of $\mathbb{R}^2 \backslash D$ zero. Does there exist an additive $A : \mathbb{R} \to \mathbb{R}$ satisfying (A) for all $x, y \in \mathbb{R}$ such that $A = f$ a.e.; that is, for all $x \in D_x$, where the Lebesgue measure of $\mathbb{R} \backslash D_x$ is zero? The answer turns out to be positive (see [136, 406]). Earlier Hartman [345] had proved that if $f(x + y) = f(x) + f(y)$ holds for $(x, y) \in C \times C$, where the Lebesgue measure of $\mathbb{R} \backslash C$ is zero, then there is an additive $A : \mathbb{R} \to \mathbb{R}$ with $A = f$ a.e. on $\mathbb{R}$.

This statement fails to hold for arbitrary measures [562]. Further, that the additive A need not be an extension of f was shown by C.T. Ng and J. Aczél based on the following a counterexample by V. Zirak Walsh (see [55, 44, 45]).

Take $D = \{(x, y) : x, y, x+y \neq 3, 6\} \cup (3, 6)$. Obviously, the Lebesgue measure of $\mathbb{R}^2 \backslash D$ is zero and $D_x \cup D_y \cup D_0 = \mathbb{R}$. Define

$$f(x) = \begin{cases} 0, & \text{for } x \in \mathbb{R}\backslash\{3, 6\}, \\ 3, & \text{for } x = 3, \\ 6, & \text{for } x = 6. \end{cases}$$

Clearly f is additive on D, but f is not additive on $\mathbb{R}$.

If a function $f : \mathbb{R} \to \mathbb{C}$ satisfies (E) a.e. on $\mathbb{R}^2$, does there exist an exponential $E : \mathbb{R} \to \mathbb{C}$ such that $E = f$ a.e. on $\mathbb{R}$? The answer turns out to be positive again (see [44]).

1.8 Applications

We now present some applications based on the functional equations studied so far. We will return to more applications later, in Chapter 17. We start off with two examples from economics given by W. Eichhorn [250].

1.8.1 Economics

Example 1. Price and Advertising Policy

Consider a firm making a single product. It is natural to assume that the sales S depend on the price p of its product and on the advertising expenditure λ (a simple model); that is, $S : \mathbb{R}_+ \times \mathbb{R}_+^* \to \mathbb{R}$, $S = S(p, \lambda)$. What properties has one to impose on S to determine S? It is normal to expect the sales to go down with an increase in price and go up with an increase in advertising expenditure. Keeping these in mind, the following postulates were given by Eichhorn [250]. Let the sales function S satisfy

$$(C_1) \qquad S(p + q, \lambda) = \psi(q, \lambda) S(p, \lambda), \quad \text{for } p, q \in \mathbb{R}_+,\ \lambda \in \mathbb{R}_+^*,$$

where $\psi : \mathbb{R}_+ \times \mathbb{R}_+^* \to \mathbb{R}$ with $\psi(0, w) = 1$ for $w > 0$, $S(0, 1) > 0$, and $\psi(\cdot, \lambda)$ is a (nonconstant) decreasing function in the first variable (for fixed λ); that is, $p \to \psi(p, \lambda)$ is a nonconstant decreasing function.

(C_1) says that the sales at price $p + q$ (for an increment q in price) equal the previous sales times a real number ≤ 1 that is a nonincreasing function of q; that is, for fixed λ, $S(p, \lambda)$ considered as a function of p is convex and decreasing. The assumption $S(0, 1) > 0$ implies there are sales if the price is zero and the advertising expenditure is 1.

$$(C_2) \qquad S(p, \lambda\mu) = S(p, \lambda) + T(p, \mu), \quad \text{for } p \in \mathbb{R}_+,\ \lambda, \mu \in \mathbb{R}_+^*,$$

where $T : \mathbb{R}_+ \times \mathbb{R}_+^* \to \mathbb{R}$ with $T(p, 1) = 0$ for all $p \geq 0$ and $T(p, \cdot)$ is an increasing (nonconstant) function of the second variable (for fixed p); that is, $\lambda \to T(p, \lambda)$ is a nonconstant increasing function.

(C_2) says that a multiplicative change in advertising expenditure results in an additive change in sales; that is, for a given p, S considered as a function of λ is concave and increasing. For instance, sales increase with an increase in advertising expenditure.

Now we determine S subject to the conditions (C_1) and (C_2). For fixed λ, (C_1) is a Pexider equation (PE) of exponential type. Hence

$$S(p,\cdot) = a_1 E(p,\cdot),\ \psi(p,\cdot) = E(p,\cdot),$$

where $E(p,\cdot)$ satisfies (E) on $\mathbb{R}_+$ and a_1 is a constant depending on λ. Since $\psi(p,\cdot)$ is monotonically decreasing in p, so is $E(p,\cdot)$ and $E(p,\lambda) = e^{b_1(\lambda)p}$. Thus

$$(C_{11}) \qquad S(p,\lambda) = a_1(\lambda)e^{b_1(\lambda)p},\ \psi(p,\lambda) = e^{b_1(\lambda)p},\ a_1(\lambda) = S(0,\lambda).$$

For fixed p, (C_2) is a Pexider equation (PL) of logarithmic type. Thus, there exists an L satisfying (L) on $\mathbb{R}_+^*$ such that

$$S(\cdot,\lambda) = L(\lambda,\cdot) + b_2, \quad T(\cdot,\lambda) = L(\lambda,\cdot).$$

Since T is monotonically increasing in λ, $L(\lambda, p) = a_2(p)\log\lambda$, so that

$$(C_{22}) \qquad S(p,\lambda) = a_2(p)\log\lambda + b_2(p),\ T(p,\lambda) = a_2(p)\log\lambda,$$

with $a_2(p) > 0$, $b_2(p) = S(p,1)$.

In order to determine S, either $a_1(\lambda)$ and $b_1(\lambda)$ or $a_2(p)$ and $b_2(p)$ are to be obtained. We seek $a_2(\lambda)$ and $b_2(\lambda)$.

Substitution of S from (C_{11}) in (C_2) gives

$$\begin{aligned} a_1(\lambda\mu)\cdot e^{b_1(\lambda\mu)p} &= a_1(\lambda)e^{b_1(\lambda)p} + a_2(p)\log\mu \\ &= a_1(\mu)e^{b_1(\mu)p} + a_2(p)\log\lambda \quad \text{(interchanging } \lambda \text{ and } \mu\text{)}. \end{aligned}$$

With $p = 0$, we obtain $a_1(\lambda) - a_2(0)\log\lambda = a_1(\mu) - a_2(0)\log\mu$ so that $a_1(\lambda) = a\log\lambda + b$ (by fixing μ), where a and b are constants and

$$S(p,\lambda) = (a\log\lambda + b)e^{b_1(\lambda)p},$$

with $b > 0$, since $S(0,1) > 0$ and $a = a_2(0) > 0$, $b = b_2(0) > 0$ (put $p = 0$ in the equation above and (C_{22})).

Substitution of S from (C_{22}) in (C_1) gives

$$a_2(p+q)\log\lambda + b_2(p+q) = [a_2(p)\cdot\log\lambda + b_2(p)]e^{b_1(\lambda)q}.$$

With $p = 0$, we have

$$(1.118) \qquad a_2(q)\log\lambda + b_2(q) = (a\log\lambda + b)e^{b_1(\lambda)q},$$

which by setting $\lambda = 1$ gives $b_2(q) = be^{b_1(1)q}$. Putting $\lambda = 2$ and 4 separately in (1.118) and eliminating $a_2(\lambda)$ from these equations, we get

$$(a\log 2+b)e^{b_1(2)q}-\left(a\log 2+\frac{b}{2}\right)e^{b_1(4)q}-\frac{b}{2}e^{b_1(1)q}=0.$$

Suppose $b_1(1)$, $b_1(2)$, and $b_1(4)$ are distinct. Then $\{e^{b_1(1)p}, e^{b_1(2)p}, e^{b_1(4)p}\}$ is linearly independent and $b = 0 = a$, contradicting $a > 0$ and $b > 0$. If two of $b_1(1)$, $b_1(2)$, and $b_1(4)$ are equal, since $a, b > 0$, all $b_1(1)$, $b_1(2)$, and $b_1(4)$ are equal.

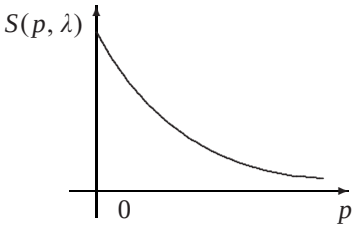
Now $\lambda = 2$ in (1.118) gives $a_2(q) = ae^{-eq}$, where $c = -b_1(1)$, and this $a_2(q)$ in (C_{22}) yields

(C_{12}) $$S(p,\lambda)=(a\log\lambda+b)e^{-cp}.$$

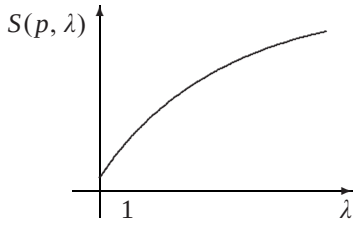
Since ψ is decreasing, from (C_1) it follows that $c > 0$, the sales function S is given by (C_{12}), and the arbitrary constants a, b, c are positive. Thus we have proved the following.

Theorem 1.91. (Eichhorn [250]). *The sales functions* $S : \mathbb{R}_+ \times \mathbb{R}_+^* \to \mathbb{R}$ *satisfying* (C_1) *and* (C_2) *are of the form* (C_{12}).

Graphs of (C_{12}) for p variation and λ variation are given by



$p \to (a + b\log\lambda)e^{-cp}$



$\lambda \to (a + b\log\lambda)e^{-cp}$

indicating a decrease in sales with an increase in price p and an increase in sales with an increase in advertisement expenditure λ. Note that basically we are using (E) and (L) equations.

Remark 1.92. (Eichhorn [250]). S given by (C_{12}) satisfies (C_1) and (C_2). For this sales function S, there is a maximum profit function P given by

$$P(p,\lambda)=pS(p,\lambda)-K(S(p,\lambda))-\lambda,$$

where the cost function $K : \mathbb{R}_+ \to \mathbb{R}_+^*$ satisfies the Pexider equation

$$K(x+y)=K(x)+N(y),\ \text{for } x,y\in\mathbb{R}_+,$$

with N not constant and $N : \mathbb{R}_+ \to \mathbb{R}_+$.

The cost function $K(x) = a_1 + b_1 x$ with $a_1 \geq 0,\ b_1 > 0$, and the profit function P is given by

$$P(p, \lambda) = (p - b_1)(a \log \lambda + b)e^{-cp} - a_1 - \lambda.$$

The maximum profit occurs at $(p^*, \lambda^*) = \left(\frac{1}{c} + b_1, \frac{a}{c}e^{-1-cb}\right)$.

Example 2. Production, Technical Progress

A production function describes how the amount of production depends on several factors, such as capital, labour, raw materials, etc. We consider the production function (given in Eichhorn and Kolm [251], and Eichhorn [250]) Φ depending on capital k, labour ℓ, and time t, which is also known as the index of the state of technology; that is, $\Phi : \mathbb{R}^3_+ \to \mathbb{R}_+$. A basic assumption on Φ is that there is at least one vector $(k^*, \ell^*), k^*, \ell^* > 0$ such that

$$t \to \Phi(k^*, \ell^*, t) \tag{p_1}$$

is a nonconstant increasing function for $t > 0$. This says that the output $\Phi(k^*, \ell^*, t)$ for a fixed input vector (k^*, ℓ^*) increases with time; that is, with technical progress.

Output augmenting. Technical progress is output augmenting if there exist $\Phi_1 : \mathbb{R}^2_+ \to \mathbb{R}_+,\ \alpha_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\Phi(k, \ell, t) = \alpha_1(t)\Phi_1(k, \ell). \tag{p_2}$$

It is clear that α_1 is a nonconstant, increasing function for $t > 0$ (use (p_1)).

Labor augmenting. Technical progress is said to be labor augmenting provided

$$\Phi(k, \ell, t) = \Phi_2(k, \alpha_2(t)\ell) \tag{p_3}$$

for some non-decreasing functions $\Phi_2 : \mathbb{R}^2_+ \to \mathbb{R}_+,\ \alpha_2 : \mathbb{R}_+ \to \mathbb{R}_+$.

Capital augmenting. Technical progress is capital augmenting provided

$$\Phi(k, \ell, t) = \Phi_3(\alpha_3(t)k, \ell) \tag{p_4}$$

for some non-decreasing functions $\Phi_3 : \mathbb{R}^2_+ \to \mathbb{R}_+,\ \alpha_3 : \mathbb{R}_+ \to \mathbb{R}_+$.

Suppose $\Phi : \mathbb{R}^3_+ \to \mathbb{R}_+$ possesses the properties (p_1) to (p_4) with Φ_2, Φ_3 nondecreasing and α_2, α_3 are continuous, strictly increasing functions from $\alpha_2(0) = 0 = \alpha_3(0)$ to infinity as $t \to \infty$; that is, the production function Φ is simultaneously output, labor, and capital augmenting. Then

$$\Phi_1(k, \ell) = c\ell^{c_1}k^{c_2},\ \Phi(k, \ell, t) = \alpha(t)k^{c_2}\ell^{c_1}, \tag{1.119}$$

where c, c_1, c_2 are positive constants.

Thus Φ_1 is the Cobb-Douglas [167, 250] function and Φ is the Cobb-Douglas function with respect to k and ℓ.

By hypothesis,

$$\Phi_2(k, \alpha_2(t)\ell) = \alpha_1(t)\Phi_1(k, \ell) = \Phi_3(\alpha_3(t)k, \ell). \tag{1.120}$$

Since α_2 is a strictly increasing, continuous, onto function, for $s > 0$, there is a t such that $\alpha_2(t) = s$. Replacing t by $\alpha_2^{-1}(s)$ in (1.120) (the first part of the equality), we have

$$\Phi_2(k, s\ell) = m(s)\Phi_1(k, \ell),$$

where $m(s) = \alpha_1(\alpha_2^{-1}(s))$ is a nonconstant monotonically increasing function.

Now, for $s, t > 0$, we get

$$\begin{aligned}\Phi_2(k, ts\ell) &= m(st)\Phi_1(k, \ell)\\ &= m(t)\Phi_1(k, s\ell) = m(s)\Phi_1(k, t\ell).\end{aligned}$$

Setting $s = 1$ in the last equality gives $m(1)\Phi_1(k, t\ell) = m(t)\Phi_1(k, \ell)$ and $m(1) \neq 0$ since otherwise $\Phi_1(k, \ell) = 0 = \Phi_2(k, \ell)$, contradicting Φ_2 being a nondecreasing function. Hence

$$m(st)\Phi_1(k, \ell) = m(t)\Phi_1(k, s\ell) = m(t)\frac{m(s)}{m(1)}\Phi_1(k, \ell).$$

Hence $m(st) = m(s)\frac{m(t)}{m(1)}$ and $\frac{m}{m(1)}$ is multiplicative (satisfies (M) on $\mathbb{R}_+$) and monotonically increasing, so $m(t) = b_1 t^{c_1}$ with $c_1 > 0$ and

$$\Phi_2(k, t\ell) = b_1 t\Phi_1(k, \ell) = b_1\ell^{c_1}\Phi_1(k, t);$$

that is,

$$\Phi_1(k, \ell) = \ell^{c_1}\Phi_1(k, 1).$$

Similarly, by considering the second part of (1.120), we get

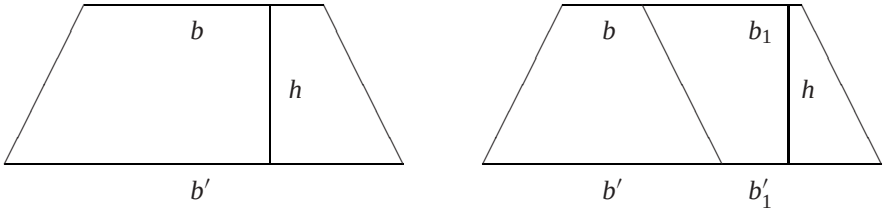
$$\Phi_1(k, \ell) = k^{c_2}\Phi_1(1, \ell),\ c_2 > 0.$$

Thus, $\Phi_1(k, \ell) = \ell^{c_1}\Phi_1(k, 1) = \ell^{c_1}k^{c_2}\Phi_1(1, 1) = c\ell^{c_1}k^{c_2}$. This proves (1.119).

1.8.2 Area

Example 3. Area of a Trapezoid

The area of a trapezoid is a function of the parallel sides and the height, say $f(b, b', h)$, and f is a nonnegative function from $\mathbb{R}_+^3 \to \mathbb{R}_+$ to be determined.



The desired properties of f are

$$(t_1) \qquad f(b+b_1, b'+b_1', h) = f(b, b', h) + f(b_1, b_1', h),$$
$$(t_2) \qquad f(b, b, h_1+h_2) = f(b, b, h_1) + f(b, b, h_2),$$
$$(t_3) \qquad f(b, b', h) = f(b', b, h).$$

From (t_1) we see that, for fixed h, f satisfies (A) on $\mathbb{R}_+^2$ and, by Theorem 1.24,

$$f(b, b', \cdot) = A_1(b, \cdot) + A_2(b', \cdot),$$

where A_1 and A_2 are additive in the first variables. Since f is nonnegative (bounded below), $A_1(b, h) = \alpha_1(h)b$ and $A_2(b', h) = \alpha_2(h)b'$, so that

$$f(b, b', h) = \alpha_1(h)b + \alpha_2(h)b'.$$

To determine α_1 and α_2, by the symmetry property of (t_3), $\alpha_1(h) = \alpha_2(h) = \alpha(h)$ (say). Substitution of this f in (t_2) yields

$$\alpha(h_1+h_2)2b = \alpha(h_1)2b + \alpha(h_2)2b;$$

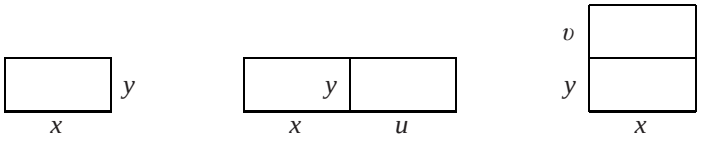
that is, α satisfies (A) on $\mathbb{R}_+$ and is nonnegative. So, $\alpha(h) = ch$ and

$$f(b, b', h) = c(b+b')h,$$

where c is a positive constant depending on the unit of area (refer to [144]).

Example 3a. Area of a Rectangle

The area $F(x, y)$ of a rectangle with sides x and y satisfies the equations



$F(x, y)$ $\qquad$ $F(x+u, y)$ $\qquad$ $F(x, y+v)$

$$F(x+u, y) = F(x, y) + F(u, y),$$
$$F(x, y+v) = F(x, y) + F(x, v), \quad x, y, u, v \in \mathbb{R}_+^*.$$

Since F is additive in the first and second variables and $F > 0$ and $F(x, y) = cxy$ with $F(1, 1) = 1$, the area is $F(x, y) = xy$ (see the interest formula in Chapter 17).

1.8.3 Allocation Problem: Characterization of Weighted Arithmetic Means

Example 4. Allocation Problem: Characterization of Weighted Arithmetic Means

A group decision-making or allocation problem treated by Aczél, Kannappan, Ng, and Wagner and others [63, 64, 53, 16, 825, 458] is the following. A fixed amount s of quantifiable goods (money or other resources, such as energy, raw materials, etc.) is to be allocated or distributed (completely) to a fixed number m (> 2) of competing projects or applications. A committee of n assessors, advisors, or decision makers is formed for this purpose. Each member of the committee recommends how the funds should be divided or allotted. The jth assessor recommends that the amount x_{ij} be allocated to the ith project in order to establish the "consensus allocation" (recommendation of the chairman) $f_i(x_i)$. The f_i may, a priori, all be different. Of course, for the chairman and each advisor, the assignment to each project should sum to s. This can be presented in the table

Project / Assessor	1	2	$\cdots$	i	$\cdots$	m	sum
1	x_{11}	x_{21}	$\cdots$	x_{i1}	$\cdots$	x_{m1}	s
.....	.....						...
j	x_{ij}	x_{2j}	$\cdots$	x_{ij}	$\cdots$	x_{mj}	s
.....	.....						...
n	x_{1n}	x_{2n}	$\cdots$	x_{in}	$\cdots$	x_{mn}	s
Chair – Aggregate Allocation	$f_1(x_1)$	$f_2(x_2)$	$\cdots$	$f_i(x_i)$	$\cdots$	$f_m(x_m)$	s

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ (ith column vector). The problem is to determine f_i subject to certain constraints. In general, the solutions are weighted arithmetic means. We now prove the following result from [53], which also shows that the conditions

$$\min\{y_j\} \le f(y_1, y_2, \ldots, y_n) \le \max\{y_j\}$$

and

$$\sum_{i=1}^{k} x_{ij} = s (j = 1, \ldots, n) \Rightarrow \sum_{i=1}^{k} f(x_{i1}, \ldots, x_{in}) = s$$

(s and $k > 2$ are fixed) for $f : [0, s]^n \to \mathbb{R}$ characterize the weighted arithmetic means. Denote $\underline{\mathbf{s}} = (s, s, \ldots, s)$, $\underline{\mathbf{0}} = (0, 0, \ldots, 0)$, $\underline{\mathbf{1}} = (1, 1, \ldots, 1)$.

Suppose $f_i : [0, s]^n \to [0, s]$ $(i = 1, 2, \ldots, m)$ satisfy

(i) $f_i(\underline{\mathbf{0}}) = 0$ (consensus of rejection; that is, if all referees assign amount 0 to the ith project, then the aggregated allocation should also allocate amount 0),

(ii) $f_i(x) \ge 0$ (aggregated allocation is nonnegative) (for $x \in [0, r]^n$, $0 < r \le s$), and

(iii) $\sum_{i=1}^{m} x_i = \underline{\mathbf{s}} \Rightarrow \sum_{i=1}^{m} f_i(x_i) = s.$

Then $f_1 = f_2 = \cdots = f_m$ and there exist weights $w_j \geq 0$ such that $\sum_{j=1}^{n} w_j = 1$ and

$$f_i(y) = f_i(y_1, y_2, \ldots, y_n) = \sum_{j=1}^{n} w_j y_j (i = 1, 2, \ldots, m)$$

for $y \in [0, s]^n$.

Setting $x_i = \underline{\mathbf{0}}$ for $(i = 2, 3, \ldots, m)$ in (iii) and using (i), we get $f_1(\underline{\mathbf{s}}) = s$. Similarly, $f_i(\underline{\mathbf{s}}) = s$ for $i = 1, 2, \ldots, m$ (consensus of overwhelming merit). First rewrite (iii) as

$$\sum_{i=2}^{m} f_i(x_i) + f_1\left(\underline{\mathbf{s}} - \sum_{i=2}^{m} x_i\right) = s,$$

and then substitute $x_3 = \underline{\mathbf{0}} = \cdots = x_m,\ x_2 = x$ to get

$$f_1(\underline{\mathbf{s}} - x) + f_2(x) = s$$

and then $x_2 = x,\ x_3 = y,\ x_i = \underline{\mathbf{0}}\ (i = 4, \ldots, m)$ to obtain

$$f_1(\underline{\mathbf{s}} - x - y) + f_2(x) + f_3(y) = s;$$

that is,

$$f_2(x + y) = f_2(x) + f_3(y) \quad \text{for } x, y, x + y \in [0, s]^n.$$

Finally, we obtain

$$f_3(x + y) = f_3(x) + f_3(y) \quad \text{for } x, y, x + y \in [0, s]^n.$$

(Set $x = 0$ in the above to have $f_2(y) = f_3(y)$. This also shows that all f_i $(i = 1, 2, \ldots, m)$ are equal.) Now f_3 is additive on $[0, s]^n$ and is nonnegative. By the extension result,

$$f_3(y) = \sum_{j=1}^{n} w_j y_j,\ w_j \geq 0, \quad \text{for } y \in [0, s]^n.$$

Since $f_3(\underline{\mathbf{s}}) = s,\ \sum_{j=1}^{n} w_j = 1$.

This proves the result.

Linear Programming

The result above extends to constraints consisting of a system of linear equations and inequalities. This may be of interest for multiobjective linear programming as an alternative to Pareto optimization. We now present the result of Radó and Baker [676].

Consider a linear programming problem with constraints of the form

$$(1.121)\quad x_j \in I_j \quad (j = 1, \ldots, m),\ x_{p+i} + \sum_{j=1}^{p} \alpha_{ij} x_j = a_i \quad (i = 1, 2, \ldots, m-p),$$

where $I_1, \ldots, I_m$ are given proper open intervals, $2 \leq p < m$, $a_1, \ldots, a_{m-p}$ are given reals, and $[\alpha_{ij}]$ is an $(m-p) \times p$ real matrix having in each column at least one and in each row at least two nonzero elements.

The problem is the following. Suppose that n individuals select solutions of (1.121), each in accordance with his own criterion. Find a consensual solution of (1.121) such that the consensual value of each variable depends only on the values assigned by individuals to that variable, the criteria being arbitrary. In other words, determine m functions (said to form an aggregation method with respect to the system (1.121))

$$f_j : I_j^m \to I_j \quad (j = 1, 2 \ldots, m)$$

in such a way that

$$(1.122)\qquad z_j \in I_j^n,\ z_{p+1} + \sum_{j=1}^{p} \alpha_{ij} z_j = \underline{a_i} \quad (i = 1, \ldots, m-p)$$

implies

$$f_{p+i}(z_{p+i}) + \sum_{j=1}^{p} \alpha_{ij} f_j(z_j) = a_i, \quad (i = 1, \ldots, m-p).$$

Let C_j $(j = 1, \ldots, m)$ and C be the sets of those $z_j \in \mathbb{R}^n$ or $(z_1, \ldots, z_p) \in (\mathbb{R}^n)^p$, respectively, that can be completed as a solution of (1.122). Note that C is an open convex set and C_j are n-dimensional intervals.

Suppose $C \neq \phi$ and f_j is bounded above or below on some nonempty open subsets of I_j^n $(j = 1, \ldots, p)$. Then f_j are given by

$$f_j(x) = \sum_{k=1}^{n} w_k x_k + \beta_j \quad (j = 1, \ldots, m),$$

where w_k and β_j are real constants subject to

$$\beta_{pi} + \sum_{j=1}^{p} \alpha_{ij} \beta_j = \left(1 - \sum_{k=1}^{n} w_k\right) a_i, \quad (i = 1, \ldots, m-p).$$

1.8.4 Sum of Powers of First n Natural Numbers and Sum of Powers on Arithmetic Progressions

Sums of Powers of Integers

There are many ways of finding the sum

$$S_k(n) = 1^k + 2^k + \cdots + n^k, \quad k \in Z_+,\ n \in Z_+^*,$$

that were developed by several fascinated mathematicians. The methods adopted were varied in nature—evaluation at $n = 1$, symmetry argument, modified Bernoulli number identity, generating functions, etc. Here the focus is going to be on the use of functional equations, especially the *additive Cauchy equation* (A) on Z_+^*, to obtain the sum. We will illustrate this in the cases $k = 3$ and 4; that is, we will find $S_3(n)$ and $S_4(n)$ assuming the sums $S_1(n)$ and $S_2(n)$ (these could also be obtained by the method to be presented). The general references ([16, 21, 122, 142, 517, 543, 607, 664, 758, 759, 836]) are only a partial list.

The sums $S_k(n)$ satisfy the system of functional equations

$$(*) \quad S_k(m+n) = S_k(m) + S_k(n) + \sum_{i=1}^{k} \binom{k}{i} n^i S_{k-i}(m), \quad k \in Z_+,\ m, n \in Z_+^*$$

(use binomial expansions in $S_k(m+n)$ after the nth term). Note that $S_0(n) = n$, $S_k(1) = 1$, and we are assuming $S_1(n) = \frac{n(n+1)}{2}$, $S_2(n) = \frac{n(n+1)(2n+1)}{6}$.

For $k = 3$, $(*)$ yields

$$S_3(m+n) = S_3(m) + S_3(n) + 3nS_2(m) + 3n^2 S_1(m) + n^3 S_0(m);$$

that is,

$$\begin{aligned} 4S_3(m+n) &= 4S_3(m) + 4S_3(n) + \{4m^3 n + 6m^2 n^2 + 4mn^3\} \\ &\quad + \{6m^2 n + 6mn^2\} + 2mn \\ &= 4S_3(m) + 4S_3(n) + \{(m+n)^4 - m^4 - n^4\} \\ &\quad + 2\{(m+n)^3 - m^3 - n^3\} + \{(m+n)^2 - m^2 - n^2\}, \end{aligned}$$

which can be put in the form

$$A_1(m+n) = A_1(m) + A_1(n) \quad \text{for } m, n \in Z_+^*,$$

which is the additive Cauchy equation, where

$$A_1(n) = 4S_3(n) - n^4 - 2n^3 - n^2.$$

Evidently, $A_1(n) = c_3 n$, so that

$$4S_3(n) = n^4 + 2n^3 + n^2 + c_3 n, \quad n \in Z_+^*.$$

Setting $n = 1$, we obtain $c_3 = 0$ and $S_3(n) = \frac{n^2(n+1)^2}{4}$.

Now we will find $S_4(n)$. For $k = 4$, $(*)$ gives

$$S_4(m+n) = S_4(m) + S_4(n) + 4nS_3(m) + 6n^2S_2(m) + 4n^3S_1(m) + n^4S_0(m);$$

that is,

$$\begin{aligned} 5S_4(m+n) &= 5S_4(m) + 5S_4(n) + \{5m^4n + 10m^3n^2 + 10m^2n^3 + 5mn^4\} \\ &\quad + \{10m^3n + 15m^2n^2 + 10mn^3\} + \{5m^2n + 5mn^2\} \\ &= 5S_4(m) + 5S_4(n) + \{(m+n)^5 - m^5 - n^5\} \\ &\quad + \frac{5}{2}\{(m+n)^4 - m^4 - n^4\} + \frac{5}{3}\{(m+n)^3 - m^3 - n^3\}, \end{aligned}$$

which goes over to

$$A_2(m+n) = A_2(m) + A_2(n) \quad \text{for } m, n \in Z_+^*,$$

where

$$A_2(n) = 5S_4(n) - n^5 - \frac{5}{2}n^4 - \frac{5}{3}n^3.$$

Thus $5S_4(n) = n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 + c_4n$ for $n \in Z_+^*$. As before, $n = 1$ gives $c_4 = -\frac{1}{6}$ and $S_4(n) = \frac{1}{30}[6n^5 + 15n^4 + 10n^3 - n] = \frac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$.

Remark. The general solutions $s_k : N \to \mathbb{R}$ $(k = 0, 1, 2, \dots)$ of $(*)$ are given by [16],

$$S_k(n) = \frac{1}{k+1}\sum_{\ell=0}^{k}\binom{k+1}{\ell}c_\ell n^{k+1-\ell}, \quad (k = 0, 1, 2, \dots),$$

where $c_0, c_1, c_2, \dots$ are arbitrary constants in $\mathbb{R}$.

The constants c_i satisfy

$$\sum_{\ell=0}^{k-1}\binom{k}{\ell}c_\ell = k \quad (k = 1, 2, 3, \dots).$$

We get

$$c_0 = 1 \qquad S_0(n) = n$$

$$
\begin{aligned}
c_0 + 2c_1 = 2 \qquad c_1 = \frac{1}{2} \qquad S_1(n) &= \frac{1}{2}(c_0 n^2 + 2c_1 n) \\
&= \frac{1}{2}(n^2 + n) \\
&= \frac{n(n+1)}{2}, \\
c_0 + 3c_1 + 3c_2 = 3 \qquad c_2 = \frac{1}{6} \qquad S_2(n) &= \frac{1}{3}(c_0 n^3 + 3c_1 n^2 + 3c_2 n) \\
&= \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \\
&= n(n+1)\frac{(2n+1)}{6}, \\
c_0 + 4c_1 + 6c_2 + 4c_3 = 4 \qquad c_3 = 0 \qquad S_3(n) &= \frac{1}{4}[c_0 n^4 + 4c_1 n^3 + 6c_2 n^2] \\
&= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \\
&= \frac{1}{4}n^2(n+1)^2,
\end{aligned}
$$

$$
\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{te^{6x}}{e^t - 1},
$$

where $B_n = B_n(1)$.

Now we will turn our attention to the *sum of powers on arithmetic progressions*:

$$
S_k(n; a, h) = a^k + (a+h)^k + \cdots + [a+(n-1)h]^k, \quad \text{for } k \in Z_+,\ n \in Z_+^*,\ a, h \in \mathbb{R}.
$$

By adopting the method used above, we will determine $S_2(n; a, h)$ by assuming $S_1(n; a, h) = \frac{h}{2}n^2 + \left(a - \frac{h}{2}\right)n$ and noting that $S_0(n; a, h) = n,\ S_k(1; a, h) = a^k$.

First of all, S_k''s satisfy the system of functional equations

$$
(**) \quad S_k(m+n; a, h) = S_k(m; a, h) + S_k(n; a, h) + \sum_{i=1}^{k} \binom{k}{i} (nh)^i S_{k-i}(m; a, h)
$$

for $k \in Z_+,\ m, n \in Z_+^*,\ a, h \in \mathbb{R}$, which for $k = 2$ results in

$$
\begin{aligned}
S_2(m+n; a, h) &= S_2(m; a, h) + S_2(n; a, h) + 2nhS_1(m; a, h) + (nh)^2 S_0(m; a, h) \\
&= S_2(m; a, h) + S_2(n; a, h) + h^2\{m^2 n + mn^2\} \\
&\quad + \left(a - \frac{h}{2}\right) h \cdot 2mn \\
&= S_2(m; a, h) + S_2(n; a, h) + \frac{h^2}{3}\{(m+n)^3 - m^3 - n^3\} \\
&\quad + h\left(a - \frac{h}{2}\right)\{(m+n)^2 - m^2 - n^2\},
\end{aligned}
$$

from which we obtain

$$A(m+n) = A(m) + A(n) \quad \text{for } m, n \in Z_+^*,$$

where

$$A(n) = S_2(n; a, h) - \frac{h^2}{3}n^3 - h\left(a - \frac{h}{2}\right)n^2.$$

Hence, $S_2(n; a, h) = \frac{h^2}{3}n^3 + h\left(a - \frac{h}{2}\right)n^2 + cn$. Using $S_2(1; a, h) = a^2$, we have $c = a^2 - \frac{h^2}{3} - h\left(a - \frac{h}{2}\right)$ and $S_2(n; a, h) = \frac{h^2}{3}n^3 + h\left(a - \frac{h}{2}\right)n^2 + \left\{a^2 - ah + \frac{h^2}{6}\right\}n$ (see [21, 758]).

$S_k(n)$ can also be given by

$$S_k(n) = \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j n^{k+1-j},$$

where B_n is the Bernoulli number defined by the generating functions.

Remark. The general solutions $s_k : N \to \mathbb{R}$ $(k = 0, 1, 2, \dots)$ satisfying $(**)$ are given by [21]

$$s_k(n; a, h) = \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j+1}n^{j+1}c_{k-j}h^j \quad (n \in \mathbb{Z}^+),$$

where $c_0, c_1 \dots$ are arbitrary constants in $\mathbb{R}$ with

$$\sum_{j=1}^{k}\binom{k}{j}c_{k-j}h^{j-1} = ka^{k-1} \quad (k = 1, 2, \dots)$$

$$c_0 = 1, \quad 2c_1 + c_0 h = 2a, \quad 3c_2 + 3c_1 h + c_0 h^2 = 3a^2;$$

that is,

$$c_0 = 1, \quad 2c_1 = 2a - h, \quad 3c_2 = 3a^2 - 3ah + \frac{1}{2}h^2, \dots$$

$$s_1(n; a, h) = \frac{1}{2}(2nc_1 + n^2 c_0 h) = \frac{h}{2}n^2 + \left(a - \frac{h}{2}\right)n,$$

$$s_2(n; a, h) = \frac{1}{3}(3nc_2 + 3n^2 c_1 h + n^3 c_0 h^2)$$

$$= \frac{h^3}{3}n^3 + h\left(a - \frac{h}{2}\right)n^2 + \left(a^2 - ah + \frac{h^2}{6}\right)n, \dots.$$

1.8.5 More Sums Using the Additive Cauchy Equation

Not only are we interested in determining the sum of (i) powers of integers and (ii) powers on arithmetic progression, but also other summations of series. Here we give an example of finding the sum of the product of integers using the Cauchy equation, namely

$$t(n) = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2), \quad n \in Z_+^*.$$

As before, first we start with

$$\begin{aligned}
t(m+n) &= t(n) + (n+1)(n+2)(n+3) \\
&\quad + \cdots + (n+m)(n+m+1)(n+m+2) \\
&= t(n) + mn^3 + n^2(6 + 9 + \cdots + 3(m+1)) \\
&\quad + n\left\{11 + 26 + \cdots + (3m^2 + 6m + 2)\right\} + t(m) \\
&= t(n) + t(m) + mn^3 + 3n^2\left\{\frac{(m+1)(m+2)}{2} - 1\right\} \\
&\quad + n\left\{3\frac{m(m+1)(2m+1)}{6} + 6 \cdot \frac{m(m+1)}{2} + 2m\right\} \\
&= t(n) + t(m) + \left\{mn^3 + m^3n + \frac{3}{2}m^2n^2\right\} \\
&\quad + \left\{\frac{9}{2}m^2n + \frac{9}{2}mn^2\right\} + \frac{11}{2}mn \\
&= t(n) + t(m) + \frac{1}{4}[(m+n)^4 - m^4 - n^4] \\
&\quad + \frac{3}{2}[(m+n)^3 - m^3 - n^3] + \frac{11}{4}[(m+n)^2 - m^2 - n^2],
\end{aligned}$$

which results in the additive Cauchy equation

$$A(m+n) = A(m) + A(n), \quad m, n \in Z_+^*,$$

where

$$A(n) = t(n) - \frac{1}{4}n^4 - \frac{3}{2}n^3 - \frac{11}{4}n^2.$$

Thus

$$t(n) = \frac{1}{4}n^4 + \frac{3}{2}n^3 + \frac{11}{4}n^2 + cn.$$

With $n = 1$, we get

$$6 = \frac{1}{4} + \frac{3}{2} + \frac{11}{4} + c;$$

that is, $c = \frac{3}{2}$, so that

$$\begin{aligned}
t(n) &= \frac{1}{4}n^4 + \frac{3}{2}n^3 + \frac{11}{4}n^2 + \frac{3}{2}n \\
&= \frac{n}{4}(n+1)(n+2)(n+3).
\end{aligned}$$

1.8.6 Application in Combinatorics and Genetics

We are all familiar with the combinatorial formula $\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!}$ = the number of ways of selecting r things at a time out of n things. We will find $\binom{n}{3}$ by using the additive Cauchy equation (A). Define $f_r(n)$ = the number of possible ways of using additive Cauchy equation (A). Define $f_r(n)$ = the number of possible ways of picking r things from n things.

In genetics, it is of interest to know the combinatorial function $g_r(n)$ = the number of ways to pick r things out of n things, this time *permitting repetitions*. We will determine $f_3(n)$ and $g_3(n)$.

Note that $f_3(3) = 1$, $g_3(1) = 1$ (there is only one way one can take one thing thrice), $f_1(n) = n = g_1(n)$. Let us assume $f_2(n) = \frac{n(n-1)}{2}$ and $g_2(n) = \frac{n(n+1)}{2}$, which can also be obtained by the method presented below (see Snow [759]).

First of all, it is easy to check that $f_r(n)$ and $g_r(n)$ satisfy the system of function equations

$$t_r(m+n) = t_r(m) + t_r(n) + \sum_{i=1}^{r-1} t_i(n)t_{r-i}(m) \tag{1.123}$$

for $m, n, k \in Z_+^*$.

To find $f_3(n)$, take $r = 3$, $t_3 = f_3$ in (1.123) to get

$$\begin{aligned} f_3(m+n) &= f_3(m) + f_3(n) + f_1(n)\,f_2(m) + f_2(n)\,f_1(m) \\ &= f_3(m) + f_3(n) + n\frac{m(m-1)}{2} + m\frac{n(n-1)}{2} \\ &= f_3(m) + f_3(n) + \frac{1}{2}(m^2n + mn^2) - mn \\ &= f_3(m) + f_3(n) + \frac{1}{6}[(m+n)^3 - m^3 - n^3] - \frac{1}{2}[(m+n)^2 - m^2 - n^2], \end{aligned}$$

which can be rewritten as

$$A_1(n+m) = A_1(n) + A_1(m) \quad \text{for } m, n \in Z_+^*, \tag{A}$$

where $A_1(n) = f_3(n) - \frac{1}{6}n^3 + \frac{1}{2}n^2$.

Since $A_1(n) = c_1 n_1$,

$$\begin{aligned} f_3(n) &= \frac{1}{6}n^3 - \frac{1}{2}n^2 + c_1 n \\ &= \frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{1}{3}n \quad (\text{use } f_3(3) = 1) \\ &= \frac{n(n-1)(n-2)}{3!} = \binom{n}{3}. \end{aligned}$$

In order to determine $g_3(n)$, take $r = 3,\ t_3 = g_3$ in (1.123) to obtain

$$\begin{aligned} g_3(m+n) &= g_3(m) + g_3(n) + g_1(n)g_2(m) + g_2(n)g_1(m) \\ &= g_3(m) + g_3(n) + n \cdot \frac{m(m+1)}{2} + m \cdot \frac{n(n+1)}{2} \\ &= g_3(m) + g_3(n) + \frac{1}{6}[(m+n)^3 - m^3 - n^3] \\ &\quad + \frac{1}{2}[(m+n)^2 - m^2 - n^2], \end{aligned}$$

which goes over to

$$\text{(A)} \qquad A_2(m+n) = A_2(m) + A_2(n), \quad m, n \in Z_+^*,$$

where $A_2(n) = g_3(n) - \frac{1}{6}n^3 - \frac{1}{2}n^2$.

As before, since $A_2(n) = c_2 n$,

$$\begin{aligned} g_3(n) &= \frac{1}{6}n^3 + \frac{1}{2}n^2 + c_2 n \\ &= \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n \quad (\text{use } g_3(1) = 1) \\ &= \frac{1}{6}n(n+1)(n+2). \end{aligned}$$

Remark. Even though the system is the same for both $f_r(n)$ and $g_r(n)$, the solutions differ because of different boundary conditions. The beauty is that the same method works (refer to Aczél [16, 21], Snow [758, 759]).

2

Matrix Equations

This chapter is devoted to functional equations whose domain, domain and range, or range are matrices. We first consider the matrix version of the Cauchy equations (A), (E), (L), and (M). Then we treat the matrix version of cosine equation (C) and other generalizations.

Let $M_n(F)$ $(n \geq 1)$ denote matrices of order n ($n \times n$ matrices) over a field F (could be $\mathbb{R}$ or $\mathbb{C}$). Indeed, $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, $M_n(\mathbb{C}) = \mathbb{C}^{n^2}$, and $GL(n, F)$ is the set of all nonsingular (regular) matrices of order n over a field F.

For $P \in M_n(\mathbb{R})$,

$$\det(P - \lambda I_n) = (-\lambda)^n I_n + (-\lambda)^{n-1} S_1(P) + \lambda^{n-2} S_2(P) + \cdots + S_n(P)$$

is the well-known characteristic polynomial of P, where I_n is the identity matrix of order n, and det stands for the determinant. It is true that the characteristic polynomial is invariant (under conjugacy) with respect to nonsingular matrices; that is,

$$\det(C^{-1}(P - \lambda I_n)C) = \det(P - \lambda I_n), \quad P, C \in M_n(\mathbb{R}),$$

for every nonsingular matrix C. This implies that the functions $S_1(P), S_2(P), \ldots, S_n(P)$ are invariant under conjugacy, where $S_k : M_n(\mathbb{R}) \to \mathbb{R}$ for $k = 1, 2, \ldots, n$.

$S_1(P)$ is called the *trace* of P (tr P) and is the sum of all eigenvalues of P and satisfies the additive functional equation (A),

$$S_1(P + S) = S_1(P) + S_1(S), \quad \text{for } P, S \in M_n(\mathbb{R}).$$

$S_n(P)$ is the product of all eigenvalues of P and is the $\det P$ and satisfies the multiplicative equation (M),

$$S_n(PS) = S_n(P) \cdot S_n(S), \quad \text{for } P, S \in M_n(\mathbb{R}).$$

The function $S_2(P)$ satisfies the quadratic functional equation (Q),

$$S_2(P + S) + S_2(P - S) = 2S_2(P) + 2S_2(S), \quad P, S \in M_n(\mathbb{R}).$$

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Some further results concerning the functions $S_1(P)$, $S_2(P)$, and $S_n(P)$ are considered here. In particular, we treat the matrix-additive (mA), exponential (mE), multiplicative (mM), and logarithmic (mL) functional equations

(mA) $$A(P+S) = A(P) + A(S),$$

(mE) $$E(P+S) = E(P) \cdot E(S),$$

(mM) $$M(PS) = M(P)M(S),$$

(mL) $$L(PS) = L(P) + L(S),$$

where $A, E, M, L : M_n(F) \to M_k(F)$ for $n, k \geq 1$, $P, S \in M_n(F)$.

Definition. A mapping $\Omega : M_n(F) \to M_k(F)$, where $F = \mathbb{R}$ or $\mathbb{C}$, is said to be *unitarily invariant* provided

$$\Omega(U^* P U) = \Omega(P), \quad P \in M_n(F),$$

holds for every unitary matrix U.

Ω is said to be *orthogonally invariant* if

$$\Omega(O^t P O) = \Omega(P), \quad \text{for } P \in M_n(\mathbb{R}),$$

holds for every orthogonal matrix $O \in M_n(\mathbb{R})$ and O^t is the transpose of O.

Result 2.1. (Kurepa [575]). *Let T, Φ, and Ψ be unitarily invariant functions that are defined on $M_n(\mathbb{C}) \to M_k(\mathbb{C})$. Let*

(2.1) $$T(P+S) = G[\Phi(P)\Phi(S), \Psi(P) + \Psi(S)] \quad \text{for } P, S \in M_n(\mathbb{C}),$$

where $G : M_m(\mathbb{C}) \to M_k(\mathbb{C})$.

If we denote by α_1 a matrix of order n with matrix elements

$$(\alpha_1)_{ij} = \delta_{1i}\delta_{1j} \quad (i, j = 1, 2, \ldots, n),$$

and if we define $f, \phi, \psi : \mathbb{C} \to M_k(\mathbb{C})$ by

$$f(u) = F(u\,\alpha_1), \phi(u) = \Phi(u\,\alpha_1) \quad \text{and} \quad \psi(u) = \Psi(u\,\alpha_1),$$

then

(2.2) $$f(u+v) = G[\phi(u)\phi(v), \psi(u) + \psi(v)]$$

and

(2.3) $$T(P) = f(\operatorname{tr} P)$$

for every pair of complex numbers u, v and for every matrix P.

Now we obtain the unitarily invariant solutions of (mA) and (mE).

Theorem 2.2. [575]. *Let T_1 be a unitarily invariant function and satisfy* (mA) *from $M_n(\mathbb{C})$ to $M_m(\mathbb{C})$,*

$$T_1(P+S) = T_1(P) + T_1(S), \tag{mA}$$

for every pair of matrices P and S. Then

$$T_1(P) = f_1(\operatorname{tr} P), \quad P \in M_n(\mathbb{C}),$$

for all P. If the function T_1 is continuous, then

$$T_1(P) = a_1 \operatorname{Re} \operatorname{tr} P + a_2 \operatorname{Im} \operatorname{tr} P$$

for every P, where a_1 and $a_2 \in M_m(\mathbb{C})$ do not depend on P.

Proof. If in Result 2.1 we put $\Phi = 0$, $\Psi = T$, and $G(0,u) = u$, then we get $F_1(P) = f_1(\operatorname{tr} P)$, where the function $f_1 : \mathbb{C} \to M_m(\mathbb{C})$ satisfies the functional equation (mA) $f_1(u+v) = f_1(u) + f_1(v)$.

The continuity of T_1 and the fact that $\operatorname{tr} P$ is a continuous function of P imply that $f_1(u)$ is a continuous function. It is well known that a continuous function that satisfies the functional equation (mA) has the form

$$f_1(u) = a_1 \operatorname{Re} u + a_2 \operatorname{Im} u$$

or

$$T_1(P) = a_1 \operatorname{Re} \operatorname{tr} P + a_2 \operatorname{Im} \operatorname{tr} P$$

for every P.

Theorem 2.3. [575]. *Let T_2 be a unitarily invariant function from $M_n(\mathbb{C})$ to $M_m(\mathbb{C})$ and let*

$$T_2(P+S) = T_2(P)T_2(S)$$

hold true for all P and S in $M_n(\mathbb{C})$.

Then

$$T_2(P) = f_2(\operatorname{tr} P),$$

where the function $f_2 : \mathbb{C} \to M_m(\mathbb{C})$ satisfies the functional equation

$$f_2(u+v) = f_2(u)\, f_2(v) \tag{mE}$$

for all complex numbers u and v.

If $\det T_2(P) \neq 0$ for every matrix P and if $T_2(P)$ is a continuous function, then

$$T_2(P) = \exp[a_1 \operatorname{Re} \operatorname{tr} P + i a_2 \operatorname{Im} \operatorname{tr} P].$$

Proof. If in Result 2.1 we put $\Psi = 0$, $G(u,0) = u$, and $\Phi = T$, then we get

$$T_2(P) = f_2(\operatorname{tr} P),$$

where the function f_2 satisfies the functional equation (mE). Since $\det T_2(P) \neq 0$, we have $\det f_2(u) \neq 0$ for any complex number u. The continuity of $T_2(P)$ and the continuity of tr P imply that the function f_2 is a continuous function of u. From (mE), we find

$$f_2(t+is) = f_2(t)\, f_2(is) = f_2(is)\, f_2(t) \tag{2.4}$$

with $t = \operatorname{Re} u$ and $s = \operatorname{Im} u$. Further,

$$f_2(t+s) = f_2(t)\, f_2(s)$$

and

$$f_2(it+is) = f_2(it)\, f_2(is)$$

for all real numbers t and s. Thus

$$f_2(t) = \exp(a_1 t)$$

and

$$f_2(is) = \exp(i a_2 \cdot s)$$

for all real numbers t and s. Thus, from (2.4), we have

$$\exp a_1 t \cdot \exp i a_2 s = \exp i a_2 s \cdot \exp a_1 t$$

for all real numbers t and s, where a_1 and $a_2 \in M_k(\mathbb{C})$. This implies that a_1 and a_2 commute. Thus

$$f_2(u) = f_2(t+is) = \exp(a_1 t) \cdot \exp(i a_2 s) = \exp(a_1 t + i a_2 s)$$

or

$$T_2(P) = \exp(a_1 \operatorname{Re} \operatorname{tr} P + i a_2 \operatorname{Im} \operatorname{tr} P)$$

for every matrix P.

Result 2.4. [575]. *Let T_1 and T_2 be orthogonally invariant functions from $M_n(\mathbb{R})$ to $M_k(\mathbb{R})$. If T_1 satisfies* (mA) *and T_2 satisfies* (mE) *for all matrices P, S in $M_n(\mathbb{R})$, then*

$$T_1(P) = f_1(\operatorname{tr} P) \quad \textit{and} \quad T_2(P) = f_2(\operatorname{tr} P),$$

where f_1 and f_2 are functions of real variables to $M_k(\mathbb{R})$ and satisfy the corresponding functional equations

$$f_1(x+y) = f_1(x) + f_1(y), \qquad f_2(x+y) = f_2(x)\, f_2(y), \qquad \textit{for } x, y \in \mathbb{R}.$$

Now we consider the remaining two equations (mL) and (mM).

Result 2.5. [575]. *Let T, Φ, and Ψ be unitarily invariant functions that are defined on $M_n(\mathbb{C})$ and have values in $M_k(\mathbb{C})$. Let*

$$T(P \cdot S) = G[\Phi(P)\Phi(S), \Psi(P) + \Psi(S)] \tag{2.5}$$

hold true for all matrices P and S, where $G \cdot M_k(\mathbb{C}) \to M_k(\mathbb{C})$.

Let $\alpha_1(u)$ denote a diagonal matrix with matrix elements $(\alpha_1(u))_{jj} = 1$ if $j \neq 1$ and $[\alpha_1(u)]_{11} = u$. If we set

$$f(u) = T[\alpha_1(u)], \quad \varphi(u) = \Phi[\alpha_1(u)], \quad \text{and} \quad \psi(u) = \Psi[\alpha_1(u)], \quad \text{for } u \in \mathbb{R},$$

then we get

$$f(uv) = G[\phi(u)\phi(v), \psi(u) + \psi(v)] \tag{2.6}$$

for any couple of real numbers u, v, and

$$T(P) = f(\det P)$$

for every matrix P.

Theorem 2.6. [575]. *Let T_3 be a function that is defined on the set of nonsingular matrices of order n $(M_n(\mathbb{C}))$ and takes values in $M_k(\mathbb{C})$. Let*

$$T_3(PS) = T_3(P) + T_3(S) \quad \text{(mL)} \tag{2.7}$$

for all nonsingular $P, S \in M_n(\mathbb{C})$. Then

$$T_3(P) = f_3(\det P),$$

where the function $f_3 : \mathbb{C} \to M_m(P)$ satisfies the functional equation

$$f_3(uv) = f_3(u) + f_3(v) \quad (u, v\ (\neq 0) \in \mathbb{C}), \tag{2.8}$$

which is logarithmic.

If $T_3(P)$ is a continuous function, then

$$T_3(P) = a \log |\det P|$$

for every nonsingular matrix P, where the matrix $a \in M_k(\mathbb{C})$ does not depend on P.

Proof. For $P = S = I_n$, (2.7) implies $T_3(I_n) = 0$. This and (2.7) lead to

$$T_3(S^{-1}PS) = T_3(P)$$

for any nonsingular matrix S. Thus $T_2(P)$ is a unitarily invariant function. In the same way as in Result 2.5 ($\Phi = 0$, $\Psi = T$, $G(0, u) = u$), we get $T_3(P) = f_3(\det P)$, where the function f_3 satisfies the functional equation (2.8). We thus have that a continuous function f_3 that satisfies (2.8) has the form

$$f_3(u) = a \log u \tag{2.9}$$

for $u > 0$ (see Theorem 1.2 and Corollary 1.22). Suppose that $u = \exp(it)$ and $v = \exp(is)$, where t and s are real numbers. Then the continuous function $h(t) = f_3(\exp it)$ has the properties

$$h(t + s) = h(t) \cdot h(s) \quad \text{and} \quad h(t + 2\pi) = h(t)$$

for all real numbers t and s. Thus

$$h(t) \equiv 0 \tag{2.10}$$

for all real numbers t. Now (2.8), (2.9), and (2.10) lead to

$$f_3(u) = f_3(|u| \exp i \arg u) = f_3(|u|) + f_3(\exp i \arg u) = f_3(|u|);$$

that is,

$$f_3(u) = a \log |u|$$

for every complex number $u \neq 0$. Thus $T_3(P) = a \ln |\det P|$ for any nonsingular matrix P. This concludes the proof.

Result 2.7. [575]. *Let T_4 be a unitarily invariant function defined on $M_n(\mathbb{C})$ that has values in $M_k(\mathbb{C})$. Let*

$$T_4(PS) = T_4(P) \cdot T_4(S) \quad \text{(mM)} \tag{2.11}$$

for every pair of matrices P and S. Then

$$T_4(P) = f_4(\det P),$$

where the function f_4 of a complex variable u satisfies the functional equation

$$f_4(uv) = f_4(u) f_4(v) \tag{2.12}$$

for all complex numbers u and v.

If the function T_4 is continuous and if $\det T(P) \neq 0$ for every matrix P, then

$$T_4(P) = \exp(a \log |\det P| + i N \arg \det P)$$

for every nonsingular matrix P, where a and N are square matrices of order m. Matrix N is similar to the diagonal matrix that has integers on the main diagonal.

Result 2.8. [575]. *Let T_3 and T_4 be orthogonally invariant functions defined on $M_n(\mathbb{R})$ with ranges in $M_k(\mathbb{R})$.*

If

$$T_3(PS) = T_3(P) + T_3(S) \quad \text{and} \quad T_4(PS) = T_4(P) \cdot T_4(S)$$

for every pair of matrices P and S in $M_n(\mathbb{R})$, then

$$T_3(P) = f_3(|\det P|) \quad \text{and} \quad T_4(P) = f_4(\det P).$$

Result 2.9. [575]. *Let T be defined on $M_n(\mathbb{R})$ and have values in $M_k(\mathbb{R})$. Further, let T satisfy* (Q).

$$T(P + S) + T(P - S) = 2T(P) + 2T(S) \tag{2.13}$$

holds true for every pair of matrices P and S.

If the function T is invariant with respect to a group of all nonsingular matrices (i.e., if

$$T(S^{-1}PS) = T(P)$$

for every nonsingular real matrix S and if T is a continuous function), then

$$T(P) = a\left(\sum_{j=1}^{n} P_{jj}\right)^2 + b\sum_{1\le i<j\le n}\begin{vmatrix} P_{ii} & P_{ij} \\ P_{ji} & P_{jj}\end{vmatrix} = aS_1^2(P) + bS_2(P)$$

for every matrix P where real numbers a and b do not depend on the matrix P.

The proof of this result is based on the following theorem (see Chapter 5).

Let H be a real Hilbert space and let $\alpha : H \to \mathbb{R}$ satisfy that

$$\alpha(x+y) + \alpha(x-y) = 2\alpha(x) + 2\alpha(y)$$

holds true for $x, y \in H$. If the functional α is continuous in one point or bounded on one sphere, then

$$\alpha(x) = (T'x, x)$$

for every $x \in H$, where the bounded and symmetric linear transformation T' does not depend on x. The transformation T' is uniquely determined by the functional α.

2.1 Multiplicative Equation

Now we determine the general solution of

$$f(XY) = f(X)f(Y), \quad X, Y \in M_n(\mathbb{R}), \tag{mM}$$

where $f : M_n(\mathbb{R}) \to M_k(\mathbb{R})$, $n, k \ge 1$.

The equation (mM) plays an important role in the theory of geometric objects and in the theory of invariants. This equation has been studied by many authors. For some values of n and k, it has been solved under strong assumptions of regularity of the function f (see, for example, Kucharzewski [544, 545, 546], Kuczma and Zajtz [564]).

Without any regularity supposition about the function f, equation (mM) has been solved for $n = 2$, $k = 1$ in Golab [322]. This result has been generalized to the case of $k = 1$ and an arbitrary n in Kuczma and Zajtz [565]; see also Aczél [12].

In the general case (m, n arbitrary), equation (mM) has been solved in Kurepa [575]. However, though the author makes no assumptions concerning the regularity of the function f, he imposes on f conditions of another kind (invariance under some operations) (see Theorems 2.2 and 2.6).

The equation (mM) for $k = n = 2$ is treated in Kuczma [548] in Result 2.10.

They find all solutions of the equation making no suppositions about the required function f and assume that relation (mM) *holds only for nonsingular matrices X, Y.*

For $n = 2$, $k = 1$, the solution of (mM) without any additional condition is determined in Golab [322].

Theorem 2.10. (Golab [322]). *Suppose* $f : M_2(\mathbb{R}) \to \mathbb{R}$ *satisfies* (mM). *Then*

$$f(X) = M(\det X), \tag{2.14}$$

where $M : \mathbb{R} \to \mathbb{R}$ *satisfies* (M).

Proof. Based on the proof in [322], we present the following proof.

It is obvious that $f = 0$ and $f = 1$ are solutions of (mM). We look for nontrivial solutions.

First the idea is to express f, a function of four variables, as a product of four functions of single variables $\alpha, \theta, \beta, \phi : \mathbb{R} \to \mathbb{R}$, two of which satisfy (E) on $\mathbb{R}$ that are constants and the two others satisfy (M) on $\mathbb{R}$. Define (introduce)

$$\alpha(x) = f(x, 0, 0, 1) = f\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix},$$
$$\theta(x) = f(1, 1, 0, x) = f\begin{pmatrix} 1 & 1 \\ 0 & x \end{pmatrix},$$
$$\beta(x) = f(1, 0, x, 1) = f\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix},$$

and

$$\phi(x) = f(1, x, 0, 1) = f\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \text{for } x \in \mathbb{R}.$$

α and θ satisfy (M), and β and ϕ satisfy (E). Indeed, use (mM) to get

$$\alpha(xy) = f\begin{pmatrix} xy & 0 \\ 0 & 1 \end{pmatrix} = f\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right] = \alpha(x)\alpha(y),$$
$$\beta(x+y) = f\begin{pmatrix} 1 & 0 \\ x+y & 1 \end{pmatrix} = f\left[\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\right] = \beta(x)\beta(y),$$

for $x, y \in \mathbb{R}$.

Similarly, we can show that θ is multiplicative and ϕ is exponential. If $a_{11} \neq 0$, then

$$X = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{a_{21}}{a_{11}} & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \frac{\Delta}{a_{11}} \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{a_{12}}{a_{11}} \\ 0 & 1 \end{pmatrix},$$

where Δ denotes the determinant of the matrix X. Thus

$$f(a, b, c, d) = \alpha(a)\beta\left(\frac{c}{a}\right)\theta\left(\frac{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}{a}\right)\phi\left(\frac{b}{a}\right), \tag{2.15}$$

for $a \neq 0$.

Whenever $M(a) = 0$ for some a $(\neq 0) \in \mathbb{R}$, then $M \equiv 0$ (for $M(x) = M\left(\frac{x}{a}a\right) = M\left(\frac{x}{a}\right) \cdot M(a) = 0$). Since $f \neq 0$, α and θ are never zero and

$\alpha(1) = \theta(1)$. If $E(a) = 0$ for some $a \in \mathbb{R}$, then $E \equiv 0$. As before, since $f \neq 0$, β and ϕ are nonzero and $\beta(0) = 1 = \phi(0)$ (see also Chapter 1). First we show that $\beta = 1 = \phi$.

With $X = \begin{pmatrix} a & 1 \\ a & 2 \end{pmatrix}$, $Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, (mM) and (2.15) yield

$$\alpha(a)\beta(1)\theta\left(\frac{a}{a}\right)\phi\left(\frac{a+1}{a}\right) = \alpha(a)\beta\left(\frac{a}{a}\right)\theta\left(\frac{a}{a}\right)\phi(1)\alpha(1)\beta(0)\theta(1)\phi(1);$$

that is, $\phi(a) = 1$ or $\phi \equiv 1$, or $\phi\left(\frac{a+1}{a}\right) = 1$ or $\phi(x) = 1$ for all $x \in \mathbb{R}$, except at $x = 1$. But then, since ϕ satisfies (E), $\phi(1) = \phi\left(\frac{1}{2}\right)^2 = 1$. Hence $\phi \equiv 1$.

Set $X = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$, $Y = \begin{pmatrix} 1 & 0 \\ 1 & b \end{pmatrix}$, and use (mM), (2.15), and $\phi = 1$ to obtain

$$\alpha(1)\beta(d)\theta(bd) = \alpha(1)\beta(0)\theta(d) \cdot \alpha(1)\beta(1)\theta(b),$$

which gives $\beta(d) = 1$ (use θ is multiplicative, $\alpha(1) = 1$), so that

$$f(a,b,c,d) = \alpha(a)\theta\left(\frac{ad-bc}{a}\right) \quad \text{for } a \neq 0. \tag{2.15a}$$

Finally, we show that $\alpha = \theta$, so that $f(a,b,c,d) = \phi\left(\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ with ϕ multiplicative, which is (2.13).

Let $X = \begin{pmatrix} 1 & b-1 \\ 0 & d \end{pmatrix}$, $Y = \begin{pmatrix} 1 & 0 \\ 1 & c \end{pmatrix}$ in (mM). Use (2.15a) to have

$$\alpha(b)\theta\left(\frac{dc}{b}\right) = \alpha(1)\theta(d) \cdot \alpha(1)\theta(c);$$

that is, $\alpha(b)\phi\left(\frac{1}{b}\right) = 1$ (since θ is multiplicative, $\alpha(1) = 1$). Thus, $\alpha(b) = \theta(b)$ for all b ($\neq 0$), and (2.15a) yields

$$f(a,b,c,d) = \phi\left(\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right), \tag{2.14}$$

where ϕ satisfies (M). This proves the theorem. □

The case $n = 2 = k$

Result 2.11. (Kuczma [548]). *Let $f : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ satisfy the functional equation* (mM) *for all nonsingular matrices X and Y. Then we have*

$$f(X) = 0 \tag{2.16}$$

or

$$f(X) = \phi(\det X) \cdot C \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} C^{-1} \tag{2.16a}$$

or

$$f(X) = \phi(\det X) \cdot C \cdot X \cdot C^{-1} \tag{2.17}$$

or

$$f(X) = G(X), \tag{2.17a}$$

where ϕ is given by

$$\phi(x) = \begin{Vmatrix} m(x) & 0 \\ 0 & m(x) \end{Vmatrix},$$

where $m \not\equiv 0$ is an arbitrary multiplicative function on $\mathbb{R}$, G is an arbitrary (matrix-valued) function satisfying the equation (M) *and the condition $G(1) = I_2$, and C is an arbitrary nonsingular matrix (and thus playing the role of a parameter in formulas* (2.16a) *and* (2.17)). *Formulas* (2.16) *and* (2.16a) *give the singular solutions of* (mM)*, while solutions* (2.17) *and* (2.17a) *are nonsingular.*

Case n arbitrary, $k = 1$

Let $f : M_n(F) \to F$ ($F = \mathbb{R}$ or $\mathbb{C}$) satisfy (mM). It was shown that f has the form (2.14). A simplified proof is given in Kucharzewski and Kuczma [547]. Here we present the proof in Hosszú [379] (also see Karenska [527]).

It is a well-known theorem [299] that every matrix X has a (polar decomposition) factorization $X = HU$, where H is Hermitian and U is unitary, and hence both factors are equivalent to diagonal matrices. On the other hand, the value of f is the same for equivalent matrices, just like the value of the determinant, since

$$f(CXC^{-1}) = f(C)\,f(X)\,f(C^{-1}) = f(C)\,f(C^{-1})\,f(X) = f(CC^1X) = f(X).$$

So X is a product of two function values depending on diagonal matrices, and hence it depends only on a diagonal matrix D having the same determinant as X since the determinant is also a multiplicative function. Therefore, considering the factorization

$$\begin{aligned} D &= \begin{pmatrix} d_1 & 0 & \cdots & & \\ 0 & d_2 & 0 & \cdots & \\ 0 & 0 & d_3 & 0 & \cdots \\ 0 & 0 & \cdots & \cdots & d_n \end{pmatrix} \\ &= \begin{pmatrix} d_1 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ & \cdots & & \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & & \\ 0 & d_2 & 0 & \cdots & \\ 0 & 0 & 1 & 0 & \cdots \\ & \cdots & & & \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 & \cdots & & \\ 0 & 1 & 0 & \cdots & \\ & \cdots & & & \\ 0 & 0 & \cdots & 0 & d_n \end{pmatrix} \\ &= \prod_{k=1}^{n} P_k \begin{pmatrix} d_k & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ & \cdots & & \end{pmatrix} P_k^{-1}, \end{aligned}$$

where P_k consists of the elements of the unit matrix but the first and kth rows are permuted, we get

$$\begin{aligned} f(X) = f(D) &= \prod_{k=1}^{n} f \begin{pmatrix} d_k & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ & \cdots & & \end{pmatrix} \\ &= f \begin{pmatrix} \prod_{k=1}^{n} d_k & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ & \cdots & & \end{pmatrix} \\ &= f(\det D) = f(\det X) \end{aligned}$$

for every $X \in M_n(F)$.

The theorem proved above gives the possibility of axiomatizing determinants without coordinates by the multiplicativity and homogeneity:

$$f(\lambda X) = \lambda^n f(X), \quad \lambda \in F,\ X \in M_n(F).$$

Result 2.12. [565]. *Let* $f : \mathbb{R}^* \to M_3(\mathbb{R})$ *satisfy* (mM). *Then* f *has one of the forms*

$$f(x) = C \begin{pmatrix} M_1(x) & 0 & 0 \\ 0 & M_2(x) & 0 \\ 0 & 0 & M_3(x) \end{pmatrix} C^{-1},$$

$$f(x) = C \begin{pmatrix} M(x) & M(x)L(x) & 0 \\ 0 & M(x) & 0 \\ 0 & 0 & M_3(x) \end{pmatrix} C^{-1},$$

$$f(x) = C \begin{pmatrix} g(x) & -\sigma(x) & 0 \\ \sigma(x) & g(x) & 0 \\ 0 & 0 & M(x) \end{pmatrix} C^{-1},$$

$$f(x) = C \begin{pmatrix} M(x) & M(x)L_1(x) & M(x)L_2(x) \\ 0 & M(x) & 0 \\ 0 & 0 & M(x) \end{pmatrix} C^{-1},$$

$$f(x) = C \begin{pmatrix} M(x) & M(x)L_1(x) & 0 \\ 0 & M(x) & 0 \\ 0 & M(x)L_2(x) & M(x) \end{pmatrix} C^{-1},$$

$$f(x) = C \begin{pmatrix} M(x) & M(x)L_1(x) & M(x)[L_2(x) + \frac{1}{2}L_1^2(x)] \\ 0 & M(x) & M(x)L_1(x) \\ 0 & 0 & M(x) \end{pmatrix} C^{-1},$$

where M, M_1, M_2, M_3 *satisfy* (M), L, L_1, L_2 *satisfy* (L), C *is a nonsingular matrix, and* g *and* σ *satisfy the system of functional equations*

$$g(xy) = g(x)g(y) - \sigma(x)\sigma(y), \quad \sigma(xy) = \sigma(x)g(y) + g(x)\sigma(y), \quad x, y \neq 0.$$

Case $n = 1$, k arbitrary

See Kuczma and Zajtz [564] and Karenska [527] for the solutions $f : \mathbb{R} \to GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ of (mM), where $GL(n, \mathbb{R})$ (linear group) is the set of all nonsingular matrices of order n over $\mathbb{R}$.

Case $n = 2$, $k = 3$

We now consider the functional equation

$$M(XY) = M(X) \cdot M(Y), \tag{mM}$$

where X and Y are nonsingular 2×2 real matrices (i.e., $x, y \in GL(2, \mathbb{R})$) and M is an unknown function whose values are 3×3 real matrices. We do not make any assumptions concerning the regularity of the matrix function M.

Result 2.13. *A function M satisfying functional equation* (mM) *for every $X, Y \in GL(2, \mathbb{R})$ is given by the formulas*

$$M(X) = C \cdot m(\det X) \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & m_0(\det X) \end{pmatrix} \cdot C^{-1},$$

$$M(X) = C \cdot m(\det X) \begin{pmatrix} x_{11}^2 & 2x_{11}x_{12} & x_{12}^2 \\ x_{11}x_{21} & x_{11}x_{22} + x_{21}x_{12} & x_{12}x_{22} \\ x_{21}^2 & 2x_{21}x_{22} & x_{22}^2 \end{pmatrix} \cdot C^{-1},$$

$$M(X) = G(\det X) = -C \cdot G^*(\det X) \cdot C^{-1},$$

where $G^*(\det X)$ has one of the six forms

$$G^*(\det X) = \begin{pmatrix} m_1(\det X) & 0 & 0 \\ 0 & m_2(\det X) & 0 \\ 0 & 0 & m_3(\det X) \end{pmatrix},$$

$$G^*(\det X) = m(\det X) \begin{pmatrix} 1 & L(\det X) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & m_0(\det X) \end{pmatrix},$$

$$G^*(\det X) = \begin{pmatrix} \chi(\det X) & -\sigma(\det X) & 0 \\ \sigma(\det X) & \chi(\det X) & 0 \\ 0 & 0 & m_3(\det X) \end{pmatrix},$$

$$G^*(\det X) = m(\det X) \begin{pmatrix} 1 & L_1(\det X) & \frac{1}{2}\alpha_1^2(\det X) + L_2(\det X) \\ 0 & 1 & L_1(\det X) \\ 0 & 0 & 1 \end{pmatrix},$$

$$G^*(\det X) = m(\det X) \begin{pmatrix} 1 & L(\det X) & L_2(\det X) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$G^*(\det X) = m(\det X) \begin{pmatrix} 1 & 0 & L_1(\det X) \\ 0 & 1 & L_2(\det X) \\ 0 & 0 & 1 \end{pmatrix},$$

$$M(X) = 0,$$

$$M(X) = C \cdot \begin{pmatrix} m(\det X) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot C^{-1},$$

$$M(X) = C \cdot \begin{pmatrix} m(\det X) & 0 & 0 \\ 0 & m(\det X) & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot C^{-1},$$

$$M(X) = C \cdot \begin{pmatrix} m_1(\det X) & 0 & 0 \\ 0 & m_2(\det X) & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot C^{-1},$$

$$M(X) = C \cdot \begin{pmatrix} m(\det X) & m(\det X)L(\det X) & 0 \\ 0 & m(\det X) & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot C^{-1},$$

$$M(X) = C \cdot \begin{pmatrix} \chi(\det X) & -\sigma(\det X) & 0 \\ \sigma(\det X) & \chi(\det X) & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot C^{-1},$$

where m and m_i $(i = 0, 1, 2, 3)$ are arbitrary multiplicative functions not vanishing identically (i.e., satisfying the equation (M)*), L and L_i $(i = 1, 2)$ are arbitrary functions satisfying the equation* (L)*, and the functions χ and σ are a solution of the system of functional equations*

$$\begin{aligned} \chi(\xi\eta) &= \chi(\xi)\chi(\eta) - \sigma(\xi)\sigma(\eta), \\ \sigma(\xi\eta) &= \chi(\xi)\sigma(\eta) + \chi(\eta)\sigma(\xi), \end{aligned} \qquad \xi\eta \neq 0,$$

fulfilling the condition

$$\sigma(\xi) \not\equiv 0, \quad X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

The solutions of the function equation (mM) *given by the first nine are nonsingular, and the rest are singular.*

Remark. The general nonsingular solution of the functional equation represents all homomorphisms $M : GL(2, \mathbb{R}) \to GL(3, \mathbb{R})$. It means all real linear representations 3×3 of the group $GL(2, \mathbb{R})$.

In McKiernan [630] (see also [631]), the following system of equations was considered, where summation on integer n is assumed.

Find all $a_j^i : G \to F$ satisfying

$$a_j^i(x \circ y) = a_j^i(x) + a_n^i(x)a_j^n(y) + a_j^i(y), \tag{2.18}$$

together with

$$a_j^i(x) = 0 \quad \text{for all } i \leq j, \text{ for all } x, y \in G, \tag{2.18a}$$

where G is a groupoid with 0 and F is a ring.

In view of (2.18a), summation on all n in (2.18) reduces to summation on $j < \alpha < i$ for each fixed i, j. In the event that a^i_j depend only on the difference $i - j$, this problem reduces to (2.19) with $g_{i-j} := a^i_j$,

$$g_k(x+y) = g_k(x) + \sum_{\alpha=1}^{k-1} g_\alpha(x) g_{k-\alpha}(y) + g_k(y), \tag{2.19}$$

where $g_i : \mathbb{R}$ or $\mathbb{C} \to F$ is connected with a characterization of "polynomials of binomial type" [690] to a composition law for Poisson distribution [49].

Result 2.14. (McKiernan [630]). *Suppose G is a commutative groupoid with F a commutative and associative algebra over the rationals and F_∞ the infinite matrices with elements in F. Then a necessary and sufficient condition that there exist $a : G \to F_\infty$ satisfying*

$$\begin{aligned} &a(x \circ y) = a(x) + a(x)a(y) + a(y) && \text{for all } x, y \in G, \\ &a(x) \simeq 0 && \text{for } 1 - j \leq 0 \text{ and for all } x \in G, \end{aligned}$$

is that there exist $L : G \to F_\infty$ satisfying

$$\begin{aligned} &L(x) = 0 && \text{for } i - j \leq 0 \text{ and all } x \in G, \\ &L(x \circ y) = L(x) + L(y) && \text{for all } x, y \in G, \\ &L(x)L(y) = L(y)L(x) && \text{for all } x, y \in G, \end{aligned}$$

and given such a, L, then

$$a(x) = \sum_{k=1}^{\infty} \frac{1}{k!} L(x)^k \qquad \text{for all } x \in G.$$

2.2 Cosine Matrix Equation

Let G be an Abelian group, and consider the functional equations

$$f(x+y) = \sum_{i=1}^{n} h_i(x) k_i(y) \tag{2.20}$$

and

$$f(x+y) + g(x-y) = \sum_{i=1}^{n} h_i(x) k_i(y), \tag{2.21}$$

where n is a fixed positive integer and f, g, h_i, k_i are complex-valued functions on the group G. Equations of the form (2.20) and (2.21) have been treated by many authors (see Chapter 3).

Aczél raised the question, what is the general form of the solutions of (2.20) and (2.21)? In some cases, the general solution is well known even in the case of Abelian semigroups. For instance, (2.20) is a common generalization of the additive and multiplicative Pexider equations, and (2.21) is a generalization of the d'Alembert equation.

Concerning (2.20), in Kuczma and Zajtz [565] and McKiernan [630] it is observed that it is closely related to the matrix equation

$$E(x+y) = E(x)E(y) \tag{mE}$$

and can be solved by the simultaneous diagonalization of the matrices $E(x)$. Using this method, in [630, 631] is given the general solution of (2.20) under some simple conditions.

In [787], the author extends the above-mentioned method for solving (2.21) by reducing it to the matrix equation (mE) and

$$C(x+y) + C(x-y) = 2C(x)C(y), \tag{C}$$

and all solutions f, g of (2.21) are exponential polynomials and so are the functions h_i, k_i under some conditions on their linear independence.

The Matrix Equation (C)

We show now that, under the assumption $C(0) = I_n$, the components of any solution C of (C) are exponential polynomials.

Theorem 2.15. (Székelyhidi [787]). *Let G be a topological Abelian group, n a positive integer, and $C : G \to M(\mathbb{C}^n)$ a continuous function for which the functional equation* (C) *holds for all x and y in G. If $C(0) = I_n$ and $C(x_0)^2 - I_n$ is regular for some x_0 in G, then there exists a continuous function $E : G \to M_n(\mathbb{C}^n)$ satisfying* (mE) *such that*

$$C(x) = \frac{1}{2}(E(x) + E(-x))$$

holds for all x in G. (*See* [424] *and* Chapter 3, Theorem 3.21.)

Proof. The proof follows the method in [424]. As $C(0) = I_n$, we see from (C) that C is even. Then, by (C),

$$C(x)C(y) = \frac{1}{2}[C(x+y) + C(x-y)] = \frac{1}{2}[C(y+x) + C(y-x)] = C(y)C(x);$$

that is, the matrices $C(x)$ commute. On the other hand, it follows that $C(2x) + I = 2C(x)^2$ for all x in G. Denote the matrix $C(x_0)$ by P. It is well known [376] that there exists a regular matrix S that commutes, with every matrix commuting with $P^2 - I_n$, and S satisfies the equation $S^2 = P^2 - I_n$. But

$$P^2 - I = C(x_0)^2 - I_n = \frac{1}{2}(C(2x_0) - I_n)$$

commutes with every value of C, and hence so does S. By (C) we have the following equations for x and y in G:

$$\begin{aligned} 2C(x+y)C(x-y) &= C(2x)+C(2y), \\ 2[C(x_0+x)C(y)+C(x_0+y)C(x)] &= C(x_0+x+y)+C(x_0+x-y) \\ &\quad +C(x_0+x+y)+C(x_0+y-x) \\ &= 2C(x_0+x+y)+2C(x_0)C(x-y) \\ &= 2[C(x_0+x+y)+P(2C(x)C(y) \\ &\quad -C(x+y))], \end{aligned}$$

and

$$\begin{aligned} 2C(x_0+x)C(x_0+y) &= C(2x_0+x+y)+C(x-y) \\ &= [2C(x_0)C(x_0+x+y)-C(x+y)] \\ &\quad +[2C(x)C(y)-C(x+y)] \\ &= 2[C(x)C(y)+C(x_0)C(x_0+x+y)-C(x+y)], \end{aligned}$$

and finally

$$\begin{aligned} [C(x+y)-C(x-y)]^2 &= [C(x+y)+C(x-y)]^2-4C(x+y)C(x-y) \\ &= 4C(x)^2C(y)^2-4C(x)^2-4C(y)^2+4I \\ &= 4(C(x)^2-I)(C(y)^2-I). \end{aligned}$$

Let

$$\begin{aligned} E(x) &= C(x)+S^{-1}[C(x_0+x)-C(x_0)C(x)] \\ &= S^{-1}[C(x_0+x)+(S-P)C(x)] \end{aligned}$$

for all x in G. Then an easy computation shows that E satisfies (mE) and by definition

$$\frac{1}{2}(E(x)+E(-x))=\frac{1}{2}S^{-1}[C(x_0+x)+C(x_0-x)+2(S-P)C(x)]=C(x),$$

which was to be proved.

Result 2.16. [787]. *Let G be an infinite topological Abelian group in which division by 2 is defined, n is a positive integer, and $C : G \to M_n(\mathbb{C})$ is a continuous function for which the functional equation* (C) *holds for all x and y in G. If $C(0) = I_n$, then there exist continuous functions $E : G \to M_n(\mathbb{C})$ satisfying* (mE) *and $N : G \to M_{n-n_1}(\mathbb{C})$ satisfying* (C) *such that $E(x)$ is triangular from above for all x in G, $N(x) - I_n$ is strictly triangular from above for all x in G, and further there is a regular matrix T in $M_n(\mathbb{C})$ for which the representation*

$$C(x)=T^{-1}\mathrm{diag}\left\{N(x),\frac{1}{2}(E(x)+E(-x))\right\}T$$

holds for all x in G.

For the following result only, we denote the elements of $\mathbb{R}^n$ columnwise. Suppose $f, g, h, k : \mathbb{R}^n \to \mathbb{R}^*$ satisfy the functional equation

$$f(\alpha x + \beta y)g(\gamma x + \delta y) = h(x)k(y), \quad \text{for } x, y \in \mathbb{R}^n, \tag{2.22}$$

and $\alpha,\ \beta,\ \gamma,\ \delta$ are nonsingular square matrices of order n; that is, in $GL(n, \mathbb{R})$. Note that $f,\ g,\ h,\ k$ are never zero.

Let $F, G, H, K : \mathbb{R}^n \to \mathbb{R}^*$ be defined by

$$F(x) = \frac{f(x)}{f(0)}, \quad G(x) = \frac{g(x)}{g(0)}, \quad H(x) = \frac{h(x)}{h(0)}, \quad K(x) = \frac{k(x)}{k(0)}, \quad \text{for } x \in \mathbb{R}^n,$$

and note that $F(0) = G(0) = H(0) = K(0) = 1$ and (2.22) becomes

$$F(\alpha x + \beta y)G(\gamma x + \delta y) = H(x)K(y), \quad \text{for } x, y \in \mathbb{R}^n. \tag{2.23}$$

The following theorem is obvious.

Theorem 2.17. *Under the condition $f, g, h, k \neq 0$, the solutions of the functional equation* (2.22) *are given by the functions*

$$f(x) = aF(x), \quad g(x) = bG(x), \quad h(x) = cH(x), \quad k(x) = dK(x),$$

where $F(x),\ G(x),\ H(x),\ K(x)$ are the solutions of the functional equation (2.23) *under conditions $F(0) = H(0) = G(0) = K(0) = 1$ and* (2.23)*, and $a,\ b,\ c,\ d$ are arbitrary constants satisfying*

$$ab - cd = 0, \quad abcd \neq 0.$$

Proof. By putting $x = 0$ and $y = 0$ into (2.23) separately, we have

$$F(\alpha x)G(\gamma x) = H(x), \tag{2.24}$$

$$F(\beta y)G(\delta y) = K(y). \tag{2.24a}$$

Substituting (2.24) and (2.24a) in (2.23), we obtain

$$F(\alpha x + \beta y)G(\gamma x + \delta y) = F(\alpha x)F(\beta y)G(\gamma x)G(\delta y). \tag{2.25}$$

For $u, v \in \mathbb{R}$, there are $x, y \in \mathbb{R}^n$ such that

$$u = Ax \quad \text{and} \quad v = By,$$

and (2.25) can be rewritten as

$$F(u + v)G(Pu + Sv) = F(u)F(v)G(Pu)G(Sv), \tag{2.26}$$

where

$$P = \gamma\alpha^{-1}, \quad S = \delta\beta^{-1}.$$

Case 1. Let $P = S$. Then (2.26) reduces to

$$E(u+v) = E(u)E(v), \tag{E}$$

where

$$E(u) = F(u)G(Pu),$$

and from (2.24) and (2.24a) we obtain

$$H(u) = E(\alpha u), \quad K(u) = E(\beta u).$$

Thus, in this case we get

$$h(x) = cE(\alpha x), \quad k(x) = dE(\beta x), \tag{2.27}$$

f, g arbitrary with

$$f(x)g(\gamma\alpha^{-1}x) = abE(x),$$

where E satisfies (E), $a = f(0),\ b = g(0)$.

Case 2. Let $P \neq S$.
The proof involves the functional equation

$$\phi(x+y) + \phi(x-y) = 2(\phi(x) + \phi(y) + \phi(-y) \tag{4.22}$$

studied in Chapter 4.

If one of the inequalities $\rho(T) < 1$ or $\rho(T^{-1}) < 1$ ($T = \delta\beta^{-1}\alpha\gamma^{-1}$), where $\rho(T)$ is the spectral radius of T, then the continuous solution of the equation (2.22) is given by

$$\begin{aligned} f(x) &= a\exp(b_1^t x + x^t N_1 x), \quad g(x) = b\exp(b_2^t x + x^t N_2 x),\\ h(x) &= c\exp(b_3^t x + x^t N_3 x), \quad k(x) = d\exp(b_4^t x + x^t N_4 x), \end{aligned} \tag{2.28}$$

where

$$N_3 = \alpha^t N_1 \alpha + \gamma^t N_2 \gamma, \quad N_4 = \beta^t N_1 \beta + \delta^t N_2 \delta, \tag{2.29}$$

with

$$\alpha^t N_1 \beta + \gamma^t N_2 \delta = 0,\ b_3^t = b_1^t\alpha + b_2\gamma,\ b_4^t = b_1^t\beta + b_2\delta,\ c = h(0),\ d = k(0),$$

$b_1, b_2, b_3, b_4 \in \mathbb{R}^n$.

When $P = -S$ and $\alpha\gamma^{-1} + \beta\delta^{-1} = 0$, the continuous solutions of the equation (2.22) are given by

$$\begin{aligned} f(x) &= a\exp(x^t N_1 x), \quad g(x) = b\exp(x^t N_2 x),\\ h(x) &= c\exp(x^t N_3 x), \quad k(x) = d\exp(x^t N_4 x), \end{aligned} \tag{2.30}$$

where N_3, N_4 satisfy (2.28) and $\alpha^t N_1 \beta + \gamma^t N_2 \delta = 0$.

Result 2.18. (Ecsedi [248]). *The continuous solutions of* (2.22) *are given by either* (2.27), (2.28), *or* (2.29).

For the solution of the matrix equation

$$\sum_{\rho=1}^{r} f_\rho(\alpha) X g_\rho(\beta) = C,$$

where $\alpha \in M_n(\mathbb{C})$, $\beta \in M_m(\mathbb{C})$, $C \in M_{mn}(\mathbb{C})$, $X \in M_{mn}(\mathbb{C})$, see Wimmer and Ziebur [832]. A special case of this is the well-known equation $\alpha X + X\beta = C$.

3

Trigonometric Functional Equations

In this chapter, trigonometric functional equations are studied on reals, groups, Hilbert space, etc., and cosine equations on reals, non-Abelian groups, and Hilbert space are studied. Sine equations, analytic solutions, addition and subtraction formulas, operator values, and some generalizations are studied also, and counterexamples are provided.

One of the important applications of functional equations is a functional characterization of various functions such as Euler's gamma function, Lebesgue's singular function, cyclic functions, polynomials, exponential and logarithmic functions, etc. (the last three are treated in Chapters 7 and 1). The most extensively studied problem of this type is the functional characterization of the trigonometric functions. Trigonometric functions—the sine and cosine functions—can be introduced in many ways: in terms of infinite series, as solutions of differential equations, by exponential functions, or by geometry. In this chapter, the stress is going to be on functional equations. In this section, we will treat many functional equations connected with trigonometric functions, especially sine and cosine. We are all familiar with so many trigonometric formulas (identities). From the (addition) formulas

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x, \quad x, y \in \mathbb{R}, \tag{3.1}$$

one obtains

$$\sin(x + y) + \sin(x - y) = 2 \sin x \cos y, \quad \text{for } x, y \in \mathbb{R}, \tag{3.2}$$

$$\sin(x + y) - \sin(x - y) = 2 \sin y \cos x, \quad x, y \in \mathbb{R}, \tag{3.3}$$

$$\sin(x + y) \sin(x - y) = \sin^2 x - \sin^2 y, \quad x, y \in \mathbb{R}, \tag{3.4}$$

and (3.4) leads to the study of the *sine functional equation*

$$g(x + y)g(x - y) = g(x)^2 - g(y)^2 \quad \text{for } x, y \in \mathbb{R}. \tag{S}$$

From the (addition) formulas

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y, \quad \text{for } x, y \in \mathbb{R}, \tag{3.5}$$

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one derives

$$\cos(x-y)-\cos(x+y)=2\sin x\sin y,\quad x,y\in\mathbb{R}, \tag{3.6}$$

$$\cos(x+y)+\cos(x-y)=2\cos x\cos y,\quad \text{for } x,y\in\mathbb{R}, \tag{3.7}$$

and (3.7) leads to the study of the well-known functional equation known in the literature (Aczél and Dhombres [44], Aczél [12], Davison [211], Kannappan [426], Kurepa [577], and Stetkaer [772]) as the *cosine equation*, *d'Alembert equation*, or *Poisson equation*,

$$f(x+y)+f(x-y)=2f(x)f(y),\quad \text{for } x,y\in\mathbb{R}. \tag{C}$$

Functional equations arise from applications—the parallelogram law of forces or the rule for addition of vectors leading to the study of (C) and the problem of vibration of strings leading to the study of

$$f(x+y)-f(x-y)=g(x)h(y),\quad \text{for } x,y\in\mathbb{R}, \tag{VS}$$

were solved by d'Alembert in three papers in 1747, 1750, and 1769 [189] (see among others Poisson [668, 669], Aczél [12], Aczél and Dhombres [44], Kannappan [426] and Kurepa [580]). The solution of (C) was obtained by d'Alembert by reducing (C) to a differential equation, a method quite common until the middle of the last century. One of the interesting aspects of functional equations is that, unlike in differential equations, one equation can determine many unknown functions. (VS) contains three unknown functions.

We are going to look at these identities and others from the point of view of functional equations and obtain some interesting results.

We will obtain general solutions of these and many more functional equations satisfied by trigonometric functions in this chapter. We will solve these equations without assuming any regularity conditions by sometimes reducing them to Cauchy and Pexider equations. We will also treat these equations on different structures for domain and range; that is, on abstract spaces. The literature in this area is voluminous—for a partial list, see [12, 40, 44, 88, 147, 162, 189, 211, 282, 348, 422, 552, 426, 424, 577, 582, 579, 580, 477, 419, 717, 729, 772, 750, 795, 776, 767, 830, 831].

Sometimes the established solutions of the unknown functions result in the usual trigonometric identity, but sometimes they do not. When the unknown functions are specialized as sine or cosine, the results obtained turn out to be the familiar trigonometric identity. But the unknown functions need not be sine or cosine functions. We illustrate these in Section 3.1.

First we will consider simple equations (mixed) containing trigonometric and general unknown functions and obtain some unexpected results. Then we will treat general trigonometric equations that do not contain trigonometric functions.

3.1 Mixed Trigonometric Equations

We start off with the equation

$$f(x+y)+f(x-y)=2\cos x\cos y \quad \text{for } x,y\in\mathbb{R}, \tag{3.8}$$

where $f:\mathbb{R}\to\mathbb{R}$ (cf. (3.7)).

$y=0$ in (3.8) yields $f(x)=\cos x$ (no surprise), giving the following result.

Result 3.1. *A function $f:\mathbb{R}\to\mathbb{R}$ satisfies* (3.8) *if and only if $f(x)=\cos x$.*

Now we replace the cosine by the sine and consider

$$f(x+y)+f(x-y)=2\sin x\sin y \quad \text{for } x,y\in\mathbb{R}, \tag{3.9}$$

where $f:\mathbb{R}\to\mathbb{R}$.

Equation (3.9) has no solution (not surprising); put $y=0$ to get $f(x)=0$, which is not a solution of (3.9). But if we consider for $f,g:\mathbb{R}\to\mathbb{R}$

$$f(x+y)+g(x-y)=2\sin x\sin y, \quad \text{for } x,y\in\mathbb{R}, \tag{3.10}$$

we get a nontrivial solution.

Equation (3.10) can be rewritten as

$$f(x+y)+g(x-y)=\cos(x-y)-\cos(x+y)$$

or

$$f(u)+\cos u=\cos v-g(v)=c$$

(set $x+y=u,\ x-y=v$). That is, $f(u)=c-\cos u,\ g(u)=\cos u-c$. Hence we get the following result.

Result 3.2. *The only solution $f,g:\mathbb{R}\to\mathbb{R}$ of* (3.10) *is given by $f(x)=c-\cos x,\ g(x)=\cos x-c$ (cf.* (3.6)).

Result 3.1a. *Let $f:\mathbb{R}\to\mathbb{R}$ be a solution of the equation*

$$f(x+y)-f(x-y)=2\sin x\sin y \quad \textit{for } x,y\in\mathbb{R}. \tag{3.9a}$$

Then $f(x)=-\cos x+c$, where c is a constant (cf. (3.6)).

Proof. Let f satisfy (3.9a). Then

$$f(x+y)-f(x-y)=\cos(x-y)-\cos(x+y);$$

that is,

$$f(x+y)+\cos(x+y)-f(x-y)+\cos(x-y)$$

(with $x + y = u,\ x - y = v$) shows that

$$f(u) = -\cos u + c,$$

which proves the result. □

Whereas for $f, g : \mathbb{R} \to \mathbb{R}$

$$f(x + y) + f(x - y) = 2\cos x \sin y, \quad \text{for } x, y \in \mathbb{R}, \tag{3.11}$$

has no solution,

$$f(x + y) + g(x - y) = 2\cos x \sin y, \quad \text{for } x, y \in \mathbb{R}, \tag{3.12}$$

has the solution $f(x) = \sin x + c,\ g(x) = -\sin x - c$ (cf. (3.10), (3.3)).

In (3.11), let $y = 0$ to have $f(x) = 0$, which cannot be.

As for (3.12), write the right side of (3.12) as $\sin(x + y) - \sin(x - y)$ (then use $x + y = u,\ x - y = v$) to get the solution sought.

Result 3.2a. *The only function $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation*

$$f(x + y) + f(x - y) = 2\sin x \cos y, \quad \textit{for } x, y \in \mathbb{R}, \tag{3.11a}$$

is $f(x) = \sin x$ (cf. (3.2)).

Proof. Put $y = x$ in (3.11a) to have

$$f(2x) + f(0) = 2\sin x \cos x = \sin 2x,$$

which gives the desired result (put $x = 0$ to get $f(0) = 0$). □

Now we will solve

$$f(x + y) + g(x - y) = 2\sin x \cos y \quad \text{for } x, y \in \mathbb{R}, \tag{3.13}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ (cf. (3.2)).

Let $y = 0$ in (3.13) to obtain $f(x) + g(x) = 2\sin x$. Replace y by $-y$ in (3.13) and subtract the resultant from (3.13) to have

$$f(x + y) + g(x - y) - f(x - y) - g(x + y) = 0$$

or

$$f(u) - g(u) = f(v) - g(v) = c$$

(set $x + y = u,\ x - y = v$). Hence $f(x) = \sin x + \frac{c}{2}$ and $g(x) = \sin x - \frac{c}{2}$. Thus we have proved the following result.

Result 3.3. *The only general solutions $f, g : \mathbb{R} \to \mathbb{R}$ of* (3.13) *are $f(x) = \sin x + c,\ g(x) = \sin x - c$, where c is a constant.*

Associativity is a useful tool. We have used it before in Chapter 1. We will highlight its usefulness in the next result and at many places in this chapter.

Result 3.4. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a solution of the equation*

$$f(x+y) + f(x-y) = 2f(x)\cos y \quad \text{for } x, y \in \mathbb{R} \tag{3.14}$$

(*cf.* (3.6)) (*see* [12, 44, 144, 426, 580]). *Then* $g : \mathbb{R} \to \mathbb{R}$ *defined by* $g(x) = f(x) - d\cos x$ *satisfies the functional equation*

$$g(x+y) = g(x)\cos y + g(y)\cos x \quad \text{for } x, y \in \mathbb{R} \tag{3.15}$$

(*cf.* (3.5)). *The general solution of* (3.15) *is given by* $g(x) = b\sin x$, *and that of* (3.14) *is given by* $f(x) = d\cos x + b\sin x$.

Proof. Let $x = 0$ in (3.14) to have $f(y) + f(-y) = 2d\cos y$.

Interchange x and y in (3.14) and add the resultant to obtain

$$2f(x+y) + f(x-y) + f(y-x) = 2f(x)\cos y + 2f(y)\cos x,$$

which by the relation above becomes

$$2f(x+y) + 2d\cos(x-y) = 2f(x)\cos y + 2f(y)\cos x;$$

that is, with $g(x) = f(x) - d\cos x$, the equation above can be written as (3.15). Now we solve (3.15), using the associativity of $(+)$ and (3.15),

$$\begin{aligned} g(x+y+z) &= (g(x)\cos y + g(y)\cos x)\cos z + g(z)\cos(x+y) \\ &= g(x)\cos(y+z) + (g(y)\cos z + g(z)\cos y)\cos x, \end{aligned}$$

which by equating the right sides yields

$$g(x)\sin y \sin z = g(z)\sin y \sin x \quad \text{for } x, y, z \in \mathbb{R}.$$

Fix y, z with $\sin y \neq 0$, $\sin z \neq 0$ to obtain $g(x) = b\sin x$ and then $f(x) = d\cos x + b\sin x$. This proves the result. □

Remark. Whereas $g(x) = b\sin x$ in (3.15) gives the addition formula for sine (3.1) and the solution also is no surprise, $f(x) = d\cos x$ in (3.14) gives the familiar addition formula for cosine (3.5) but $f(x)$ need not be cosine. Further, the same conclusion and result hold even for $f : \mathbb{R} \to \mathbb{C}$, satisfying (3.14). Note that the solutions of (3.14) and (3.15) are obtained without assuming any regularity condition.

Similar to equation (3.14), we consider (3.16) in Theorem 3.5.

Theorem 3.5. *For* $f : \mathbb{R} \to \mathbb{R}$, *there is a* $g : \mathbb{R} \to \mathbb{R}$ *satisfying the functional equation*

$$f(x-y) - f(x+y) = 2g(x)\sin y, \quad \text{for } x, y \in \mathbb{R}, \tag{3.16}$$

if and only if $f(x) = a\cos x - d\sin x + c$, $g(x) = a\sin x + d\cos x$.

Proof. Set $x = 0$ in (3.16) to get $f(-y) - f(y) = 2d \sin y$.

Interchange x and y in (3.16) and subtract the resultant from (3.16) to have

$$f(x-y) - f(y-x) = 2g(x)\sin y - 2g(y)\sin x;$$

that is,

$$-2d\sin(x-y) = 2g(x)\sin y - 2g(y)\sin x$$

or

$$(g(x) - d\cos x)\sin y = (g(y) - d\cos y)\sin x,$$

so that

$$g(x) = d\cos x + a\sin x$$

(fix y with $\sin y \neq 0$). Substitution of this $g(x)$ in (3.16) results in

$$\begin{aligned} f(x-y) - f(x+y) &= 2(d\cos x + a\sin x)\sin y \\ &= d(\sin(x+y) - \sin(x-y)) + a(\cos(x-y) - \cos(x+y)) \end{aligned}$$

or

$$f(u) - a\cos u + d\sin u = c$$

(set $x + y = u,\ x - y = v$). This proves the result. □

Remark. $f(x) = \cos x,\ g(x) = \sin x$ results in the trigonometric identity (3.6), but f need not be cosine and g need not be sine.

Now we will consider a few equations related to sine before treating general functional equations connected with trigonometric functions and on different structures. First we consider the three simple equations

$$\sin(x+y) = f(x)\sin y + f(y)\sin x, \quad \text{for } x, y \in \mathbb{R} \tag{3.17}$$

(cf. (3.1), [118]), a slight generalization of (3.17),

$$\sin(x+y) = f(x)\sin y + g(y)\sin x, \quad \text{for } x, y \in \mathbb{R}, \tag{3.18}$$

and a generalization of both (3.17) and (3.18),

$$f(x+y) = g(x)\sin y + g(y)\sin x, \quad \text{for } x, y \in \mathbb{R}. \tag{3.19}$$

As for (3.17), set $y = \frac{\pi}{2}$ to have $\cos x = f(x) + f\left(\frac{\pi}{2}\right)\sin x$, which with $x = \frac{\pi}{2}$ yields $f\left(\frac{\pi}{2}\right) = 0$ and $f(x) = \cos x$ (no surprise).

As for (3.18), as before, let $y = \frac{\pi}{2}$ to get $\cos x = f(x) + g\left(\frac{\pi}{2}\right)\sin x$ or $f(x) = \cos x + b\sin x$, and substitution of this $f(x)$ in (3.18) shows $\sin x \cos y = b\sin x \sin y + g(y)\sin x$; that is, $g(y) = \cos y - b\sin y$ (choose x such that $\sin x \neq 0$). Both equations can also be solved by expanding $\sin(x+y)$.

Finally, put $y = 0$ in (3.19) to obtain $f(x) = g(0)\sin x = a\sin x$. Then $y = \frac{\pi}{2}$ in (3.19) yields

$$f\left(x+\frac{\pi}{2}\right)=g\left(\frac{\pi}{2}\right)\sin x+g(x);$$

that is,

$$a\cos x=g\left(\frac{\pi}{2}\right)\sin x+g(x).$$

Setting $x=\frac{\pi}{2}$, we see that $g\left(\frac{\pi}{2}\right)=0$ and $g(x)=a\cos x$ (no surprise in (3.19)). Thus we have proved the following result.

Result 3.6. *A function* $f:\mathbb{R}\to\mathbb{R}$ *satisfying* (3.17) *is given by* $f(x)=\cos x$*; general solutions* $f,g:\mathbb{R}\to\mathbb{R}$ *of the equation* (3.18) *are given by* $f(x)=\cos x+b\sin x$, $g(x)=\cos x-b\sin x$*; and functions* $f,g:\mathbb{R}\to\mathbb{R}$ *satisfy* (3.19) *if and only if* $f(x)=a\cos x=g(x)$.

Remark. There are no surprises in (3.17) and (3.19). Whereas $f(x)=\cos x$ and $g(x)=\cos x$ yield the familiar formula for the addition of sine, f and g need not be cosine.

Next we treat the following equations and solve the second equation using associativity, which comes in handy in many places:

$$f(x+y)f(x-y)=\sin^2 x-\sin^2 y,\quad \text{for } x,y\in\mathbb{R}, \tag{3.20}$$

$$f(x+y)f(x-y)=f(x)^2-\sin^2 y,\quad \text{for } x,y\in\mathbb{R} \tag{3.21}$$

(replacing f on the right side of (3.21) with g makes no difference).

Now (3.20) can be rewritten as

$$f(x+y)f(x-y)=\sin(x+y)\sin(x-y)$$

or

$$f(u)f(v)=\sin u\sin v\quad (x+y=u,\ x-y=v).$$

Noting that $f=0$ is not a solution of (3.20), we obtain $f(u)=k\sin u$ (fix v). This f is a solution of (3.20) provided $k^2=1$. Thus we have proved the following result.

Result 3.7. *The only solution of* (3.20) *for* $f:\mathbb{R}\to\mathbb{R}$ *has the form* $f(x)=k\sin x$ *with* $k^2=1$ (*no surprise—cf.* (3.4)).

Theorem 3.7a. *The most general solution* $f:\mathbb{R}\to\mathbb{R}$ *satisfying the functional equation* (3.21) *has the form* $f(x)=a\sin x$ *or* $f(x)=b\cos x+d\sin x$, *and* a,b,d *are real constants with* $a^2=1=b^2+d^2$.

Proof. Set $y=x$ in (3.21) to get

$$f(2x)f(0)=f(x)^2-\sin^2 x\quad \text{for } x\in\mathbb{R}. \tag{3.21a}$$

We distinguish two cases according to whether $f(0)=0$ or $f(0)\neq 0$. First suppose $f(0)=0$. Equation (3.21a) yields $f(x)^2=\sin^2 x$, (3.21) becomes (3.20), and by Result 3.7 we obtain $f(x)=a\sin x$ with $a^2=1$. Now assume $f(0)\neq 0$. Let $f(0)=b$. From (3.21) and (3.21a), we have

$$f(x+y)f(x-y) = f(2x)f(0) + \sin^2 x - \sin^2 y$$
$$= bf(2x) + \sin(x+y)\sin(x-y)$$

or

$$f(u)f(v) = bf(u+v) + \sin u \sin v,$$

which is

(3.21b) $$bf(u+v) = f(u)f(v) - \sin u \sin v \quad \text{for } u, v \in \mathbb{R}.$$

Applying associativity, (3.21b) gives

$$bf(u+v+w) = f(u+v)f(w) - \sin(u+v)\sin w$$
$$= \frac{1}{b}(f(u)f(v) - \sin u \sin v)f(w) - (\sin u \cos v + \cos u \sin v)\sin w$$
$$= \frac{1}{b}f(u)(f(v)f(w) - \sin v \sin w) - \sin u(\sin v \cos w + \cos v \sin w);$$

that is,

$$\frac{1}{b}f(w)\sin u + \cos u \sin w = \frac{1}{b}f(u)\sin w + \sin u \cos w$$

or

$$\left(\frac{1}{b}f(w) - \cos w\right)\sin u - \left(\frac{1}{b}f(u) - \cos u\right)\sin w.$$

If $f(u) = b\cos u$ for all $u \in \mathbb{R}$, we are done or divide by $\sin w$ (choose w with $\sin w \neq 0$). Then we obtain

$$f(u) = b(\cos u + d \sin u) = b\cos u + d \sin u \quad \text{for constants } d, b.$$

Substitution of this f in (3.21) gives $d^2 + b^2 = 1$. This proves the result.

Remark 3.8. Most of the functional equations studied in Results 3.1 to 3.7 come under the general form

$$H(f(x+y),\ f(x-y),\ f(x),\ f(y), x, y) = 0$$

(Aczél [12], Castillo and Ruiz-Cobo [144]), where $H : \mathbb{R}^6 \to \mathbb{R}$ is a known function and $f : \mathbb{R} \to \mathbb{R}$ is the unknown function to be determined. Note that the equations above are solved without assuming any additional conditions such as continuity, differentiability, etc., except that the functions satisfy the functional equations. Note also that the unknown functions are continuous (even differentiable).

Now we turn our attention to general functional equations derived from several trigonometric identities. This domain is very vast. We consider a substantial part of it (see also Aczél [12], Aczél and Dhombres [44], Kannappan [426, 477]).

Usually these functional equations have been studied from $\mathbb{R} \to \mathbb{R}$, $\mathbb{R} \to \mathbb{C}$, $\mathbb{C} \to \mathbb{C}$, except by Kurepa and a few others, who studied different structures. We study these equations on reals, complex numbers, groups, Hilbert space, Banach algebra, etc. We are going to consider functional equations characterizing both cosine and sine separately and together and in conjunction with other functions. First we treat the d'Alembert cosine equation.

3.2 Cosine Equation on Number Systems

The cosine function satisfies the functional equation (C), so (C) is known as the *cosine equation*. The functional equation (C) has been extensively studied by many authors under various hypotheses. Naturally one expects the literature to be large. We have provided a good partial list. The proof of the problem of the parallelogram law of forces was reduced to the solution of (C) by D'Alembert [189], as seen in Chapter 1. Of course, the problem of the parallelogram law of forces is one of the oldest problems studied by means of functional equations. This equation was considered for the same purpose by Poisson [668, 669] with the hypothesis of analyticity. In the literature, (C) is known as *d'Alembert's equation* and *Poisson's equation*. The equation (C) finds applications in non-Euclidean mechanics and geometry (Aczél [12]). The functional equation (C) is closely connected with Cauchy's exponential equation (E), as we will see later.

Functional equations are solved by various methods such as reducing to differential equations, using integral transformations, reducing, to integral equations, the method of determinants, and so on. We illustrate these methods with examples.

Usually the study of (C) has been done on $\mathbb{R} \to \mathbb{R}$, $\mathbb{R} \to \mathbb{C}$, $\mathbb{C} \to \mathbb{C}$. We will treat (C) on other spaces, too. We start off with the study of (C) on reals.

Theorem 3.9. (Aczél [12], Aczél and Dhombres [44], Cauchy [147], Castillo and Ruiz-Cobo [144], Kannappan [426], Kurepa [580]). *When f is a real-valued continuous function of a real variable satisfying* (C) *(that is, $f : \mathbb{R} \to \mathbb{R}$ is a continuous solution of the cosine equation* (C)*), then f has one of the forms*

$$f(x) = 0 \quad \text{or} \quad \cos bx, \quad \text{for } x \in \mathbb{R}, \tag{3.22}$$

where $b \in \mathbb{R}$ or purely imaginary; that is, $f(x) = \cos bx$ or $\cosh bx$ for b real.

Proof. Several solutions are available in the literature. We give here two proofs—one based on differentiability and differential equations and the other involving the integral transformation. Later we will see in Section 3.3 that there is no need to assume any regularity condition to solve (C). But to get a form like (3.22), it is necessary to assume some regularity condition.

$f(x) = d$, a constant, is a solution of (C) when $d = 0$ or 1. So, hereafter we consider nonconstant solutions (that is, solutions not of the form $f(x) = 0$, $f(x) = 1$).

First we start with some preliminaries that will be of use in what follows.

Change y to $-y$ in (C) or interchange x and y in (C) to have $f(x) = f(-x)$ for all $x \in \mathbb{R}$; that is, f is even.

Set $x = 0$ (or $y = 0$) in (C) to get $f(0) = 1$.

With $y = x$, (C) becomes

$$f(2x) + 1 = 2f(x)^2, \quad \text{for } x \in \mathbb{R}, \tag{3.23}$$

or

$$f\left(\frac{x}{2}\right)^2 = \frac{1 + f(x)}{2}, \quad \text{for } x \in \mathbb{R}. \tag{3.23a}$$

From (C) and (3.23), we obtain

$$\begin{aligned} 2f(x+y)f(x-y) &= f(x+y+x-y)+f(x+y-(x-y)) \\ &= f(2x)+f(2y) \\ &= 2(f(x)^2+f(y)^2-1) \quad \text{for } x,y\in\mathbb{R}. \end{aligned} \tag{3.24}$$

Hence, by (C) and (3.24) there results

$$\begin{aligned} [f(x+y)-f(x-y)]^2 &= [f(x+y)+f(x-y)]^2-4f(x+y)f(x-y) \\ &= 4f(x)^2f(y)^2-4(f(x)^2+f(y)^2-1) \\ &= 4[f(x)^2-1][f(y)^2-1] \end{aligned} \tag{3.25}$$

or

$$[f(x+y)-f(x)f(y)]^2 = 4[f(x)^2-1][f(y)^2-1] \quad \text{for } x,y\in\mathbb{R}. \tag{3.25a}$$

From continuity follows the differentiability of f, from which follows that f has derivatives of all orders. We require here derivatives of order two only. Differentiating (C) with respect to y twice, we have

$$f'(x+y)-f'(x-y) = 2f(x)f'(y)$$

and

$$f''(x+y)+f''(x-y) = 2f(x)f''(y), \quad \text{for } x,y\in\mathbb{R},$$

which, with $y=0$, yields

$$f'(0) = 0,$$

and

$$f''(x) = kf(x), \quad \text{for } x\in\mathbb{R},\ k \text{ constant}.$$

Solution of this differential equation depends on k.

$k=0$ results in $f''(x)=0$ or $f(x)=ax+b$. This f is a solution of (C) when $a=0$ and $b=0$ or 1.

When $k>0$, there is a b in $\mathbb{R}$ such that $k=b^2$ and

$$f(x) = c\sin bx + d\cos bx, \quad c,d \text{ constants}.$$

Conditions $f(0)=1,\ f'(0)=0$ imply $d=1$ and $bc=0$. $b=0$ gives f constant, which is not the case. So, $c=0$ and $f(x)=\cos bx,\ b\in\mathbb{R}$, which is (3.22). Finally, $k<0$ leads to

$$f(x) = c\sinh bx + d\cosh bx,$$

where b is real and $b^2=-k$. Conditions $f(0)=1,\ f'(0)=0$ lead to the solution $f(x)=\cosh bx,\ b\in\mathbb{R}$, which is (3.22). □

Now we will use integral transformation to obtain the same conclusion (3.22). Carstoiu (1947) remarked that integral transformation can be used to reduce functional equations. We explain his method by using the example of the d'Alembert equation (C). If $\phi(p)$ denotes the transform of $f(x)$, it follows from (C) that

$$\frac{q\phi(p) - p\phi(q)}{q - p} + \frac{q\phi(p) + p\phi(q)}{q + p} = 2\phi(p)\phi(q);$$

that is,

$$\phi(p)q^2[\phi(q) - 1] = \phi(q)p^2[\phi(p) - 1],$$

and so

$$p^2\left(1 - \frac{1}{\phi(p)}\right) = q^2\left(1 - \frac{1}{\phi(q)}\right) = c,$$

$$\phi(p) = \frac{p^2}{p^2 - c} \quad \text{or} \quad \phi(p) = 0.$$

Hence $f(x) = 1$ or $\cos\sqrt{-c}x$ or $\cos\sqrt{c}x$, according to whether $c = 0, < 0$, or > 0 or $f(x) = 0$, which is (3.22) (see [12]). □

Remark 3.10. It is easy to see from Theorem 3.9 that the only continuous solution $f : \mathbb{R} \to \mathbb{C}$ of the equation (C) is (3.22), where $b \in \mathbb{C}$.

This result (3.22) has been extended by Kaczmarz [409] to the case in which f is a complex-valued, measurable function of a real variable (weaker hypothesis, measurability implies continuity), and he studied the functional equation

$$f(x) + f(x + y) = \varphi(y) f\left(x + \frac{y}{2}\right) \quad \text{for } x, y \in \mathbb{R}. \tag{3.26}$$

Now we consider the functional equation

$$f(x + y) + f(x - y) = 2f(x)\varphi(y) \quad \text{for } x, y \in \mathbb{R}, \tag{3.27}$$

which is the same as (3.26) (Kannappan [439], Van der Lyn [810], Wilson [830, 831], Kurepa [580]), where $f, \varphi : \mathbb{R} \to \mathbb{C}$, and prove the following theorem.

Theorem 3.11. *Suppose $f, \varphi : \mathbb{R} \to \mathbb{C}$ satisfy* (3.27) *with φ continuous. Then φ is a solution of* (C)*, and the solutions of* (3.27) *are given by*

$$f \equiv 0,\ \varphi \text{ arbitrary},\ \varphi(x) = 1,\ f(x) = A(x) + d,$$
$$f(x) = c\sin bx + d\cos bx,\ \varphi(x) = \cos bx,$$

where A is additive and b, c, d are complex constants.

Remark. That equation (3.26) is the same as (3.27) can be seen by substituting in (3.26) $x - y$ for x and $2y$ for y to get $f(x - y) + f(x + y) = \varphi(2y)f(x)$.

So we will solve (3.27) with φ continuous. We will determine the general solution of (3.27) in Theorem 3.46.

Proof. When $f = 0$, φ is arbitrary. When $\varphi = 0$, so does f.

We consider when $f \neq 0$. Choose $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$ and (3.27) becomes

$$2f(x_0)\varphi(y) = f(x_0 + y) + f(x_0 - y) \quad \text{for } y \in \mathbb{R}. \tag{3.28}$$

Claim that φ is a solution of (C). For $x, y \in \mathbb{R}$, utilizing (3.28) and (3.27), we have

$$\begin{aligned} 2f(x_0)[\varphi(x+y)+\varphi(x-y)] &= f(x_0+x+y)+f(x_0-x-y) \\ &\quad + f(x_0+x-y)+f(x_0-x+y) \\ &= 2f(x_0+x)\varphi(y)+2f(x_0-x)\varphi(y) \\ &= 2\varphi(y)\cdot 2f(x_0)\varphi(x); \end{aligned}$$

that is, φ satisfies (C). So far, no regularity condition has been used.

Suppose φ is continuous. Then, by Remark 3.10, $\varphi = 1$ or $\varphi(x) = \cos bx$, $b \in \mathbb{C}^*$.

$\varphi = 1$ in (3.27) yields $f(x+y)+f(x-y) = 2f(x)$, which is Jensen's equation (J) and $f(x) = A(x)+d$, for $x \in \mathbb{R}$, where A is additive and $d \in \mathbb{C}$.

With $\varphi(x) = \cos bx$, (3.27) becomes

$$f(x+y)+f(x-y) = 2f(x)\cos by,$$

the equation studied in Result 3.4. Hence

$$f(x) = c\sin bx + d\cos bx \quad \text{for } b, c, d \in \mathbb{C}.$$

This proves the theorem. □

Remark 3.12. The result of Kaczmarz was extended by Flett [280]. If $f : \mathbb{C} \to \mathbb{C}$ is a nonzero continuous solution of (C), then it is no longer true that the solution of (C) is of the form (3.22); that is, $f(z) = \cosh bz$, b is a complex constant, and $z \in \mathbb{C}$. What is true is the following. Let f ($\not\equiv 0$ or 1) be a complex-valued function of a complex variable continuous at least at one point and satisfying (C). Then

$$f(z) = \cosh(\alpha x + \beta y), \quad z = x+iy,\ \alpha, \beta \in \mathbb{C},$$

not both zero. If f is differentiable at at least one point that is not a zero of $\sinh(\alpha x + \beta y)$, then $\beta = i\alpha$ and f indeed has the form (3.22), $f(z) = \cos \alpha z$, $\alpha \in \mathbb{C}$.

This result was further extended to $\mathbb{R}^n$ in [572]. Let $f : \mathbb{R}^n \to \mathbb{C}$, continuous at some point, be a solution of (C). If $f \not\equiv 0$, then

$$f(x_1, x_2, \ldots, x_n) = \cosh\left(\sum_{i=1}^{n} \alpha_i x_i\right), \quad \alpha_i \in \mathbb{C},\ x_i \in \mathbb{R}.$$

□

We now consider a generalization of (3.27), namely

$$f(x+y)+f(x-y) = g(x)h(y), \quad \text{for } x, y \in \mathbb{R}. \tag{3.29}$$

We will determine the general solution of a generalization of (3.29) later, in Section 3.8.

If $g = 0$ or $h = 0$, then $f = 0$. If $f = 0$, then either $g = 0$, h arbitrary or $h = 0$, g arbitrary. So we consider only nonzero solutions of (3.29) and prove the following theorem.

Theorem 3.13. *Nonzero, continuous* $f, g, h : \mathbb{R} \to \mathbb{C}$ *satisfy* (3.29) *if and only if*

$$h(x) = a,\ g(x) = bx + c,\ f(x) = \frac{1}{2}a(bx + c) \tag{3.30}$$

or

$$\begin{aligned} h(x) =& a\cos bx,\ g(x) = d\cos bx + c\sin bx, \\ & f(x) = \frac{1}{2}a(d\cos bx + c\sin bx), \end{aligned} \tag{3.30a}$$

where a, b, c, d *are complex constants.*

Proof. Set $y = 0$ in (3.29) to get

$$2f(x) = h(0)g(x) = ag(x) \quad \text{for } x \in \mathbb{R}. \tag{3.31}$$

$h(0) = a = 0$ implies $f = 0$, which is not the case, so $a \neq 0$. Using (3.31), (3.29) becomes

$$g(x + y) + g(x - y) = 2g(x) \cdot \frac{h(y)}{a},$$

which is the same as (3.27). Thus $\frac{h}{a}$ satisfies (C) and, by applying Theorem 3.11, we obtain (3.30) and (3.30a). □

If one is interested in obtaining only the cosine solution (not the hyperbolic cosine), one can consider the functional equation

$$f(x + y + 2d) + f(x - y + 2d) = 2f(x)f(y) \quad \text{for } x, y \in \mathbb{R}, \tag{3.32}$$

which is a modified form of (C).

Theorem 3.14. (Kannappan [420]). *Let* $f : \mathbb{R} \to \mathbb{C}$ *be a nonzero solution of* (3.32)*, where* $d \neq 0$*. Then* f *is periodic with period* (i) d, (ii) $2d$, *or* (iii) $4d$, *depending on* $f(0) = 1$ (*with* $f(d) = 1$ *or* $f(d) = -1$) *or* $f(0) = -1$ (*with* $f(d) = 0$), *and* f *satisfies* (C) *in cases* (i) *and* (ii) *and* $g(x) = f(x + 2d)$ *is a solution of* (C) *in case* (iii).

Proof. We will consider any nonzero solution f of (3.32).

Let $y = 0$ in (3.32) to get

$$f(x + 2d) = f(x)f(0) \quad \text{for } x \in \mathbb{R}. \tag{3.33}$$

$f(0) = 0$ would imply $f = 0$, which is not the case, so $f(0) \neq 0$. Equation (3.33) gives $f(x + 4d) = f(x + 2d)f(0) = f(x)f(0)^2$.

Now, replacing y by $y + 2d$ in (3.32) (using (3.33)),

$$f(x + y)f(0)^2 + f(x - y) = 2f(x)f(y)f(0). \tag{3.34}$$

Putting $x = 0 = y$ in (3.34) shows that $f(0)^2 = 1$, so that $f(0) = 1$ or $f(0) = -1$.

Case 1. Suppose $f(0) = 1$.

Equation (3.33) shows that $f(x+2d) = f(x)$ and then (3.32) (or (3.34)) becomes (C).

Now $f(2d) = f(0) = 1$ and $f(2d) = 2f(d)^2 - 1$ (see (3.23)) yields $f(d) = 1$ or -1.

Suppose $f(d) = 1$. Then (C) with $y = d$ and x replaced by $x + d$ gives that (i) $f(x+d) = f(x)$; that is, f has period d.

When $f(d) = -1$, (C) with $y = d$ and x replaced by $x + d$ yields (ii) $f(x) = -f(x+d) = f(x+2d)$; that is, f has period $2d$.

Case 2. Suppose $f(0) = -1$.

Then (3.33) shows that (iii) $f(x) = -f(x+2d) = f(x+4d)$; that is, f has period $4d$. Set $x = 0$ in (3.33) to have $f(2d) = 1$, and $f(4d) = -1$ and let $x = -d$ to get $f(d) = -f(d)$. Now $x = d$, $y = -d$ in (3.32) to have $f(2d) + f(4d) = 2f(d)f(-d)$; that is, $f(d) = 0$.

The periodicity in (i), (ii), and (iii) eliminates a hyperbolic solution.

In Case 2, define $g(x) = f(x+2d)$, for $x \in \mathbb{R}$, so that (3.33) gives $g(x) = f(x)f(0) = -f(x)$ and (3.32) gives

$$g(x+y) + g(x-y) = 2f(x)f(y) = 2g(x)g(y),$$

which is (C). Note that g also has period $4d$. This proves the theorem. □

Corollary 3.14a. *Nonzero continuous solutions of* (3.32) *are given by*

$$f(x) = \cos\frac{2n\pi x}{d} \quad or \quad \cos\frac{(2n+1)\pi x}{d} \quad or \quad -\cos\frac{(2n+1)\pi x}{2d}$$

for $x \in \mathbb{R}$, $n \in Z$.

Proof. Use Theorem 3.9. In case (i), f is a solution of (C) with period d so that $f(x) = \cos cx$ for some constant c, with $f(d) = \cos cd = 1$; that is, $cd = 2n\pi$. Hence $f(x) = \cos\frac{2n\pi x}{d}$.

In case (ii), f again is a solution of (C) with period $2d$, so that $f(x) = \cos cx$ for some constant c, with $f(d) = \cos cd = -1$; that is, $cd = (2n+1)\pi$ and $f(x) = \cos\frac{(2n+1)\pi x}{d}$.

In case (iii), $f(x) = g(x-2d) = \cos c(x-2d)$ for some constant c with period $4d$ and $f(d) = \cos cd = 0$ or $cd = (2n+1)\frac{\pi}{2}$ and

$$\begin{aligned} f(x) &= \cos\frac{(2n+1)\pi}{2d}(x-2d) = \cos\left(\frac{(2n+1)\pi x}{2} - (2n+1)\pi\right) \\ &= -\cos\frac{(2n+1)\pi x}{2d}. \end{aligned}$$

□

Now we determine the general solution of (3.32) using Theorem 3.21 and Theorem 3.14.

Remark 3.14b. The general solutions f (nonconstant) of (3.32) are given by

$$f(x) = \pm\frac{E(x) + E^*(x)}{2},$$

where E satisfies (E), with (i) $f(d) = 1 = f(0) = E(d)$, f period d; (ii) $f(d) = -1 = -f(0) = E(d)$, f period $2d$; (iii) $f(d) = 0$, $f(0) = -1 = E(d)^2$, f period $4d$. The proof follows easily from Theorem 3.21. □

Before considering (C) on groups and abstract spaces, we treat the functional equations

$$f(x+y)f(x-y) = f(x)^2 + f(y)^2 - b^2, \quad \text{for } x, y \in \mathbb{C}, \tag{3.35}$$

$$f(x+y)f(x-y) = f(x)^2 - f'(y)^2, \quad \text{for } x, y \in \mathbb{C}, \tag{3.35a}$$

arising from the identities

$$\cos(x+y)\cos(x-y) = \cos^2 x + \cos^2 y - 1,$$
$$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y, \quad \text{for } x, y \in \mathbb{C}.$$

Theorem 3.15. *Suppose* $f : \mathbb{C} \to \mathbb{C}$ *satisfies* (3.35) *for some constant* b. *Then* $f(x) = 0$ *when* $b = 0$ *or* $g(x) = \pm\frac{1}{b} f(x)$ *is a solution of* (C) *when* $b \neq 0$. *Further, if* f *is a nonzero analytic function satisfying* (C), *then* $f(z) = \pm b \cos dz$, *for* $z \in \mathbb{C}$, *where* $d \in \mathbb{C}$.

Proof. Let $b = 0$. Then $y = 0$ in (3.35) shows that $f(0) = 0$. Then $y = x$ in (3.35) shows that $f(x) = 0$.

Now let $b \neq 0$. Set $y = 0$ in (3.35) to get $f(0)^2 = b^2$. First $y = x$ in (3.35) gives $\pm b f(2x) = 2f(x)^2 - b^2$ and then

$$\begin{aligned} f(x+y)f(x-y) &= \frac{1}{2}[b^2 \pm bf(2x) + b^2 \pm bf(2y)] - b^2 \\ &= \pm\frac{b}{2}[f(2x) + f(2y)] \end{aligned}$$

or

$$\pm\frac{2}{b}[f(u)f(v)] = f(u+v) + f(u-v),$$

where $u = x + y$, $v = x - y$.

Define $g(x) = \pm\frac{1}{b} f(x)$ so that g satisfies (C). When f ($\not\equiv 0$) is analytic, from Remark 3.12 it follows that $f(z) = a\cosh \alpha z,\ a, \alpha, z \in \mathbb{C}$. □

Theorem 3.15a. *Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant analytic function that satisfies* (3.35a)*. Then f is a solution of* (3.35)*.*

Proof. Setting $y = 0$ in (3.35a), we see that $f'(0) = 0$. Now let $y = x$ in (3.35a) and then differentiate with respect to x to have

$$f'(2x) f(0) = f(x) f'(x) - f'(x) f''(x) \quad \text{for } x \in \mathbb{C}.$$

Differentiating (3.35a) with respect to x and letting $y = x$ in the resultant equation, we have

$$f'(2x) f(0) = 2 f(x) f'(x)$$

so that

$$f(x) f'(x) = -f'(x) f''(x), \quad \text{for } x \in \mathbb{C},$$

which by integration yields

$$f(x)^2 = -f'(x)^2 + b^2,$$

where b is a constant. Hence (3.35a) becomes (3.35). □

3.3 (C) on Groups and Vector Spaces

Certainly the functional equation (C) has a meaning on any group G. We take G to be a multiplicative group. Then (C) takes the form

(CE) $$f(xy) + f(xy^{-1}) = 2 f(x) f(y) \quad \text{for } x, y \in G.$$

One way to study (CE) on a purely algebraic structure is through the homomorphism $g : G \to \mathbb{C}^*$

(3.36) $$g(xy) = g(x)g(y) \quad \text{for } x, y \in G.$$

For example, if G is a group, then the function defined by

(3.37) $$f(x) = \frac{1}{2}[g(x) + g^*(x)], \quad \text{for } x \in G,$$

is a solution of (CE), where $g^*(x) = g(x)^{-1}$.

Naturally, one would like to find out whether or not every solution of (CE) on an arbitrary group has the form (3.37). When G is a topological group and f is a continuous solution of (CE), is g also continuous? If this g exists, is it unique? Given g, f is determined by (3.37). Given f, is it possible to determine g through f? The answers to these questions are provided in this section and Section 3.4.

The answer to the form (3.37) is affirmative for Abelian groups (Kannappan [424, 477, 422]); that is, every nonzero solution of (CE) has the form (3.37). As for non-Abelian groups, if f satisfies the additional condition

$$\text{(K)} \qquad f(xyz) = f(xzy), \quad \text{for all } x, y, z \in G,$$

then indeed every solution of (CE) is of the form (3.37) (see Section 3.4). On some special groups, f is of the form (3.37) without the use of the additional condition (K). We will also see examples of solutions of (CE) on a non-Abelian group not of the form (3.37).

Lemma 3.16. [424, 477]. *Let G be any group and $g_1, g_2 : G \to \mathbb{C}^*$ be homomorphisms* (3.36) *such that*

$$g_1 + g_1^* = g_2 + g_2^*.$$

Then $g_2 = g_1$ or $g_2 = g_1^*$.

Proof. Suppose $g_1(x) + g_1^*(x) = g_2(x) + g_2^*(x)$ for $x \in G$. Then

$$g_1(x) - g_2^*(x) = g_2(x) - \frac{1}{g_1(x)};$$

that is,

$$\frac{g_1(x)g_2(x) - 1}{g_2(x)} = \frac{g_1(x)g_2(x) - 1}{g_1(x)}$$

or

$$(g_1(x)g_2(x) - 1)\left(\frac{1}{g_2(x)} - \frac{1}{g_1(x)}\right) = 0$$

or

$$[g_1(x)g_2(x) - 1][g_1(x)g_2^*(x) - 1] = 0.$$

Let $G_1 = \{x \in G : g_1(x)g_2(x) = 1\}$ and $G_2 = \{x \in G : g_1(x)g_2^*(x) = 1\}$. Then G_1 and G_2 being the kernels of the homomorphisms, g_1g_2 and $g_1g_2^*$ are (normal) subgroups, and $G = G_1 \cup G_2$. But a group is never the union of two subgroups, for if $x_1 \notin G_1,\ x_2 \notin G_2$, then one of x_1, x_2, x_1x_2 is not in $G_1 \cup G_2$. So either $G_1 = G$, in which case $g_2 = g_1^*$, or $G_2 = G$, in which case $g_2 = g_1$. This proves the lemma. □

Lemma 3.16a. *Let G be any group. Let $f : G \to \mathbb{C}$ satisfy* (CE), *and assume the values ± 1 only. Then f has the form* (3.37).

Proof. Set $y = e$ in (CE) to have $f(e) = 1$. Then $x = e$ in (CE) yields $f(y) = f(y^{-1})$.

Since $f(x) = \pm 1$ for all $x \in G$,

$$f(xy) + f(xy^{-1}) = \pm 2 \quad \text{(from (CE))}$$

and

$$f(xy)\,f(xy^{-1}) = \pm 1.$$

If $f(xy)f(xy^{-1}) = -1$, then either $f(xy) = 1$ and $f(xy^{-1}) = -1$ or $f(xy) = -1$ and $f(xy^{-1}) = 1$. In this case, $f(xy) + f(xy^{-1}) = 0$, contradicting $f(xy) + f(xy^{-1}) = \pm 2$. Hence $f(xy)f(xy^{-1}) = 1$ and

$$[f(xy) - f(xy^{-1})]^2 = [f(xy) + f(xy^{-1})]^2 - 4f(xy)f(xy^{-1}) = 0;$$

that is, $f(xy) = f(xy^{-1})$ and $f(xy) = f(x)f(y)$ (from (CE)). Then $f(x) = \frac{f(x)+f^*(x)}{2}$ and has the form (3.37). □

Lemma 3.16b. [477]. *Let G be any cyclic group (finite or infinite). Then every nonzero solution $f : G \to \mathbb{C}$ of* (CE) *is of the form* (3.37).

Lemma 3.16c. [477]. *Let G_1 and G_2 be two Abelian groups and every solution of* (CE) *on G_i $(i = 1, 2)$ have the form* (3.37). *Then the same is true of all the solutions of* (CE) *on the direct product $G_1 \times G_2$.*

Lemma 3.16d. [477]. *Let G_i $(i \in I$, an index set$)$ be a family of Abelian groups such that every solution of* (CE) *on G_i has the form* (3.37). *Then the same is true of all solutions of* (CE) *on the weak direct product P^*G_i ($x \in P^*G_i$ means $x = (x_i)$ with $x_i = e_i$ for all but a finite number of indices).*

Lemma 3.16e. [477]. *Let G be any group and H be a homomorphic image of G. If every solution of* (CE) *on G is of the form* (3.37), *then every solution of* (CE) *on H is also of the form* (3.37).

Piecing together the results above, we can prove the following theorem.

Theorem 3.17. (Kannappan [477]). *Let G be any Abelian group. Then every nonzero solution of* (CE) *on G has the form* (3.37).

Proof. It is well known that every Abelian group G with m generators is a homomorphic image of the weak direct product P^*Z^m, Z being an infinite cyclic group of integers. But every solution of (CE) on P^*Z^m is of the form (3.37) by Lemmas 3.16b and 3.16d. Now an application of Lemma 3.16e shows that all solutions of (CE) on G are of the form (3.37). Since there is no restriction whatsoever on the cardinality m for the validity of Lemma 3.16d, we conclude that every function that satisfies (CE) on an arbitrary Abelian group G has the form (3.37). □

Now we prove that if f of (CE) is continuous, so is the corresponding g in (3.37).

Theorem 3.18. [424, 477, 422]. *Let G be a topological group (Abelian or not) and let f be a complex-valued, continuous function defined on G. Further, let g be a homomorphism of G into $\mathbb{C}^*$ such that f has the form* (3.37). *Then g is also continuous.*

Proof. By hypothesis,

$$f(xy) = \frac{g(xy) + g^*(xy)}{2}, \quad \text{for all } x, y \in G,$$
$$= \frac{g(x)g(y) + g^*(x)g^*(y)}{2},$$

and

$$f(x)f(y) = \frac{g(x)+g^*(x)}{2}\cdot\frac{g(y)+g^*(y)}{2}$$
$$= \frac{1}{4}[g(x)g(y)+g(x)g^*(y)+g^*(x)g(y)+g^*(x)g^*(y)],$$

and so

$$f(xy)-f(x)f(y) = \frac{g(x)g(y)+g^*(x)g^*(y)-g(x)g^*(y)-g^*(x)g(y)}{4}$$
$$= \frac{g(x)-g^*(x)}{2}\cdot\frac{g(y)-g^*(y)}{2}. \tag{3.38}$$

But

$$f(x) = \frac{g(x)+g^*(x)}{2},$$

and hence

$$g(x)-f(x) = \frac{g(x)-g^*(x)}{2}. \tag{3.39}$$

From (3.38) and (3.39), we obtain

$$f(xy)-f(x)f(y) = [g(x)-f(x)][g(y)-f(y)]. \tag{3.40}$$

Case I. Suppose that $g(x) = f(x)$ for every x in G. Then, obviously, g is continuous.

Case II. Otherwise, there is an x_0 in G such that $\delta = g(x_0) - f(x_0) \neq 0$. With x_0 for y in (3.40), we get

$$g(x) = f(x) + \frac{1}{\delta}[f(xx_0) - f(x)f(x_0)]. \tag{3.41}$$

From (3.41), it follows that, if f is continuous, then so is g. This completes the proof of the theorem. □

Remark. Note that g is given in terms of f, which will be used later.

Remark 3.19. Let G be a topological group (Abelian). Let f be a nonzero continuous solution of (CE) on G. Then f is of the form (3.37), where g is a continuous homomorphism of G into $\mathbb{C}^*$.

Suppose G above is locally compact and f is (Haar) measurable. Then it is easy to see that g is also measurable [369]. But since g is a measurable homomorphism, it is also continuous. Hence f is continuous. So, every measurable solution of (CE) is continuous.

(C) on a Vector Space

Result 3.20. [427]. *Let V be a real vector space. Suppose a nonzero $f : V \to \mathbb{C}$ satisfying* (C) *is continuous along rays. Then there exists a linear function $L : V \to \mathbb{C}$ such that $f(x) = \cosh L(x)$. If V is a real linear topological space, then the corresponding L is also continuous. If $f : V \to \mathbb{R}$ is continuous along rays, $f \neq 0$, then every solution of* (C) *is of the form*

$$f(x) = \cos L(x) \quad \textit{or} \quad \cosh L(x).$$

3.4 Solution of (CE) on a Non-Abelian Group

Let $f : G \to \mathbb{C}$ be a nonzero solution of (CE) satisfying (K), where G is a non-Abelian group.

First we develop some preliminaries that will be useful in what follows [424]. First put $y = e$ (identity of G) in (CE) to have $f(e) = 1$ (use $f \not\equiv 0$). Then $x = e$ in (CE) yields

$$f(y^{-1}) = f(y) \quad \text{for } y \in G;$$

that is, f is even.

It is easy to see that (interchanging x and y in (CE)) $f(xy) = f(yx)$ for all $x, y \in G$.

Letting $y = x$ in (CE), we have

$$f(x^2) + 1 = 2f(x)^2 \quad \text{for } x \in G \tag{3.42}$$

(cf. (3.23), known as the duplication formula). Using (CE) and (K), we obtain

$$\begin{aligned} 2f(xy)f(xy^{-1}) &= f(xy \cdot xy^{-1}) + f(xy \cdot yx^{-1}) \\ &= f(xy \cdot y^{-1}x) + f(xx^{-1} \cdot y^2) \\ &= f(x^2) + f(y^2), \quad \text{for } x, y \in G. \end{aligned} \tag{3.43}$$

From (CE), (3.42), and (3.43), we find that

$$\begin{aligned} [f(xy) - f(xy^{-1})]^2 &= [f(xy) + f(xy^{-1})]^2 - 4f(xy)f(xy^{-1}) \\ &= 4f(x)^2 f(y)^2 - 2(f(x^2) + f(y^2)) \\ &= 4f(x)^2 f(y)^2 - 4f(x)^2 - 4f(y)^2 + 4 \\ &= 4[f(x)^2 - 1][f(y)^2 - 1] \end{aligned} \tag{3.44}$$

or

$$[2f(xy) - 2f(x)f(y)]^2 = 4[f(x)^2 - 1][f(y)^2 - 1];$$

that is,

$$[f(xy) - f(x)f(y)]^2 = [f(x)^2 - 1][f(y)^2 - 1] \quad \text{for } x, y \in G \tag{3.44a}$$

(cf. (3.25) and (3.25a)).

Now we are ready to prove the main theorem.

Theorem 3.21. (Kannappan [424]). *Let G be a non-Abelian group and $f : G \to \mathbb{C}$ be a nonzero solution of* (CE) *with* (K). *Then f has the form* (3.37).

Proof. Lemma 3.16a shows that if $f(G) \subset \{-1, 1\}$, then f has the form (3.37). Suppose there exists an $x_0 \in G$ such that $f(x_0)^2 \neq 1$. Let $\alpha = f(x_0)$ and β be a square root of $(\alpha^2 - 1)$; that is, $\beta^2 = \alpha^2 - 1$.

Now define $g : G \to \mathbb{C}$ by (cf. (3.41))

$$g(x) = f(x) + \frac{1}{\beta}[f(x_0x) - f(x_0)f(x)], \quad x \in G,$$

$$= \frac{1}{\beta}[f(x_0x) + (\beta - \alpha)f(x)]. \tag{3.45}$$

We will prove that g is a homomorphism; that is, g satisfies (3.36) and f has the form (3.37). First we will show that f is of the form (3.37). From (3.44a) and (3.45), it follows that

$$[g(x) - f(x)]^2 = \frac{1}{\beta^2}[f(x_0)^2 - 1][f(x)^2 - 1]$$
$$= f(x)^2 - 1$$

or

$$g(x)^2 - 2f(x)g(x) + 1 = 0.$$

Hence, $g(x) \neq 0$ and moreover

$$f(x) = \frac{g(x) + g^*(x)}{2}, \quad \text{for } x \in G.$$

This proves (3.37). It remains to show that g is a homomorphism.

By the repeated use of (CE) and (K), we have

$$\begin{aligned} 2[f(x_0x)f(y) + f(x_0y)f(x)] &= f(x_0xy) + f(x_0xy^{-1}) + f(x_0yx) + f(x_0yx^{-1}) \\ &= 2f(x_0xy) + 2f(x_0)f(xy^{-1}) \\ &= 2\{f(x_0xy) + \alpha[2f(x)f(y) - f(xy)]\} \end{aligned}$$

and

$$\begin{aligned} 2f(x_0x)f(x_0y) &= f(x_0^2xy) + f(xy^{-1}) \\ &= 2f(x_0xy)f(x_0) - f(xy) + (2f(x)f(y) - f(xy)) \\ &= 2[\alpha f(x_0xy) - f(xy) + f(x)f(y)]. \end{aligned}$$

In view of these, finally we get

$$\begin{aligned}
g(x)g(y) &= \frac{1}{\beta^2}[f(x_0x)+(\beta-\alpha)f(x)][f(x_0y)+(\beta-\alpha)f(y)]\\
&= \frac{1}{\beta^2}[f(x_0x)f(x_0y)+(\beta-\alpha)\{f(x_0x)f(y)+f(x_0y)f(x)\}\\
&\quad +(\beta-\alpha)^2 f(x)f(y)]\\
&= \frac{1}{\beta^2}[\{\alpha f(x_0xy)-f(xy)+f(x)f(y)\}\\
&\quad +(\beta-\alpha)\{f(x_0xy)+2\alpha f(x)f(y)-\alpha f(xy)\}\\
&\quad +(\beta^2-2\alpha\beta+\alpha^2)f(x)f(y)]\\
&= \frac{1}{\beta^2}[\beta f(x_0xy)+(2\alpha\beta-2\alpha^2+1+\beta^2-2\alpha\beta+\alpha^2)f(x)f(y)\\
&\quad +(\alpha^2-\alpha\beta-1)f(xy)]\\
&= \frac{1}{\beta}[f(x_0xy)+(\beta-\alpha)f(xy)]\\
&= g(xy).
\end{aligned}$$

This proves the theorem. □

The condition (K) in a way reduces the study of (CE) to the Abelian case [429, 588]; that is, the condition (K) reduces the study of (CE) to the study of the corresponding equation over the commutative group $\frac{G}{G'}$, where G' is the commutator group of G.

Remark 3.22. [429, 588]. Let $f : G \to \mathbb{C}$ satisfy (CE) and (K). Use (K) to get

$$f(x) = f(xx^{-1}y^{-1}xy) = f(xu) \quad \text{for } x \in G, \tag{3.46}$$

where u is any element of the commutator subgroup G' of G. Note that $f(u) = f(zyz^{-1}y^{-1}) = 1$ by (K).

Now define $F : \frac{G}{G'} \to \mathbb{C}$ by

$$F(\overline{x}) = f(x) \quad \text{for } x \text{ in } \overline{x} \in \frac{G}{G'}. \tag{3.47}$$

The function F is well defined because of (3.46), for, if $y \in \overline{x}$, then $xy^{-1} \in G'$. Then, by (3.46),

$$f(x) = f(xy^{-1}\cdot y) = f(y)$$

since $xy^{-1} \in G'$. (Note that $f = F \circ \ell$, where $\ell : G \to \frac{G}{G'}$ is the canonical homomorphism.)

It is easy to verify that F is a solution of (CE) on $\frac{G}{G'}$. Then, by Theorem 3.17, there is a homomorphism $k : \frac{G}{G'} \to \mathbb{C}^*$ such that

$$F(\overline{x}) = \frac{k(\overline{x}) + k^*(\overline{x})}{2} \quad \text{for } \overline{x} \in \frac{G}{G'}.$$

Now, if we define $g : G \to \mathbb{C}^*$ by $g(x) = k(\overline{x})$, where x is in $\overline{x} \in \frac{G}{G'}$, g is well defined and a homomorphism, and f has the form (3.37). □

As a matter of fact, the following is true.

Result 3.17a. (Kannappan [429], Lawrence [601]). *Let $f : G \to \mathbb{C}$ be a solution of* (CE) *and $f \not\equiv 0$. Then, that f has the form* (3.37) *implies $f(G') = 1$, and if G' is a torsion group and $f(u) = 1$ for every $u \in G'$, then f is of the standard form* (3.37).

Proof. Suppose f has the form (3.37). Then

$$\begin{aligned} f(xyx^{-1}y^{-1}) &= \frac{g(xyx^{-1}y^{-1}) + g^*(xyx^{-1}y^{-1})}{2} \\ &= \frac{g(x)g(y)g(x^{-1})g(y^{-1}) + g^*(x)g^*(y)g^*(x^{-1})g^*(y^{-1})}{2} \\ &= 1. \end{aligned}$$

Conversely, let $f(G') = 1$ and G' be a torsion group.

We will prove that f satisfies (K). First we will prove that if there is an element $b \in G$ such that $b^n = e$, for some $n \in Z_+$ and $f(b) = 1$, then $f(xb) = f(x)$ for all $x \in G$.

From (CE), we have

$$f(xb \cdot b) + f(xb \cdot b^{-1}) = 2f(xb)f(b) \quad \text{for } x \in G;$$

that is,

$$f(xb^2) = 2f(xb) - f(x).$$

Replace x by xb in the equation above to obtain

$$f(xb^3) = 2f(xb^2) - f(xb) = 3f(xb) - 2f(x).$$

Another replacement of x by xb will result in

$$f(xb^4) = 4f(xb) - 3f(x),$$

and, continuing in a similar manner, we have, in general,

$$f(xb^k) = kf(xb) - (k-1)f(x) \quad \text{for } x \in G,\ k \in Z_+.$$

In particular,

$$f(xb^n) = nf(xb) - (n-1)f(x);$$

hence $f(xb) = f(x)$, for $x \in G$, since $b^n = e$.

Since $f(G') = 1$ and G' is a torsion group,

$$f(xu) = f(x) \quad \text{for } x \in G,\ u \in G'.$$

For $x, y, z \in G$,

$$f(x \cdot y^{-1}z^{-1}yz) = f(x),$$

or changing x by xzy, we get

$$f(xyz) = f(xzy),$$

which is (K). Hence f has the form (3.37). □

This could also be proved as follows [601]. Suppose a and b are in the same coset of G modulo G'. Then $a = bc$ for some $c \in G'$ and $c^n = e$ for some $n \in Z_+$ and $f(c) = 1$. So, $f(a) = f(bc) = f(b)$. The function f therefore factors through $\frac{G}{G'}$ (see Remark 3.22) and is thus of the standard form (3.37).

Solutions of (CE) on a Non-Abelian Group not of the Form (3.37)

Example 1. Let G be the quaternion group. Instead of writing $G = \{e, a, a^2, a^3, b, ab, ba, ba^2\}$ with $a^4 = e,\ a^2 = b^2,\ ba = a^3b$, we write $G = \{\pm 1, \pm i, \pm j, \pm k\}$ with $i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i,\ ki = j$.

Let $f : G \to \mathbb{C}$ be a nonzero solution of (CE). We will find all f and show that all but one has the form (3.37). There is only one nonzero solution f of (CE) that is not of the form (3.37) (see also [163, 170, 429]).

Set $x = 1 = y$ in (CE). Then $f(1) = 0$ or 1. But $f(1) = 0$ leads to $f \equiv 0$. So, $f(1) = 1$. Recall $f(xy) = f(yx)$ and $f(y^{-1}) = f(y)$ for $x, y \in G$.

Let $f(-1) = \delta,\ f(i) = f(-i) = \alpha,\ f(j) = f(-j) = \beta,\ f(k) = f(-k) = \gamma$. Now $x = -1 = y$ in (CE) gives $2 = 2\delta^2$, so $\delta = \pm 1$.

First suppose $\delta = -1$. $x = i,\ y = -i$ in (CE) yields $\alpha = 0$. Similarly, $\beta = 0 = \gamma$. This f ($f(1) = 1,\ f(-1) = -1,\ f(\pm i) = 0 = f(\pm j) = f(\pm k)$) is not of the form (3.37). Indeed, if f has the form (3.37), (K) holds. But $f(ijk) = f(k^2) = f(-1) = -1$ and $f(ikj) = f(-jj) = f(1) = 1$.

Also, if there is a homomorphism g associated to f, then $g(x)^2 = -1$ for $x \in G,\ x \neq \pm 1$. But $-1 = g(k)^2 = g(ij)^2 = g(i)^2 g(j)^2 = 1$, a contradiction.

The case $\delta = 1$ leads f to have the form (3.37). For $x = i,\ y = -i$ in (CE) shows that $\alpha^2 = 1$. Similarly, $\beta^2 = 1 = \gamma^2$; that is, $f(G) = \{1, -1\}$. By Lemma 3.16a, f has the form (3.37) (in this case, f itself is the homomorphism, $f(ijk) = f(-1) = 1$, and $f(ikj) = f(-j^2) = f(1) = 1$).

Remark. In [188], (CE) is considered without condition (K) but with the conditions that (i) there exists an $e_0 \in G$ (group) such that $f(e_0) = 0$ and (ii) $f(xye_0) = f(xe_0y)$ for $x, y \in G$, where $f : G \to \mathbb{C}$, and it is shown that f has the form (3.37).

Conditions (K) and (i) and (ii) are not equivalent. Conditions (i) and (ii) imply (K). The example of a quaternion group above illustrates the other way that implication is not always true—namely, $f(G) = \{-1, 1\}$ satisfies (K) but not (i) and (ii).

Now, to show that (i) and (ii) imply (K), first set x and y by xe_0 and ye_0, respectively, in (CE) to have

$$2f(xe_0)f(ye_0) = f(xe_0ye_0) + f(xy^{-1}) = f(xye_0^2) + f(xy^{-1}).$$

Then replace x and y by xye_0 and e_0 in (CE) to get $f(xye_0^2) + f(xy) = 0$. Use these two equations with (CE) to obtain $f(xy) = f(x)f(y) - f(xe_0)f(ye_0)$ for $x, y \in G$,

$$\begin{aligned} f(xyz) &= f(x)f(yz) - f(xe_0)f(yze_0) \\ &= f(x)f(yz) - f(xe_0)f(ze_0y) \\ &= f(x)f(zy) - f(xe_0)f(zye_0) \\ &= f(xzy), \end{aligned}$$

which is (K).

Example 2. (Aczél, Chung, and Ng [40]). Suppose H is the skew field of quaternions over $\mathbb{R}$, and G is the unit sphere in H under multiplication; that is,

$$\begin{aligned} H &= \{a + bi + cj + dk : \ a, b, c, d \in \mathbb{R}\}, \\ G &= \{x \in H : \ a^2 + b^2 + c^2 + d^2 = \|x\|^2 = 1\}. \end{aligned}$$

If we define $f : G \to \mathbb{C}$ by $f(a + bi + cj + dk) = a$, then f is a solution of (CE), $f(xy) = f(yx)$ for $x, y \in G$, but (K) is not satisfied in general, for $f(ikj) = f(-j \cdot j) = 1$ and $f(ijk) = f(k \cdot k) = f(-1) = -1$. So f cannot be of the form (3.37).

Example 3. [601]. Let $G = \langle a, b \mid a^{-1}b = ba \rangle$ be a torsion-free group. Define $f : G \to \mathbb{C}$ by

$$\begin{aligned} f(a^t b^s) &= 1 \qquad \text{if } 4 \mid t + 2s, \\ f(a^t b^s) &= -1 \quad \text{if } 2 \mid t + 2s; \ 4 \nmid t + 2s, \\ f(a^t b^s) &= 0 \qquad \text{in all other cases,} \end{aligned}$$

where f is a solution of (CE) and is not of the form (3.37). Otherwise, we would have $f(aba^{-1}b^{-1}) = 1$. But $aba^{-1}b^{-1} = a^2$ and $f(a^2) = -1$, so f is not of the form (3.37).

Example 4. In some special cases, it is not necessary to assume restriction or condition (K) to obtain the standard form (3.37). We will provide many results in this direction. It is necessary to assume certain conditions satisfied by G.

Theorem 3.23. (Corovei [170]). *Let G be a group and F a field with* char $F \neq 2$ *and $f : G \to F$ be a nonzero solution of* (CE). *Suppose that* (i) *there is an $x_0 \in Z(G)$ (center of G) such that $f(x_0)^2 \neq 1$ and* (ii) *there is a $b \in F$ such that $b^2 = f(x_0)^2 - 1$. Then f has the form* (3.37).

Proof. (See [424]). The proof in [424, pp. 72–73] works since $x_0 \in Z(G)$.

Let $f(x_0) = a$. First we show that

(3.44b) $[f(x_0x) - f(x_0)f(x)]^2 = [f^2(x) - 1][f^2(x_0) - 1] \quad \text{for } x \in G,\ x_0 \in Z(G)$

(cf. (3.44a)). Replace x by xx_0 and y by xx_0^{-1} in (CE), and use (3.42) to obtain

$$f(x^2) + f(x_0^2) = 2f(xx_0)f(xx_0^{-1});$$

that is,

$$\begin{aligned} f(x)^2 + f(x_0)^2 - 1 &= f(xx_0)f(xx_0^{-1}) \\ &= f(xx_0)[2f(x)f(x_0) - f(xx_0)] \\ &= -[\{f(xx_0) - f(x)f(x_0)\}^2 - f(x)^2 f(x_0)^2] \end{aligned}$$

or

$$\begin{aligned} [f(xx_0) - f(x)f(x_0)]^2 &= f(x)^2 f(x_0)^2 - f(x)^2 - f(x_0)^2 + 1 \\ &= [f(x)^2 - 1][f(x_0)^2 - 1], \end{aligned}$$

which is (3.44b).

Define $g : G \to F^*$ by

$$\begin{aligned} g(x) &= f(x) + \frac{1}{b}[f(xx_0) - f(x)f(x_0)] \\ &= \frac{1}{b}[f(xx_0) + (b - a)f(x)] \quad \text{for } x \in G. \end{aligned}$$

Utilizing (3.42), we have

$$\begin{aligned} [g(x) - f(x)]^2 &= \frac{1}{b^2}[f(xx_0) - f(x)f(x_0)]^2 \\ &= \frac{1}{b^2}(f(x)^2 - 1)(f(x_0)^2 - 1); \end{aligned}$$

that is,

$$g(x)^2 - 2g(x)f(x) + 1 = 0.$$

Now we can conclude that $g(x) \neq 0$ and f has the form (3.37). The only thing remaining is to show that g is a homomorphism.

Instead of using condition (K), use $x_0 \in Z(G)$. Then, as in Theorem 3.21 (see also [429]),

$$\begin{aligned} f(x_0x)f(y) + f(x_0y)f(x) &= f(x_0xy) + a[2f(x)f(y) - f(xy)] \\ f(x_0x)f(x_0y) &= af(x_0xy) + f(x)f(y) - f(xy) \end{aligned}$$

and

$$g(x)g(y) = \frac{1}{b^2}[f(x_0x)f(x_0y) + (b-a)\{f(x)f(x_0y) + f(y)f(x_0x)\} + (b-a)^2 f(x)f(y)] = g(xy).$$

This proves the result. □

Result 3.24. [170]. *Let G be a nilpotent group whose elements are of odd order and F a quadratically closed field with* char $F \neq 2$. *If $f : G \to F$ is a solution of* (CE), *then f has the form* (3.37).

Definition. A group G is said to be 3-*rewritable* if $x_1x_2x_3 = x_{\varphi(1)}x_{\varphi(2)}x_{\varphi(3)}$ for some φ, where φ is a nonidentity permutation on $\{1, 2, 3\}$.

Definition. A group G is said to be *metabelian* provided $G' \subseteq Z(G)$.

Definition. Let $f : G \to F$, G be a group, F be a field, and S be a subset of G. Then f is called an *S-odd function* provided $f(xy^{-1}) = -f(xy)$ for all $x, y \in S$ (a generalization of odd function).

Result 3.25. [174]. *Let G be a* 3-*rewritable group and F a quadratically closed field with* char $F \neq 2$. *A mapping $f : G \to F$ is a solution of* (CE) *if and only if* (i) *f has the form* (3.37) *or* (ii) *there exists a subset S of G such that $H = G\backslash S$ is a subgroup of G and f is an S-odd function such that $f|H$ is a homomorphism of H into F.*

Result 3.25(a). [172]. *Let G be a metabelian group and F a quadratically closed field with* char $F \neq 2$. *The function $f : G \to F$ satisfies* (CE) *if and only if* (i) *f has the form* (3.37) *or* (ii) *there exists a subset S of G such that $H = G\backslash S$ is a subgroup of G, f is an S-odd function, and $f|H$ is of the form* (3.37), *where g is a homomorphism of H into F^*.*

Result 3.26. [601]. *Let $f : G \to \mathbb{C}$, be a solution of* (CE), *where the group G is generated by two elements a and b. Suppose that $f(aba^{-1}b^{-1}) = 1$. Then f has the standard form* (3.37).

3.4.1 Discussion

Our results permit us to compute in detail all solutions of (CE) on an arbitrary Abelian group G. A function f on G satisfying (CE) satisfies (K) if and only if f is of the form (3.37). Now, g can be written in one and only one way in the form $g = \mathcal{X}\exp(\varphi)$, where $\mathcal{X}$ is a character of G [369] and φ is a homomorphism of G into the additive reals. Further, if g is continuous, then so are $\mathcal{X}$ and φ.

Indeed, since g is a homomorphism into $\mathbb{C}^*$, $g/|g|$ is a character of G, say $\mathcal{X}$. If g is continuous, then so is $\mathcal{X}$. Now let $\varphi = \log|g|$. Then plainly enough φ

is a homomorphism of G into the additive reals, called a real character [369], and $|g| = \exp(\varphi)$. Further, it is evident that φ is continuous when g is. Hence, we have $g = \mathcal{X}\exp(\varphi)$. So, computing g's (or f's) is reduced to computing characters and real characters of G.

Let G be a locally compact Abelian group. The homomorphisms g of (3.37) all have the form $\mathcal{X}\exp(\varphi)$, where $\mathcal{X}$ is a continuous character of G and φ is a continuous real character of G (in the sense of [369, p. 389]). The group G has the form $R^n \times G_0$, where G_0 contains a compact subgroup (open) J_0 and n is any positive integer [369, p. 389]. The form of φ on R^n is obvious, and φ is identically zero on every element of G some power of which lies in J_0. Otherwise φ on G_0 is easily described by a Zorn's Lemma argument since the image group R of φ is divisible.

Example. Let $1 \leq n < \infty$. Let f be a continuous solution of (CE) on R^n, where R^n is a topological space that is a Cartesian product of the reals R. By Lemma 3.16c, it is enough to know the homomorphisms on the component groups; that is, on R with the usual topology. So if f is a continuous solution of (CE) on R, then f/Q is also a continuous solution of (CE) on Q, where Q is the additive group of rational numbers. The continuity of f ensures the continuity of the corresponding g on Q satisfying (3.37). Further, $g(x) = g(1)^x$ with x in Q. So $|g(x)| = |g(1)|^x = \exp(\alpha x)$, where $\alpha = \log|g(1)|$ ($|g|$ is real and greater than zero). Now $g/|g|$ is a continuous character of Q and therefore has the form $\exp(i\beta x)$ with β real (see [369, p. 414]). So $g(x) = \exp[(\alpha+i\beta)x]$ on Q. $f = (g+g^*)/2$ on Q and therefore on R. Hence, all continuous solutions f of (A) on R have the form $f(x) = \cosh(\alpha x)$, where α is any complex number. Now applying Lemma 3.16c, we conclude that every continuous solution of (CE) on R^n is given by

$$f(x_1, x_2, \ldots, x_n) = \frac{g(x_1, x_2, \ldots, x_n) + g^*(x_1, x_2, \ldots, x_n)}{2},$$

where

$$\begin{aligned} g(x_1, x_2, \ldots, x_n) &= \prod_{p=1}^{n} \exp[(\alpha_p + i\beta_p)x_p] \\ &= \exp\left(\sum_{p=1}^{n}[(\alpha_p + i\beta_p)x_p]\right). \end{aligned}$$

So

$$f(x_1, x_2, \ldots, x_n) = \cosh\left[\sum_{p=1}^{n}(\alpha_p + i\beta_p)x_p\right].$$

This generalizes Flett's results and earlier ones [147, 280, 439].

There do exist unbounded, discontinuous solutions of (CE). Consider the p-adic integers Δ_p [369]. $\Delta_p \supset Z^{c*}$, where c is the continuum. Since there are many unbounded, discontinuous real characters of Z and each can be extended to

a real character on Δ_p, it is evident that there are many unbounded, discontinuous solutions of (CE) on Δ_p.

3.4.2 (C) on Abstract Spaces

The equation (C) on abstract spaces such as Hilbert space, Banach space, and Banach algebra have been treated. We will consider some of them (see [581, 582, 118, 94, 621, 409, 44, 577, 574, 760, 772]).

Result 3.27. (Kurepa [581]). *Let X be a Banach space and f be a complex-valued, continuous function on X satisfying* (C) *with $f(0) = 1$. Then there exists an $A : X \to \mathbb{C}$ that is continuous and additive* (A) *such that $f(x) = \cos A(x)$ for $x \in X$.*

Corollary 3.27a. *Suppose X in Result* 3.27 *is a Hilbert space. Then $f(x) = \cos\langle x, x_0\rangle$ for a unique $x_0 \in X$ dependent only on f.*

Result 3.27b. [581]. *Let M_n be the set of all $n \times n$ matrices over $\mathbb{R}$ and f be a complex-valued, continuous function on M_n such that f is a solution of* (C) *with $f(0) = 1$ and $f(s^{-1}xs) = f(x)$ for all $x \in M_n$ and for every nonsingular matrix $s \in M_n$. Then $f(x) = \cos(a \operatorname{tr} x)$ for $x \in M_n$, where $a \in \mathbb{C}$ and* tr $x =$ trace of $x =$ sum of the diagonal elements of x.

Result 3.28. [582]. *Let X be a real or complex Banach algebra with unit e and $f : \mathbb{R} \to X$ be a measurable solution of* (C) *with $f(0) = e$. Then there exists a unique $a \in X$ such that*

$$f(x) = a + \frac{ax^2}{2!} + \frac{a^2x^4}{4!} + \cdots = \sum_0^\infty \frac{a^n x^{2n}}{(2n)!} \quad \text{for } x \in \mathbb{R}. \tag{3.48}$$

If in X there is a b such that $a = b^2$ (that is, a has a square root), then

$$f(x) = \sum_0^\infty \frac{(bx)^{2n}}{(2n)!} = \frac{\exp bx + \exp(-bx)}{2} = \cosh bx; \tag{3.48a}$$

that is, f has the form (3.37)*. However, if b exists, it is not unique.*

Generally the square root of a does not exist in X, and the solution (3.48) cannot be written in the form (3.48a). It is illustrated by the following example.

Example 3.28a. [582]. Let $X = M_2$, 2×2 matrices over $\mathbb{C}$. Let $f : \mathbb{R} \to M_2$ be defined by $f(t) = \begin{pmatrix} 1 & 0 \\ t^2 & 1 \end{pmatrix}$ for $t \in \mathbb{R}$. Clearly f is a solution of (CE) and is not of the form (3.48a). Indeed, in this case $a = f''(0) = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ and there is no $b \in M_2$ such that $b^2 = a$. On the other hand, consider $\hat{a} \in M_4$, where

$$\hat{a} = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} & 1 \\ \hline -1 & 0 & -1 & 0 \\ \frac{1}{2} & -1 & -\frac{1}{2} & -1 \end{array}\right).$$

Then

$$\hat{a}^2 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) = \left(\begin{array}{c|c} a & 0 \\ \hline 0 & a \end{array}\right).$$

In this case, f on M_2 in (3.48) can be written by using $\hat{f}$ on M_4 in the form

$$\begin{aligned} \hat{f}(t) &= \begin{pmatrix} f(t) & 0 \\ 0 & f(t) \end{pmatrix} \\ &= \hat{e} + \frac{\hat{a}^2 t^2}{2!} + \frac{\hat{a}^4 t^4}{4!} + \cdots \\ &= \cosh \hat{a} t. \end{aligned}$$

This example suggests the idea of imbedding the Banach algebra X in another Banach algebra $\hat{X}$ that has the property that any $a \in X$ as an element in $\hat{X}$ has a square root in $\hat{X}$; i.e., there is at least one element $\hat{b} \in B$ such that $\hat{b}^2 = \hat{a}$. The construction of such a Banach algebra $\hat{X}$ is very simple. It is sufficient to consider all 2×2 matrices $\hat{x}, \hat{y}$ the elements x_{ij}, y_{ij} $(i, j = 1, 2)$ of which are elements of X and to define the usual matrix operations between such matrices. Introducing the norm in $\hat{X}$ by the formula

$$\|\hat{x}\| = \sum_{i,j=1}^{2} \|x_{ij}\|,$$

one easily verifies that $\hat{X}$ is a Banach algebra. In the Banach algebra $\hat{X}$, we imbed (isomorphically but not isometrically) the Banach algebra X by the corresponding

$$a \to \hat{a} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad \text{for } a \in X.$$

Now, simple calculation shows that

$$\begin{pmatrix} e + \frac{a}{4} & e - \frac{a}{4} \\ -\left(e - \frac{a}{4}\right) & -\left(e + \frac{a}{4}\right) \end{pmatrix}^2 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

for every $a \in X$; i.e., in $\hat{X}$, every element $a \in X$ has a square root.

The function

$$\hat{f}(t) = \begin{pmatrix} f(t) & 0 \\ 0 & f(t) \end{pmatrix}$$

satisfies all conditions of Result 3.28, and

$$\hat{f}(t) = \begin{pmatrix} f(t) & 0 \\ 0 & f(t) \end{pmatrix} = \cosh t \begin{pmatrix} e + \frac{a}{4} & e - \frac{a}{4} \\ -\left(e - \frac{a}{4}\right) & -\left(e + \frac{a}{4}\right) \end{pmatrix}$$

holds for every t, where the hyperbolic cosine is defined by the series. Thus we have the following result.

Result 3.28b. [582]. *Let $f : \mathbb{R} \to X$ (a Banach algebra with unit e) be a measurable solution of* (CE) *with $f(0) = e$ and let $\hat{X}$ be a Banach algebra of all 2×2 matrices the elements of which are elements of X and for which the norm of an $\hat{x} \in \hat{X}$ is defined by the formula*

$$\|\hat{x}\| = \sum_{i,j=1}^{2} \|x_{ij}\|.$$

If we imbed X in $\hat{X}$ by the correspondence

$$a \to \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

then

$$\begin{pmatrix} f(t) & 0 \\ 0 & f(t) \end{pmatrix} = [\exp t\hat{a} + \exp(-t\hat{a})]/2,$$

for every $t \in R$, where $\hat{a}$ can be taken as

$$\hat{a} = \begin{pmatrix} e + \frac{a}{4} & e - \frac{a}{4} \\ -\left(e - \frac{a}{4}\right) & -\left(e + \frac{a}{4}\right) \end{pmatrix} \quad \text{for } a \in X,$$

and it does not depend on t.

Baker [88] has shown with an example that the measurability assumption in Result 3.28 [582] is necessary. First we give the following result.

Result 3.28c. [88]. *Suppose G is an Abelian group, X a complex Banach algebra with identity e, and $f : G \to X$ a solution of* (C). *If, for some $a \in G$, the spectrum of $f(2a)$, $\sigma(f(2a))$ is contained in a simply connected open subset of $\mathbb{C}$ that does not contain* 1 *or $\|f(2a)\| < 1$, then f has the form* (3.37).

Example 3.28a of Kurepa can be generalized as follows [88]. Suppose N is nilpotent in X of order 2 ($N \neq 0$, $N^2 = 0$), and let $q : \mathbb{R} \to \mathbb{R}$ satisfy (Q). It is easy to verify that $f(t) = e + q(t)N$, for $t \in \mathbb{R}$, satisfies (C), where $f : \mathbb{R} \to X$.

Theorem 3.28d. [88]. *Let N be nilpotent of order* 2 *in X, $q : \mathbb{R} \to \mathbb{R}$ satisfy* (Q), *and $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a symmetric, biadditive function such that $q(x) = B(x, x)$ for $x \in \mathbb{R}$ (see* Theorem 4.1). *The function $f : \mathbb{R} \to X$ defined by $f(t) = e + q(t)N$ satisfies* (C). *Then f has the form* (3.37), *provided there exists $K : \mathbb{R} \to [0, +\infty)$ such that*

$$|B(s, t)| < K(s)K(t) \quad \text{for all } s, t \in \mathbb{R}. \tag{3.49}$$

Proof. Suppose $E : \mathbb{R} \to X$ is exponential and such that

$$f(t) = \frac{E(t) + E(-t)}{2} \quad \text{for all } t \in \mathbb{R}.$$

Define $S : \mathbb{R} \to X$ by $S(t) = \frac{E(t)-E(-t)}{2}$ for $t \in \mathbb{R}$. Then

$$\begin{aligned} S(s)S(t) &= [E(s) - E(-s)]\frac{E(t) - E(-t)}{4} \\ &= \frac{E(s+t) - E(s-t) - E(-s+t) + E(-s-t)}{4} \\ &= \frac{f(s+t) - f(s-t)}{2} \\ &= [q(s+t) - q(s-t)]\frac{N}{2} \\ &= 2B(s,t)N \quad \text{(by (4.2))} \end{aligned}$$

and hence

$$B(s,t) \le \|S(s)\|\frac{\|S(t)\|}{2}\|N\| \quad \text{for all } s, t \in \mathbb{R}.$$

The proof is complete if we let $K(t) = \frac{\|S(t)\|}{(2\|N\|)^{1/2}}$ for $t \in \mathbb{R}$.

In [207] it is shown that there exists a symmetric, biadditive $B : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that there is no $K : \mathbb{R} \to [0, +\infty)$ for which (3.49) holds. Now let q be the quadratic function associated with such a B and let $f(t) = e + q(t)N$ for $t \in \mathbb{R}$, where N is nilpotent of order 2 in X. From Result 3.28d, it follows that it is impossible to embed X in a larger algebra $\hat{X}$ in such a way that f could be of the form (3.37) from $\mathbb{R}$ to $\hat{X}$.

If the function K of Result 3.28d exists, it is not known if X can be embedded in a larger algebra $\hat{X}$ so that $\hat{f}(t)$ is an exponential cosine of the form (3.37). However, the following partial converse is true.

Result 3.28e. *Suppose $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is symmetric, biadditive, and such that* (3.49) *holds for some $K : \mathbb{R} \to [0, +\infty)$, and let $q(t) = B(t, t)$ for $t \in \mathbb{R}$. Then there exists a Banach algebra X and nilpotent N of X such that $f(t) = e + q(t)N$, for $t \in \mathbb{R}$ (an exponential cosine function), has the form* (3.37).

Result 3.28f. [88]. *Suppose X is a Banach algebra and $f :]0, \infty[\to X$ satisfies* (C) *whenever $x > y > 0$. If $\lim_{t\to 0+} f(t) = j$ exists, then $j^2 = j$ and there exist $b, c \in X$ such that $jb = bj = b$, $cj = c$, $jc = 0$, and*

$$f(x) = \left[j + \frac{x^2 b}{2!} + \frac{x^4 b^2}{4!} + \cdots\right] + c\left[xj + \frac{x^3 b}{3!} + \frac{x^5 b^2}{5!} + \cdots\right] \tag{3.48b}$$

for all $x > 0$. Conversely, with such j, b, and c, if f is defined by (3.48b) *for all $x > 0$, then f satisfies* (C).

Result 3.29. [582]. *Suppose X is a Banach space and Y a Banach algebra with identity e, that $f : X \to Y$ satisfies* (C) *with $f(0) = e$ for all $x, y \in X$, and that f is measurable on every ray; i.e., $f(tx)$ is measurable as a function of $t \in \mathbb{R}$ for every $x \in X$. Then there exists a quadratic function $q : X \to Y$ satisfying* (Q) *such that*

$$f(x) = \sum_0^\infty \frac{[q(x)]^n}{(2n)!} \quad \text{for } x \in X.$$

Further, q is continuous if and only if $\lim_{t\to 0} f(tx) = e$ uniformly for x in some sphere. If $\|e - q(x_0)\| < 1$ for at least one $x_0 \in X$, then an additive $A : X \to Y$ exists such that $q(x) = [A(x)]^2$ and therefore

$$f(x) = \frac{1}{2}[\exp(A(x)) + \exp(-A(x))] = \cosh A(x).$$

If q is continuous, so is A.

Result 3.30. [577]. *Let H be a Hilbert space and $L(H)$ the set of all linear continuous mappings of H into H endowed with the usual structure of a Banach space. Let $N : \mathbb{R} \to L(H)$ satisfy* (C) *for $x, y \in \mathbb{R}$. Suppose that* (i) *$N(x)$ is a normal transformation for every $x \in \mathbb{R}$;* (ii) *if $N(x) f = 0$ almost everywhere, then $f = 0$; and* (iii) *$N(x)$ is weakly continuous. Then a bounded self-adjoint transformation L_1 and self-adjoint transformation L_2 that commutes with L_1 can be found such that*

$$N(x) = \frac{1}{2}[\exp(ixN_1) + \exp(-ixN_1)] = \cos(xN_1)$$

holds for all x where $N_1 = L_2 + iL_1$.

More on Cosine

Theorem 3.31. (H. Haruki [352]). *Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function and satisfies*

$$f(0)[f(z+w) + f(z-w)] = 2f(z)f(w) \quad \text{for } z, w \in \mathbb{C}. \tag{3.50}$$

Then there is an entire $g : \mathbb{C} \to \mathbb{C}$ such that

$$\begin{aligned} &|f(0)|^2[|f(z+w)|^2 + |f(z-w)|^2] \\ &\quad = 2|f(z)|^2|f(w)|^2 + 2|g(z)|^2|g(w)|^2 \quad \text{for } z, w \in \mathbb{C} \end{aligned} \tag{3.51}$$

and conversely. Further, the only system of entire solutions of (3.51) *is $f(z) = a\cos bz$, $g(z) = a\exp(i\theta)\sin bz$, where a and b are arbitrary complex constants and θ is an arbitrary real constant.*

Proof. Let f satisfy (3.50). We will prove that (3.51) holds. We may assume that $f(z) \not\equiv$ constant. Otherwise the proof is clear. Note that $f(0) \neq 0$. Otherwise $f = 0$.

Differentiating both sides of (3.50) twice with respect to w and putting $w = 0$, we have

$$f(0)\,f''(z) = f''(0)\,f(z). \tag{3.52}$$

Differentiating both sides of (3.50) with respect to z and w, we have

$$f(0)(f''(z+w) - f''(z-w)) = 2f'(z)\,f'(w). \tag{3.52a}$$

We can deduce that $f''(0) \neq 0$. Otherwise, by (3.50) and (3.52), $f(z)$ is a complex constant, contradicting the assumption that $f(z) \not\equiv$ constant.

By (3.52) and (3.52a), we have

$$f(0)(f(z+w) - f(z-w)) = 2\frac{f(0)}{f''(0)}\,f'(z)\,f'(w). \tag{3.52b}$$

Equations (3.50) and (3.52b) and the parrallelogram identity $|a+b|^2 + |a-b|^2 = 2|a|^2 + 2|b|^2$ (a, b complex) yield that

$$\begin{aligned} &|f(0)|^2(|f(z+w)|^2 + |f(z-w)|^2) \\ &\quad = 2|f(z)|^2|f(w)|^2 + 2\left|\frac{f(0)}{f''(0)}\right|^2 |f'(z)|^2|f'(w)|^2. \end{aligned} \tag{3.51a}$$

By (3.51a), we see that (3.50) implies (3.51) with $g(z) = \sqrt{\frac{f(0)}{f''(0)}}\,f'(z)$, which is an entire function of a complex variable z. This proves the theorem. □

For the converse, (3.51) implies (3.50) (see [352]). By (3.50) and Remark 3.12 there follows $f(z) = a\cos bz$ and then $g(z) = a\exp(i\theta)\sin bz$.

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire solution of (C). Setting $x = \frac{s+it}{2}$ and $y = \frac{s-it}{2}$ in (C), where s and t are reals, yields

$$2f\left(\frac{s+it}{2}\right) f\left(\frac{s-it}{2}\right) = f(s) + f(it).$$

If $f(z) = \cos z$ in the equation above, by

$$f\left(\frac{s-it}{2}\right) = \cos\left(\frac{s-it}{2}\right) = \overline{\cos\left(\frac{s+it}{2}\right)} = \overline{f\left(\frac{s+it}{2}\right)}$$

we obtain

$$2f\left(\frac{s+it}{2}\right) \overline{f\left(\frac{s+it}{2}\right)} = f(s) + f(it)$$

and so

$$\left|f\left(\frac{s+it}{2}\right)\right|^2 = \frac{f(s) + f(it)}{2}$$

and then

$$\left| f\left(\frac{s+it}{2}\right)\right|^2 = \left|\frac{f(s)+f(it)}{2}\right|^2 \quad \text{for } s,t \in \mathbb{R}. \tag{3.53}$$

Thus we obtain a new cosine functional equation (3.53).

Result 3.32. [357]. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an entire function. Then the only entire solutions of* (3.53) *are* $f(z) = 0$, $f(z) = \exp(i\theta)\cos\alpha z$, $f(z) = \exp(i\theta)\cosh\alpha z$, *where θ, α are real constants.*

3.4.3 Jacobi's Elliptic Function Solution

Consider the following initial value problem, where p, q, r are meromorphic functions of a complex variable z for $|z| < \infty$ and k is an arbitrary real constant:

$$p'(z) = q(z)r(z),\ q'(z) = -p(z)r(z),\ r'(z) = -k^2 p(z)q(z),$$
$$p(0) = 0,\ q(0) = r(0) = 1.$$

A unique system of solutions for the initial value problem above is given in [375, pp. 66–75]. We write $s_n(z;k)$, $c_n(z;k)$, $d_n(z;k)$, respectively, for $p(z)$, $q(z)$, $r(z)$. They are the three kinds of Jacobi's elliptic functions.

Suppose $f : \mathbb{C} \to \mathbb{C}$ satisfies (C). Replace x and y by $x+y$ and $x-y$, respectively, in (C) to get

$$f(2x) + f(2y) = 2f(x+y)f(x-y) \quad \text{for } x, y \in \mathbb{C},$$

which with the use of the duplication formula (3.23) becomes

$$f(x)^2 + f(y)^2 = f(x+y)f(x-y) + 1$$

(e.g., (3.35)), and thus we obtain the functional equation

$$[f(x+y)+f(x-y)][f(x)^2+f(y)^2] = 2f(x)f(y)[f(x+y)f(x-y)+1]. \tag{3.54}$$

Result 3.33. [355]. *If f is a meromorphic function of a complex variable z for $|z| < \infty$, then the only solutions of* (3.54) *are $f(z) = 0$ and $f(z) = c_n(z;k)$, where α and k are arbitrary complex constants.*

Theorem 3.34. (Ger [308, 306, 307], Aczél and Dhombres [44]). *Assume G is an Abelian, connected topological group. A function $f : G \to \mathbb{R}$ is the real part of a continuous character if and only if f is not an identically zero, bounded, continuous solution of* (C).

Proof. Recall that a function $\mathcal{X} : G \to \mathbb{C}$ is called a *character*, provided $\mathcal{X}(x+y) = \mathcal{X}(x)\mathcal{X}(y)$ for all $x, y \in G$ and $|\mathcal{X}(x)| = 1$.

Let $\mathcal{X}$ be a continuous character of G. Then clearly $\mathcal{X}(0) = 1$, and $\mathcal{X}$ is bounded and $\mathcal{X}(-x) = \overline{\mathcal{X}(x)}$. So, if we set $f(x) = \mathrm{Re}\mathcal{X}(x)$, then $f(x) \neq 0$ and we get

$$\begin{aligned} f(x+y)+f(x-y) &= \mathrm{Re}\mathcal{X}(x+y)+\mathrm{Re}\mathcal{X}(x-y) \\ &= \mathrm{Re}[\mathcal{X}(x)\mathcal{X}(y)+\mathcal{X}(x)\mathcal{X}(-y)] \\ &= \mathrm{Re}[\mathcal{X}(x)(\mathcal{X}(y)+\overline{\mathcal{X}(y)})] \\ &= 2\mathrm{Re}\mathcal{X}(x)\cdot\mathrm{Re}\mathcal{X}(y) = 2f(x)f(y) \quad \text{for } x,y\in G. \end{aligned}$$

So, f satisfies (C).

Conversely, let $f : G \to \mathbb{R}$ satisfy (C) and be both bounded and continuous. Then, according to Theorem 3.17 and Theorem 3.18 [424, 477],

$$f(x) = \frac{1}{2}\left[E(x)+\frac{1}{E(x)}\right] \quad \text{for } x\in G,$$

where $E : G \to \mathbb{C}^*$ is a continuous (homomorphism) exponential function satisfying (E). Obviously, since f is real, we have

$$E(x)+\frac{1}{E(x)} = \overline{E(x)}+\frac{1}{\overline{E(x)}},$$

and since E is unique by Lemma 3.16 and $\overline{E}$ is also such a homomorphism, we must have either $E(x) = \overline{E(x)}$ or $E(x) = \frac{1}{\overline{E(x)}}$ on G.

In the first case, E is real and we shall show that $E(x) \equiv 1$. In fact, E being continuous and nonvanishing at any point, it has to be of constant sign since G is connected. Suppose that $E(x) > 0$ for $x \in G$. Since $f = \frac{1}{2}(E+\frac{1}{E})$ is bounded, this implies that E is also bounded. If there is an x_0 with $E(x_0) \neq 1$, we may assume that $E(x_0) > 1$ (otherwise take $-x_0$ instead of x_0). Then $E(nx_0) = E(x_0)^n \to \infty$ as $n \to \infty$, a contradiction. Thus $E(x) \equiv 1 = f(x) = \mathrm{Re}\ E(x)$, and $E(x)$ is a continuous character.

In the other case, $|E(x)| = 1$ for all $x \in G$, whence

$$f(x) = \frac{1}{2}(E(x)+\overline{E(x)}) = \mathrm{Re}E(x) \quad \text{for } x\in G.$$

Thus, in both cases, $f(x) = \mathrm{Re}E(x)$, where $E : G \to \{z \in \mathbb{C} : |z| = 1\}$ is a continuous homomorphism. This ends the proof (due to Ger [307]). □

A few remarks are in order.

Remark 3.34a. (Ger [308]). The assumption of the connectedness of G may be replaced by the requirement of 2-divisibility of G and the assumption that G is Abelian by the assumption that f satisfies (K). In fact, the only place connectedness is used is to show that E is positive, which can be derived from 2-divisibility as

$$E(x) = E\left(2\cdot\frac{1}{2}x\right) = E\left(\frac{1}{2}x\right)^2 > 0 \quad \text{for } x\in G$$

($\frac{1}{2}x$ need not be unique). See below in Remark 3.34c that connectedness is also not necessary.

Remark 3.34b. [308]. O'Connor in [652] proved the converse part under the additional strong assumption that G is locally compact. His proof relies on rather heavy tools from abstract harmonic analysis. The proof above shows that there is no need to assume local compactness of G (see also Ljubenova [610]).

Remark 3.34c. (Kannappan [419]). A function $f : G \to \mathbb{C}$ is nonzero, bounded, and satisfies (C) and (K) if and only if there is a character $\mathcal{X} : G \to \mathbb{C}$ such that $f(x) = \mathcal{X}(x)$ for all $x \in G$, where G is a group.

Proof. Suppose f satisfies the hypothesis. Then g given by (3.41) and Theorem 3.21 is a homomorphism from G into $\mathbb{C}^*$. Since f is bounded, so is g by (3.41) and $\frac{1}{g}$ by (3.37). Hence $|g(x)| = 1$ for all $x \in G$. Indeed, if $|g(x_0)| > 1$, consider $|g(nx_0)| = |g(x_0)|^n \to \infty$, a contradiction to g being bounded. If $|g(x_0)| < 1$, consider $\left|\frac{1}{g(nx_0)}\right| \to \infty$ as $n \to \infty$. Thus g is a character of G. Hence

$$f(x) = \frac{1}{2}\left[g(x) + \frac{1}{g(x)}\right] = \frac{1}{2}[g(x) + \overline{g(x)}] = \operatorname{Re} g(x).$$

The converse is obvious (see the proof of Theorem 3.34). □

Remark 3.34d. [419]. Suppose a nonzero $f : G \to \mathbb{R}$ satisfies (CE) and (K). Then either there is a character $\mathcal{X} : G \to \mathbb{C}$ such that $f(x) = \operatorname{Re} \mathcal{X}(x)$ for all $x \in G$ or there is a homomorphism $\varphi : G \to \{1, -1\}$ and an (additive) homomorphism $\xi : G \to \mathbb{R}$ such that $f(x) = \varphi(x) \cosh(\xi(x))$ for $x \in G$.

Proof. By Theorem 3.21, there is a homomorphism $g : G \to \mathbb{C}$ such that (3.37) holds. Let $u(x) = \operatorname{Re} g(x)$, $v(x) = \operatorname{Im} g(x)$. Then

$$2f(x) = u(x)\left[1 + \frac{1}{u(x)^2 + v(x)^2}\right] + iv(x)\left[1 - \frac{1}{u(x)^2 + v(x)^2}\right].$$

Since f is real valued, either $v(x) = 0$ or $u(x)^2 + v(x)^2 = 1$; that is, either

$$\operatorname{Im} g(x) = 0 \text{ or } |g(x)| = 1 \quad \text{for } x \in G. \tag{3.55}$$

First we will prove that $|g(x)| \equiv 1$ for all x or $\operatorname{Im} g(x) \equiv 0$ for all x. Suppose there exist $x_0, y_0 \in G$ with $|g(x_0)| \neq 1$, $\operatorname{Im} g(y_0) \neq 0$. Then, by (3.55), $\operatorname{Im} g(x_0) = 0$ and $|g(y_0)| = 1$.

Using g as a homomorphism, first we get $|g(x_0y_0)| = |g(x_0)| \neq 1$. Now claim $\operatorname{Im} g(x_0y_0) \neq 0$. If $\operatorname{Im} g(x_0y_0) = 0$, then $u(x_0)v(y_0) = 0$. Since $v(y_0) \neq 0$, $0 = u(x_0) = g(x_0)$, which is a contradiction for $g : G \to \mathbb{C}^*$. Hence neither $|g(x_0y_0)| \neq 1$ nor $\operatorname{Im} g(x_0y_0) = 0$.

Thus $|g(x)| = 1$ for all x or $\operatorname{Im} g(x) = 0$ for all $x \in G$. In the case $|g(x)| = 1$ for all x, g itself is a character of G and $f(x) = \frac{1}{2}\left[g(x) + \overline{g(x)}\right] = \operatorname{Re} g(x)$.

In the case of $\operatorname{Im} g(x) = 0$, g is a real homomorphism of G into $\mathbb{R}^*$. Set $\varphi(x) = \frac{g(x)}{|g(x)|}$ and $\xi(x) = \log|g(x)|$. Then φ is a homomorphism of G into $\{1, -1\}$,

$$\xi(xy) = \xi(x) + \xi(y),$$
$$g(x) = \varphi(x) \cdot \exp \xi(x).$$

So, for $x \in G$,

$$f(x) = \frac{1}{2}\{\varphi(x) \exp \xi(x) + [\varphi(x) \exp \xi(x)]^{-1}\}$$
$$= \varphi(x)\frac{1}{2}[\exp \xi(x) + \exp(-\xi(x))]\varphi(x) \cosh \xi(x).$$

Now it is easy to deduce that the following are equivalent when $f : G \to \mathbb{R}$ satisfies (CE) and (K): $|f| \geq 1$; f is unbounded or $|f(x)| = 1$; $f(x) = \varphi(x) \cosh \xi(x)$ for all $x \in G$, where $\varphi : G \to \{-1, 1\}$ is a homomorphism and ξ is additive on G. □

Result 3.35. [419]. *Let H be a dense subgroup of G and $f : H \to \mathbb{C}$ be a continuous solution of* (C) *with* (K). *Then f has a unique continuous extension $f^* : G \to \mathbb{C}$, which satisfies* (C) *and* (K).

Proposition 3.36. [419]. *The following are equivalent on G:*

(i) *For $f : G \to \mathbb{C}$ satisfying* (CE) *and* (K), *there exists an (additive) homomorphism $\eta : G \to \mathbb{C}(+)$ such that $f(x) = \cos \eta(x)$ for $x \in G$.*
(ii) *For any homomorphism $g : G \to \mathbb{C}^*$, there is a homomorphism $\xi : G \to \mathbb{C}$ such that $g(x) = \exp \xi(x)$ for $x \in G$.*
(iii) *For any character $\mathcal{X} : G \to \mathbb{C}$, there is a real homomorphism $\xi : G \to \mathbb{R}$ such that $\mathcal{X}(x) = \exp i\xi(x)$, $x \in G$.*

Proof. To prove (i) ⇒ (ii), suppose $g : G \to \mathbb{C}^*$ is a homomorphism. Now, define $f(x) = \frac{1}{2}[g(x) + \frac{1}{g(x)}]$. Then f satisfies (CE) and (K). So, by (i), there is an (additive) homomorphism $\eta : G \to \mathbb{C}$ such that

$$f(x) = \cos \eta(x) = \frac{\exp(i\eta(x)) + \exp(-i\eta(x))}{2} \quad \text{for } x \in G.$$

Since, by Lemma 3.16, a homomorphism associated to f is "unique", $\exp i\eta(x) = g(x)$ or $\exp(-i\eta(x)) = g(x)$ for $x \in G$. This proves (ii).

Next, to prove (ii) ⇒ (iii), we are given a character $\mathcal{X} : G \to \mathbb{C}$. By (ii), there is a homomorphism $\xi : G \to \mathbb{C}$ such that $\mathcal{X}(x) = \exp i\xi(x)$, $x \in G$. Since $|\mathcal{X}(x)| = 1$, $\xi(x)$ is real. This proves (iii).

Finally, to show that (iii) ⇒ (i), let $f : G \to \mathbb{C}$ satisfy (CE) and (K) and be nonzero. By Theorem 3.21, there is a homomorphism $g : G \to \mathbb{C}^*$ associated to f. Set $\mathcal{X}(x) = \frac{g(x)}{|g(x)|}$ and $\varphi(x) = \log|g(x)|$ for $x \in G$. Then $\mathcal{X}$ is a character, φ is a real (additive) homomorphism on G into $\mathbb{R}$, and $g(x) = \mathcal{X}(x) \exp \varphi(x)$. By (iii), there is a real homomorphism $\xi : G \to \mathbb{R}$ such that $\mathcal{X}(x) = \exp i\xi(x)$ for $x \in G$. So,

$$g(x) = \exp \varphi(x) \cdot \exp i\xi(x) = \exp i\eta(x),$$

where

$$\eta(x) = -i\varphi(x) + \xi(x).$$

Thus $f(x) = \cos\eta(x)$. This proves (i) and completes the proof of this proposition. □

If G is a topological group, each of the functions involved above can be taken to be continuous.

Characterization of the Cosine and Some Generalizations

Since Cauchy equations possess noncontinuous solutions, (C) has non-(measurable) continuous solutions. In order to obtain the cosine solution, it is necessary to assume some regularity condition on f.

As is known, the definition of the trigonometric cosine may be given using the theory of representation of groups. Such an approach is nearer to the introduction of the cosine by functional equations of two variables. The latter one can be obtained by investigating the equation (C),

$$f(x+y) = f(x)f(y) - \sqrt{1-f(x)^2}\sqrt{1-f(y)^2}, \tag{3.56a}$$

for instance, and more; this equation has in the class of continuous functions the solutions $f(x) = \cos dx$ and only these. The characterization is done also by using functional equations in a single variable (Kannappan [423], Kuczma [554], Kulkarni [568], Sharkovski [733]). It seems that the best way of characterizing the cosine (as well as the other functions) is by several (and not one) functional relations. We will see some in a later section.

Let f and g be functions satisfying

$$f(x-y) = f(x)f(y) + g(x)g(y) \tag{3.56b}$$

and

$$\lim_{x\to 0}\frac{g(x)}{x} = 1.$$

Then $f(x) = \cos x$ and $g(x) = \sin x$. We will determine the general solution of (3.56b) later in this section. If, however, one wants to define cosine alone, one may use the equation (3.56a), whose general continuous solution is $f(x) = \cos kx$ [12, 314]. Equation (3.56b) or (3.56a) certainly characterizes the cosine function. But it would perhaps be desirable and at least interesting to characterize the cosine by an equation in a single variable. In this direction, the functional equations

$$f(2x) = 2f(x)^2 - 1 \tag{3.23}$$

and

$$f\left(\frac{x}{2}\right) = s(x)\sqrt{\frac{[1+f(x)]}{2}}, \tag{3.23a}$$

where

$$s(x) = \begin{cases} 1 & \text{for } x \in [(4k-1), (4k+1)], \\ -1 & \text{for } x \in [(4k+1), (4k+3)], \quad k = 0, \pm 1, \pm 2, \end{cases}$$

can be considered.

It has been proved that $f(x) = \cos x$ is the only even solution of (3.23) that is twice differentiable at $x = 0$ and such that $f(0) = 1$ and $f''(0) = -1$. Also $f(x) = \cos x$ is the only function defined for all x, continuous in a neighbourhood of $x = 0$, satisfying (3.23a) and periodic with period 2π. These results characterize the cosine.

In (3.23), it would be nicer to be able to introduce the cosine before introducing the derivatives, with no stronger means than, for example continuity. But it has been proved that the condition that f is twice differentiable cannot be weakened, even under the additional assumption of continuity. In (3.23), it contains the sign factor, which is somewhat disagreeable.

First we give a characterization of the cosine function with the help of a functional equation in a single variable based on the duplication formula (3.23) and then another characterization related to the parallelogram law of addition of forces.

Theorem 3.37. [423]. *Let $f : \mathbb{R} \to \mathbb{R}$ be even, continuous in a neighbourhood of 0, $f(x) \geq 0$ for $0 \leq x \leq \frac{\pi}{2}$ and $f(x) \leq 0$ for $\frac{\pi}{2} \leq x \leq \pi$, and satisfy*

$$f(2x) = 1 - 2f\left(\frac{\pi}{2} - x\right)^2 \quad \text{for } x \in \mathbb{R}. \tag{3.56}$$

Then $f(x) = \cos x$ for all $x \in \mathbb{R}$.

Proof. First we show that f is periodic with period 2π. Then we prove that f is continuous everywhere. Change x into $x + \frac{\pi}{2}$ in (3.56) to get

$$f(\pi + 2x) = 1 - 2f(-x)^2 \quad \text{for } x \in \mathbb{R}.$$

Change x into $-x$ in the equation above and use evenness of f to conclude that f has period 2π. The sign of $f(x)$ is determined uniquely for each x in $\mathbb{R}$ by the periodicity 2π, evenness, and the hypothesis $f(x) \geq 0$ in $[0, \frac{\pi}{2}]$ and ≤ 0 in $[\frac{\pi}{2}, \pi]$.

Rewrite (3.56) as $f(x) = 1 - 2f(\frac{\pi}{2} - \frac{x}{2})^2$. As x varies over $]\pi - \epsilon, \pi + \epsilon[$, $\left(\frac{\pi}{2} - \frac{x}{2}\right)\epsilon] - \frac{\epsilon}{2}, \frac{\epsilon}{2}[$. This shows that f is continuous at π. When $\frac{\pi}{2} - x$ varies over $]\pi - \epsilon,\ \pi + \epsilon[,\ 2x\epsilon\ ] - \pi - 2\epsilon, -\pi + 2\epsilon[$ f is continuous in $]\pi - 2\epsilon, \pi + 2\epsilon[$. Proceeding in this way, we can conclude that f is continuous at all points. Finally, we will show that f is determined uniquely at every point.

As f is continuous everywhere, it is enough to prove that f takes unique values at every point $x = \frac{k\pi}{2^{n-1}}$, k any integer, and n any integer $\geqq 1$. The proof is based on induction on n.

Putting $x = \frac{-\pi}{2}$ in (3.56) and using f even, we get $f(\pi) = \frac{1}{2}$ or -1. But since $f(x) \leq 0$ for $x \in [\frac{\pi}{2}, \pi]$, we have $f(\pi) = -1$. Letting $x = \frac{\pi}{2}$ in (3.56), we get

$f(0) = 1$ for k an even integer and $f(k\pi) = f(\pi) = -1$, for k an odd integer. Now, suppose that f takes unique values for all x of the form $x = \frac{k\pi}{2^{n-1}}$. Then, as

$$f\left(\frac{kx}{2^n}\right) = \pm\sqrt{\frac{1}{2}\left\{1 - f\left(\frac{(2^{n-1}-k)\pi}{2^{n-1}}\right)\right\}},$$

by using the induction hypothesis and the fact that f has a fixed sign at every point, we see that f takes unique values at every point $x = \frac{k\pi}{2^n}$. Thus f takes unique values at every point. But the function $\cos x$ satisfies all the conditions of the theorem and hence $f(x) = \cos x$ for all x.

Of course, the equation (3.56) can be reduced to

$$g(2x) = 2g(x)^2 - 1, \quad x \in \mathbb{R}, \tag{3.23}$$

and then the solution of (3.56) can be found using [554] by the following procedure.

Now define a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$g(x) = -f(\pi - x), \quad x \in \mathbb{R}.$$

Then, from (3.56) and using f even, we have

$$2g(x)^2 = 2f(\pi - x)^2 = 1 - f(2x - \pi) = 1 + g(2x), \quad x \in \mathbb{R}.$$

Hence g satisfies the duplication formula (3.23). As f is continuous in a neighbourhood 0 and so in a neighbourhood of π, we see that g is continuous in a neighbourhood of 0.

Since f is periodic with period 2π, we see that $-g(x) \geq 0$ for $\frac{-\pi}{2} \leqq x \leqq \frac{\pi}{2}$ and $g(x) \leqq 0$ for $\frac{\pi}{2} \leqq x \leqq \frac{3\pi}{2}$, and g has period 2π. From [554, 733] we see that $g(x) = \cos x$ for all x in $\mathbb{R}$. Hence $f(x) = \cos x$ for all x in $\mathbb{R}$. □

Remark 3.37a. $f(x) = +\frac{1}{2}$ satisfies (3.56) and f is continuous and even. It does not satisfy $f(x) \leq 0$ in $[\frac{\pi}{2}, \pi]$.

Remark 3.37b. $f(x) = \cos(2k+1)x$, k any integer, is even, continuous, and satisfies (3.56) but not the sign condition.

The problem of finding out whether these are the only even, continuous solutions of (3.56) remains open.

3.4.4 More Characterizations in a Single Variable

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a\cos n(x + b)$ for a, b real and n an integer has the following properties:

(i) f is continuous and has period 2π.
(ii) f is differentiable, and there exist real constants c and d such that

$$f'(x) = cf(x + d) \quad \text{for all } x \in \mathbb{R}.$$

(iii) Given any real constants c, c', d, d', there exist real constants d_1, d_1' such that $cf(x+c') + df(x+d') = d_1 f(x+d_1')$ for all x.

Now, we consider the converse case by choosing from properties (i) to (iii) to determine f and then connect to the parallelogram law of addition of vectors.

Theorem 3.38. (Robbins [705]). *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous with period 2π; that is,* (i) *holds. Then, if f has either property* (ii) *or* (iii), *then $f(x) = a\cos n(x+b)$ for some constants a, b and an integer n.*

Proof. [705]. Suppose f has property (ii). Then f has derivatives of all orders with period 2π. So, f can be represented by a convergent Fourier series that can be differentiated term by term. Thus, for some constant k_n,

$$f(x) = \sum_{-\infty}^{\infty} k_n \exp inx,$$

$$f'(x) = \sum_{-\infty}^{\infty} ik_n n \exp inx.$$

Then

$$0 = f'(x) - cf(x+d) = \sum_{\infty}^{\infty} k_n(in - c\exp ind)\exp inx$$

yields, for every n, $k_n(in - \exp idn) = 0$. So, for $f \not\equiv 0$ (if $\equiv 0$ nothing to prove), there exists an n for which $k_n \neq 0$ and $in = \exp idn$. Thus $n = \pm d$. Hence there exists at most two values of n for which $k_n \neq 0$, and these values are negatives of one another. Thus, for some integer n,

$$f(x) = k_n \exp inx + k_{-n}\exp(-inx) \quad \text{for } x \in \mathbb{R}.$$

Since f is real valued, it follows that $k_{-n} = \overline{k_n}$, and the proof is complete.

For the proof when f satisfies (iii), see [705], and we apply this part to derive the law of addition of forces.

We shall now apply Theorem 3.38 (iii) to derive the law of addition of forces. For simplicity, let us consider only forces acting at a fixed point in a fixed plane in which the angular coordinate x is defined. With such a force, we identify the real-valued function $F(x)$ that specifies the scalar component of the force in the direction x; thus a force is represented by a real-valued function of period 2π. By the sum of two forces $F_1(x)$ and $F_2(x)$, we mean the function $F_1(x) + F_2(x)$. Our assumptions are the following.

All forces are geometrically similar. By this we mean that there exists a fixed function $f(x)$ of period 2π such that any force $F(x)$ can be written in the form

$$F(x) = a_F f(x + b_F), \tag{3.57}$$

where a_F and b_F are constants determined by $F(x)$. We need not assume that all values of the constants a_F and b_F can occur in (3.57), but we shall assume that there exist at least the forces $F_1(x) = f(x)$ and $F_2(x) = f(x+1)$.

The sum of two forces is a force. This implies that the function $f(x)$ has the property that, for certain constants c, d and for every x,

(iv) $$f(x+1) + f(x) = cf(x+d).$$

The function $f(x)$ is continuous, nonconstant, and vanishes for at most two values in the interval $0 \le x < 2\pi$.

Theorem 3.38 (iii) shows that the function $f(x)$, continuous, real valued, of period 2π, and satisfying (iv), must be of the form

$$f(x) = a\cos n(x+b),$$

where n is an integer. The hypotheses of (iii) ensure that n can be chosen as 1. The parallelogram law of addition of forces is an immediate consequence.

Now we treat *d'Alembert's restrictive functional equation.*

Let $\underline{X}$ be a real pre-Hilbert space with $\dim \underline{X} \ge 3$. In the class of real functionals $f : \underline{X} \to \mathbb{R}$ in $[F_{11}]$ three functional equations were considered: (C) and two conditional forms of (C), namely

$$(C_1) \qquad f(x+y) + f(x-y) = 2f(x)f(y), \quad \{\text{for } x, y \in \underline{X} : \langle x, y\rangle = 0\},$$

$$(C_2) \qquad f(x+y) + f(x-y) = 2f(x)f(y), \quad \{\text{for } x, y \in \underline{X} : ||x|| = ||y||\}.$$

Now we compare the class of solutions of (C_1) or (C_2) with that of (C). It is easy to see that functionals $f : \underline{X} \to \mathbb{R}$ defined by $f(x) = \exp A(||x||^2)$ for $x \in \underline{X}$, where $A : \mathbb{R} \to \mathbb{R}$ satisfies (A), are a solution of (C_1) but not (C). So, the class of solutions of (C) is a proper subset of the class of solutions of (C_1). But with the addition of a suitable condition on f, the two classes of solutions will be identical.

Theorem 3.39. [281]. *A nonzero function $f : \underline{X} \to \mathbb{R}$ is a solution of* (C) *if and only if f satisfies* (C_1) *and the duplication formula* (3.23).

As regards (C) and (C_2), the following is true.

Theorem 3.40a [281]. *A nonzero $f : \underline{X} \to \mathbb{R}$ is a solution of* (C) *if and only if f satisfies* (C_2) *with $f(0) = 1$.*

Proof. It is obvious to see the necessary condition. Now, to prove the sufficient condition, assume f satisfies (C_2) and $f(0) = 1$. Obviously, the duplication formula (3.23) holds. So, by Theorem 3.39, it is enough to show that (C_1) holds. Let $x, y \in \underline{X}$ with $\langle x, y\rangle = 0$. If $||x|| = ||y||$, then (C_1) holds.

Suppose $||x|| \ne ||y||$ with $\langle x, y\rangle = 0$. Then, since $||x+y|| = ||x-y||$ and $\left|\left|\frac{x+y}{2}\right|\right| = \left|\left|\frac{x-y}{2}\right|\right|$, (C_2) yields (for pairs $x+y$, $x-y$ and $\frac{x+y}{2}$, $\frac{x-y}{2}$)

$$f(2x) + f(2y) = 2f(x+y)f(x-y),$$
$$f(x) + f(y) = 2f\left(\frac{x+y}{2}\right) f\left(\frac{x-y}{2}\right).$$

Squaring the latter equation and using (3.23) in the first equation, we get

$$1 + f(x+y)f(x-y) + 2f(x)f(y) = 4f\left(\frac{x+y}{2}\right)^2 f\left(\frac{x-y}{2}\right)^2.$$

But if f satisfies (3.23), then

$$4f\left(\frac{x+y}{2}\right)^2 f\left(\frac{x-y}{2}\right)^2 = f(x+y)f(x-y) + f(x+y) + f(x-y) + 1$$

(see Fochi [281]). Thus (C) holds.

Now we determine the general solution of (3.27) [12, 830, 426, 477]. Let $f, \varphi : G \to F$, with G a group, satisfy

$$f(x+y) + f(x-y) = 2f(x)\varphi(y) \quad \text{for } x, y \in G, \tag{3.27}$$

with f satisfying (K). We have to modify the proof in Theorem 3.17 (where $f, \varphi : \mathbb{R} \to \mathbb{C}$ are continuous) to find the general solution.

When $f = 0, \varphi$ is arbitrary. When $\varphi = 0$, $f = 0$. We consider now nontrivial solutions of (3.27), $f \neq 0$, and $\varphi \neq 0$ and prove the following theorem.

Theorem 3.41. *The most general solutions $f, \varphi : G \to \mathbb{C}$ satisfying* (3.27) *with f satisfying* (K) *are $f = 0$, φ arbitrary; $\varphi = 1$ and f is Jensen's equation given by* (3.58)*; and $\varphi(x)$ given by* (3.59) *and f by* (3.60)*.*

Proof. From Theorem 3.11, it follows that φ also satisfies (K) and (C).

When $\varphi \equiv 1$, f satisfies (J) and

$$f(x) = A(x) + b \quad \text{for } x \in G, \tag{3.58}$$

where $A : G \to \mathbb{C}$ is additive.

Let $\varphi \not\equiv 1$. From Theorem 3.21, it follows that

$$\varphi(x) = \frac{g(x) + g^*(x)}{2} \quad \text{for } x \in G, \tag{3.59}$$

where g is a homomorphism G to $\mathbb{C}^*$.

Set $x = 0$ in (3.27) to obtain

$$f(y) + f(-y) = 2k\varphi(y) = k(g(y) + g^*(y)) \quad \text{for } y \in G. \tag{3.60}$$

Interchange x and y in (3.27) and add it to (3.27) and use (3.60) to have

$$2f(x+y) + f(x-y) + f(y-x) = 2f(x)\varphi(y) + 2f(y)\varphi(x);$$

that is,

$$2f(x+y)+k(g(x-y)+g^*(x-y))=f(x)(g(y)+g^*(y))+f(y)(g(x)+g^*(x))$$

or

$$\begin{aligned}f(x+y)-\frac{k}{2}(g(x+y)+g^*(x+y))&=\left[f(x)-\frac{k}{2}(g(x)+g^*(x))\right]\frac{g(y)+g^*(y)}{2}\\&\quad+\left[f(y)-\frac{k}{2}(g(y)+g^*(y))\right]\frac{g(x)+g^*(x)}{2},\end{aligned}$$

meaning

$$F(x+y)=F(x)\frac{(g(y)+g^*(y))}{2}+F(y)\frac{(g(x)+g^*(x))}{2},\quad x,y\in G, \tag{3.61}$$

where

$$F(x)=f(x)-\frac{k}{2}(g(x)+g^*(x))\quad\text{for } x\in G.$$

Apply associativity and use (3.61) to have

$$\begin{aligned}F(x+y+z)&=\frac{1}{4}[F(x)(g(y)+g^*(y))+F(y)(g(x)+g^*(x))](g(z)+g^*(z))\\&\quad+F(z)\cdot\frac{1}{2}(g(x+y)+g^*(x+y))\\&=F(x)\cdot\frac{1}{2}(g(y+z)+g^*(y+z))+\frac{1}{4}[F(y)(g(z)+g^*(z))\\&\quad+F(z)(g(y)+g^*(y))](g(x)+g^*(x)),\end{aligned}$$

which on simplification becomes

$$F(x)(g(y)-g^*(y))(g(z)-g^*(z))=F(z)(g(x)-g^*(x))(g(y)-g^*(y)).$$

Now $g(x)=g^*(x)$ for every x means $g(x)=1$. Indeed, $g(x)^2=1$. If there is an x_0 with $g(x_0)=-1$, then $-1=g(x_0)=g\left(\frac{x_0}{2}\right)^2=1$, a contradiction.

Then $\varphi(x)=1$, which is not the case. Thus

$$F(x)=b(g(x)-g^*(x))$$

and

$$f(x)=\frac{k}{2}(g(x)+g^*(x))+b(g(x)-g^*(x))\quad\text{for } x\in G. \tag{3.59a}$$

This proves the theorem. □

Result 3.41a. (Carmichael [143]). *Let G be a group whose elements are of odd order, $Z(G)$ the centre of G, F a field with* char $F \neq 2$, *and $F(b)$ the extension field of F by an element b. Suppose $f, \varphi : G \to F$ satisfy* (3.27). *Suppose there is a $u \in Z(G)$ such that $\varphi(u) \neq 1$. Then φ and f are given by* (3.59) *and* (3.59a), *respectively, where $g : G \to F(b)^*$ is a homomorphism.*

Definition. A group G is said to be a P_3-group if the order of G' (the commutator subgroup of G) is either one or two.

Definition. Let G be a group, F a field with char $F \neq 2$, and C a subset of G. Then $f : G \to F$ is called a *C-odd* function if, for all $x \in C$ and $y \in G$, $f(xy) = -f(xy^{-1})$. f is said to be *C-even* provided, for all $x \in G$ and $y \in C$, $f(xy) = f(xy^{-1})$.

Result 3.41b. (Corovei [171]). *Let G be a P_3-group and F a quadratically closed field with* char $F \neq 2$. *The pair of mappings $f, \varphi : G \to F$ is a solution of the equation* (3.27) *if and only if it has one of the following four forms:*

(i) $f = 0$ *and φ is arbitrary.*

(ii) $f(x)$ *is given by* (3.59a) *and φ is given by* (3.59), *where g is a homomorphism from G into F^*.*

(iii) $f(x) = g(x)[\varphi(x) + a]$, $\varphi(x) = g(x)$, *for all $x \in G$, where g is a homomorphism from G into F^*, φ is a homomorphism from G into the additive group of F, and a is an arbitrary element of F.*

(iv) *There exists a nonempty subset $C \subseteq G$ such that $H = G \setminus C$ is a normal subgroup of G that contains $Z(G)$ and the subset $\{x^2 \mid x \in G\}$, and*

$$f(x) = a\alpha(x) + f_2(x), \quad \varphi(x) = \alpha(x) \quad \text{for all } x \in G,$$

where α is a C-odd function, $\alpha|_H$ is a homomorphism from H into F^*, f_2 is a C-even function such that $f_2(x) = \alpha(x)\beta(x)$ for all $x \in H$, where β is a homomorphism from H into the additive group of F, and $f_2(xy) = f_2(x)\alpha(y)$ for all $x \in C$ and $y \in H$, where a is an arbitrary element of F.

In (C), replace f by $\frac{1}{2}f$ and take $g(x) = 1$ to get

$$f(x)f(y) = f(x+y) + f(x-y)g(y) \quad \text{for } x, y \in G, \tag{3.62}$$

where $f, g : G \to F$, G is a uniquely 2-divisible Abelian group, and F is a field of characteristic different from two.

A pair (f, g) is an *admissible solution* of (3.62) if $g(0) = 1$ and $f(0) = 2$. Now, we characterize the Abelian groups with the property that, in every admissible solution (f, g), the second component g must be exponential. It is easy to see that $f = E_1 + E_2$ and $g = E_1E_2$, where $E_1, E_2 : G \to F$ satisfy (E) and (3.62). In Theorem 3.42, we will prove that they are the solutions of (3.62) under certain conditions. We begin with the following example, where g is not an exponential function.

Example. If $\left|\frac{G}{2G}\right| > 2$, (3.62) admits an admissible solution (f, g), where g is not an exponential function.

Define $f, g : G \to F$ by

$$f(x) = \begin{cases} 2, & \text{for } x \in 2G \\ 0, & \text{for } x \notin 2G, \end{cases} \qquad g(x) = \begin{cases} 1, & \text{for } x \in 2G \\ -1, & \text{for } x \notin 2G. \end{cases}$$

It is easy to see that (f, g) is an admissible solution of (3.62) since $\left|\frac{G}{2G}\right| > 2$ and there exist $y, z \notin 2G$ such that $y + z \notin 2G$. Then $g(y + z) = -1$, whereas $g(y)g(z) = (-1)(-1) = 1$. So, g is not exponential.

We now proceed to show that if $\left|\frac{G}{2G}\right| \leq 2$, then every admissible (f, g) has g exponential. We consider only $f \neq 0$.

First, for all $y \in G$, $g(y)g(-y) = 1$ and $g(y) \neq 0$ for all y. Set $x = 0$ in (3.62) to get (using $f(0) = 2$)

$$f(y) = f(-y)g(y) = f(y)g(-y)g(y) \quad \text{for all } y \in G.$$

So, whenever $f(y) \neq 0, g(y)g(-y) = 1$. Suppose $f(y_0) = 0$. Then $f(-y_0) = 0$ from $f(-y) = f(y)g(-y)$.

$$f(x + y_0) + f(x - y_0)g(y_0) = 0.$$

This shows $g(y_0) \neq 0$ or otherwise $f(x + y_0) = 0$; that is, $f = 0$ and $f(2y_0) + 2g(y_0) = 0$ (let $x = y_0$). Similarly, (3.62) gives $f(x - y_0) + f(x + y_0)g(-y_0) = 0$; that is, $2 + f(2y_0)g(-y_0) = 0$. So,

$$2 + g(-y_0)(-2g(y_0)) = 0 \quad \text{or} \quad g(y_0)g(-y_0) = 1.$$

Next, define $h : G \times G \to F$ by

$$h(x, y) = g(x + y)g(x)^{-1}g(y)^{-1} = g(x + y)g(-x)g(-y) \quad \text{for } x, y \in G.$$

It is obvious that g is exponential if and only if $h(x, y) = 1$. It is easy to check that h is symmetric and satisfies [212]

(i) $\quad f(x + y)f(x + y + z)h(y, z) = f(x + y)f(x + y + z) \quad \text{for } x, y, z \in G,$

$h(-y, y) = 1 = h(0, y)$, and the cocycle equation $h(x, y)h(x + y, z) = h(x, y + z)h(y, z)$.

If f never takes the value 0, then $h = 1$ and so g is exponential. Assume $h(y, z) \neq 1$ for some $y, z \in G$. Claim $y, z \notin G$.

By (i), $f(x + y)f(x + y + z) = 0$ for all $x \in G$. With $x = -y$, we see that $2f(z) = 0$; that is, $f(z) = 0$. Similarly, $f(y) = 0$. By (3.60),

$$f(x + y + z)f(x + y) = f(2x + 2y + z) + f(z)g(x + y).$$

Hence $f(2x+2y+z)=0$ for all $x \in G$. If $z \in 2G$, choose $x \in G$ with $2x+z=0$. Then $f(2y)=0$. Equation (3.60) yields (with $x=y$) $f(y)^2 - f(2y) = 2g(y) = 0$, which cannot be. Similarly, $y \notin 2G$.

Before proving the main result (Theorem 3.42), we show that when $\left|\frac{G}{2G}\right| \leq 2$, g is exponential. From the argument above, it follows that if y or z is in $2G$, then $h(y,z)=1$. Suppose $y, z \notin 2G$. Then, since $\left|\frac{G}{2G}\right| \leq 2$, $y+z \in 2G$. In the cocycle equation, take $x=-y$ to obtain

$$h(-y,y)h(0,z) = 1 \cdot 1 = h(-y,y+z)h(y,z) = 1 \cdot h(y,z).$$

Thus $h(y,z)=1$ in all cases and g is exponential. □

Theorem 3.42. [212]. *If (f,g) is an admissible solution of* (3.62)*, then there is an extension field K of F with $[K:F] \leq 2$ and exponential functions $E_1, E_2 : G \to F$ such that $f = E_1 + E_2$, $g = E_1 E_2$.*

Proof. In [424, 477] the solution of d'Alembert's equation (C) was framed for the case $F=\mathbb{C}$; however, examination of the proof shows that the following is true.

If $f : G \to F$ is a solution of (C) with $f(0)=1$, then there is an extension field K of F with $[K:F] \leq 2$ and an exponential function $E : G \to K$ such that $2f(x) = E(x) + E(-x)$ for all $x \in G$.

By assumption, $G=2G$, so g is an exponential function. Define $\alpha : G \to F$ by $\alpha(x) = \frac{1}{2} f(2x) g(-x)$. Then α satisfies d'Alembert's equation (C) and $\alpha(0)=1$. Hence there is a field K with $F \subseteq K$, $[K:F] \leq 2$, and an exponential function $E : G \to L$ with

$$f(2x) = E(x)g(x) + E(-x)g(x).$$

Now let $x \in G$ be arbitrary. Since G is uniquely 2-divisible, there is a unique element $\frac{x}{2} \in G$. Define $E_1 = E(\frac{x}{2})g(\frac{x}{2})$, $E_2 = E(-\frac{x}{2})g(\frac{x}{2})$. Then E_1, E_2 are exponential functions with $E_1(0) = E_2(0) = 1$. Also $E_1 + E_2 = f(2 \cdot \frac{x}{2}) = f(x)$ and $E_1(x)E_2(x) = g(\frac{x}{2})g(\frac{x}{2}) = g(x)$. This completes the proof of the theorem.

□

Now we turn our attention to sine equations.

Sine Equations

This section is devoted to sine functions. We will treat several functional equations connected to sine functions. The next section is devoted to both cosine and sine functions. First we start off with the sine functional equation

(S) $$f(x+y)f(x-y) = f(x)^2 - f(y)^2 \quad \text{for } x, y \in \mathbb{R}.$$

Theorem 3.43. (Aczél [12], Aczél and Dhombres [44], Kurepa [581]). *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous* (*integrable, measurable*) *solution of* (S) *for $x, y \in \mathbb{R}$. Then $f(x) = cx$ or $c \sin bx$ or $c \sinh bx$, for $x \in \mathbb{R}$, b, c, are real constants.*

Proof. Since f is a continuous solution of (S), it can be proved that it is differentiable and has derivatives of all orders. As before, we need derivatives of order two only.

$f \equiv 0$ is a solution of (S). Let us assume that $f \not\equiv 0$. $x = 0 = y$ in (S) yields $f(0) = 0$.

Interchange x and y in (S) to have

$$f(x+y)\,f(x-y) = -\,f(x+y)\,f(y-x)$$

or

$$f(u)\,f(v) = -\,f(u)\,f(-v);$$

that is, f is odd since $f \neq 0$. Differentiate (S) first with respect to y and then with respect to x to get

$$f''(x+y)\,f(x-y) = f(x+y)\,f''(x-y)$$

or

$$f''(u)\,f(v) = f(u)\,f''(v).$$

Since $f \not\equiv a0$, we have $f''(u) = kf(u)$, a differential equation. As in Theorem 3.15, we get $f(x) = cx$ or $c_1 \cos bx + d \sin bx$ or $c_1 \cosh bx + \alpha \sinh bx$, $b, c, c_1, d \in \mathbb{R}$. $f(0) = 0$ yields $c_1 = 0$. This proves the theorem. □

Remark 3.43a. (Kannappan [429], Kurepa [581]). The only continuous solutions $f : \mathbb{R} \to \mathbb{C}$ of (S) are $f(x) = cx$ or $c \sin bx$, $b, c \in \mathbb{C}$.

Now we will determine the general solution of (S).

Theorem 3.44. (See also Kannappan [477]). Let G be a 2-divisible group. If $f : G \to \mathbb{C}$ and a complex-valued function on an arbitrary additive group G is a solution of (S) satisfying (K), then f is either additive, satisfies (A), or

$$f(x) = k(g(x) - g(-x)), \quad k \in \mathbb{C},\ x \in G, \tag{3.37a}$$

where $g : G \to \mathbb{C}^*$ is a homomorphism. Furthermore, every measurable solution of (S) on a locally compact Abelian group is continuous.

Proof. Note that $f \equiv 0$ is a solution of (S). Let us assume that $f \not\equiv 0$. First $x = 0 = y$ in (S) yields $f(0) = 0$.

Now we will prove that f is odd; that is, $f(-x) = -f(x)$ for all $x \in G$. Changing y to $-y$ in (S) yields $f(y)^2 = f(-y)^2$; that is, $f(-y) = \pm f(y)$ and $x = 0$ in (S) gives $f(y)\,f(-y) = -f(y)^2$.

If $f(y) = 0$, then $f(-y) = 0$. If $f(y) \neq 0$, then $f(-y) = -f(y)$; that is, f is odd. Or if $f(-y) = f(y)$, then $f(y)\,f(-y) = f(y)^2 = -f(y)^2$ or $f(y) = 0 = f(-y)$. Otherwise $f(-y) = -f(y)$, so f is odd.

Also $f(y)(f(y) + f(-y)) = 0$ and $f(-y)(f(y) + f(-y)) = 0$. Again $(f(y) + f(-y))^2 = 0$, yielding f odd.

(S) can be rewritten as (use 2-divisibility of G)

$$(S_1) \qquad f(u)f(v) = f\left(\frac{u+v}{2}\right)^2 - f\left(\frac{u-v}{2}\right)^2 \quad \text{for } u, v \in G.$$

Since $f \neq 0$, there is an a such that $f(a) \neq 0$. Define $h : G \to \mathbb{C}$ by

$$h(x) = \frac{1}{2f(a)}[f(x+a) - f(x-a)] \quad \text{for } x \in G. \tag{3.63}$$

By the repeated use of (S_1), (K), and (3.63), we get

$$\begin{aligned}
[f(x+y) + f(x-y)]f(a) &= f\left(\frac{x+y+a}{2}\right)^2 - f\left(\frac{x+y-a}{2}\right)^2 \\
&\quad + f\left(\frac{x-y+a}{2}\right)^2 - f\left(\frac{x-y-a}{2}\right)^2 \\
&= f\left(\frac{x+y+a}{2}\right)^2 - f\left(\frac{x-(y+a)}{2}\right)^2 \\
&\quad - f\left(\frac{x+y-a}{2}\right)^2 + f\left(\frac{x-(y-a)}{2}\right)^2 \\
&= f(x)f(y+a) - f(x)f(y-a) \\
&= 2f(a)h(y)f(x)
\end{aligned}$$

or

$$f(x+y) + f(x-y) = 2f(x)h(y) \quad \text{for } x, y \in G, \tag{3.27}$$

which is (3.27). From Theorem 3.41, we see that since f satisfies (K) and so does h, h satisfies (C).

If $h \equiv 1$, then f satisfies (J) and there is an additive $A : G \to \mathbb{C}$ such that $f(x) = A(x) + b$ for $x \in G$. Since $f(0) = 0$, $b = 0$ and f is additive.

Let $h \not\equiv 1$. From Theorem 3.41 (note that $h = 0$ implies $f = 0$), it follows that there is a homomorphism $g : G \to \mathbb{C}^*$ satisfying $g(x+y) = g(x)g(y)$ for $x, y \in G$ such that

$$h(x) = \frac{g(x) + g^*(x)}{2} \quad \text{for } x \in G. \tag{3.64}$$

Interchange x and y in (3.27) and add the resultant to (3.27) using f odd and (K) to get

$$f(x+y) = f(x)h(y) + h(x)f(y) \quad \text{for } x, y \in G. \tag{3.65}$$

Using associativity a couple of times and (3.64), we obtain, for $x, y, z \in G$,

$$\begin{aligned}
f(x+y+z) &= f(x+y)h(z) + h(x+y)f(z) \\
&= [f(x)h(y) + f(y)h(x)]h(z) + h(x+y)f(z) \\
&= f(x)h(y+z) + h(x)[f(y)h(z) + f(z)h(y)];
\end{aligned}$$

that is,

$$[h(x+y)-h(x)h(y)]f(z)=[h(y+z)-h(y)h(z)]f(x),$$

meaning

$$\left[\frac{1}{2}(g(x)g(y)+g^*(x)g^*(y))-\frac{1}{4}(g(x)+g^*(x))(g(y)+g^*(y))\right]f(z)$$
$$=\left\{\frac{1}{2}[g(y)g(z)+g^*(y)g^*(z)]-\frac{1}{4}[g(y)+g^*(y)](g(z)+g^*(z))\right\}f(x)$$

or

$$(g(x)-g^*(x))(g(y)-g^*(y))f(z)=(g(y)-g^*(y))(g(z)-g^*(z))f(x).$$

If $g(x)=g^*(x)$ for every x, then $g(x)^2=1$. Suppose there is an x_0 such that $g(x_0)=-1$. Then $-1=g(x_0)=g\left(\frac{x_0}{2}+\frac{x_0}{2}\right)=g\left(\frac{x_0}{2}\right)^2=1$, a contradiction. So $g(x)=1$ for every x, and then $h(x)=1$, which is not the case, and $f\not\equiv 0$. Thus

$$f(x)=k[g(x)-g^*(x)],\quad \text{for } x\in G.$$

From the measurability of f follows that of h in (3.63) and then that of g in (3.64). But then g is continuous (being a measurable homomorphism; see Hewitt and Ross [369]). So, f is continuous. Thus the proof of this theorem is complete. □

We will study (3.65) in a later section.

Result 3.44a. *Suppose $f:\mathbb{C}\to\mathbb{C}$ is a nonzero entire solution of the functional equation* (S). *Then f is of the form*

$$f(z)=cz \quad or \quad f(z)=c\sin bz,\quad for\ z\in\mathbb{C},\ where\ b,c\in\mathbb{C};$$

cf. Remark 3.43a. *The result follows easily from* Theorem 3.44. *Also see H. Haruki* [348].

As a special case of (S) on $\mathbb{C}$, consider

(S2) $$|f(x+iy)f(x-iy)|=|f(x)^2-f(y)^2|\quad \text{for } x,y\in\mathbb{R}.$$

What are its solutions? Are they the same as the solutions above? The answer is the following.

Result 3.44b. (Smajdor and Smajdor [754]). *The only entire solutions of order p, $p<4$ of the equation* (S2) *are the same as the solutions above.*

As in Lemma 3.16, we can show that g is unique in (3.37a) when G is a cyclic group and prove the following.

Theorem 3.44c. (Kurepa [579]). *Let G be a cyclic group. Let f be a complex-valued function on G satisfying* (S) *with the properties that* (1) *f is not identically*

zero and (2) *is of the form* (3.37a)*. Then we assert that there is one and only one homomorphism g of G into K satisfying* (S)*.*

Proof. Suppose that there are two homomorphisms $g_i : G \to \mathbb{C}$ $(i = 1, 2)$ such that

$$f(x) = \frac{g_i(x) - g_i^*(x)}{2} \quad \text{for all } x \in G, \tag{3.37a}$$

$g_i^*(x) = g_i(-x) = \frac{1}{g_i(x)}$.

Now (3.37a) gives that

$$g_1(x) + g_1^*(x) = g_2(x) + g_2^*(x)$$

for all $x \in G$, and equivalently

$$g_1(x) - g_2(x) + g_1^*(x) - g_2^*(x) = 0;$$

hence

$$g_1(x) - g_2(x) + \frac{g_1(x) - g_2(x)}{g_1(x)g_2(x)} = 0,$$

which is the same as

$$[g_1(x) - g_2(x)][g_1(x)g_2(x) + 1] = 0.$$

From the above, we conclude that, for each x, either

$$g_2(x) = g_1(x) \quad \text{or} \quad g_2(x) = -g_1^*(x).$$

Let G be generated by a. Then, we would have either

$$g_2(a) = g_1(a),$$

which in turn implies that g_2 is the same as g_1, since G is a cyclic group and there is nothing to prove, or

$$g_2(a) = -g_1^*(a).$$

Now

$$g_2(a^2) = g_2(a)^2 = (-g_1^*(a))^2 = g_1^*(a^2).$$

But

$$g_2(a^2) = -g_1^*(a^2) \quad \text{or} \quad g_1(a^2).$$

In the former case, $g_1^*(a^2) = 0$, which cannot be. So, $g_1^*(a^2) = g_1(a^2)$; that is,

$$g_1^*(a) = \pm g_1(a).$$

If $g_1^*(a) = g_1(a)$, then $f = 0$, which is not so. So, $g_1^*(a) = -g_1(a)$; that is, $g_2 = g_1$. This proves the theorem. □

Theorem 3.45. (Kurepa [581]). *Let H be a real Hilbert space. If $f : H \to \mathbb{R}$ is a nonzero continuous solution of* (S), *then there exists an $x_0 \in H$ such that*

$$f(x) = \langle x, x_0\rangle, \quad \text{or} \quad b\sin\langle x, x_0\rangle, \quad \text{or} \quad b\sinh\langle x, x_0\rangle,$$

where b is a real constant.

Proof. For a given $x \in H$, $x \neq 0$, and $t \in \mathbb{R}$, define $f_x : \mathbb{R} \to \mathbb{R}$ by $f_x(t) = f(tx)$. Clearly f_x satisfies (S) on $\mathbb{R}$ and is continuous. So Theorem 3.43 implies

$$f_x(t) = a(x)t \quad \text{or} \quad a(x)\sin b(x)t, \quad \text{for } t \in \mathbb{R},$$

where $b(x)$ is real or purely imaginary.

Let us first consider the case $f_x(t) = a(x)t$ for $x \in H$. So,

$$f(tx) = f_x(t) = a(x)t = f(x)t \quad \text{for all } x \in H. \tag{3.66}$$

Let $H_0 = \{x \in H : f(x) = 0\}$. Claim H_0 is a closed linear subspace.

Suppose $x \in H_0$. Equation (3.66) shows that $tx \in H_0$ for $t \in \mathbb{R}$. Let $x, y \in H_0$. Then (S) implies $f(x+y) = 0$ or $f(x-y) = 0$. Suppose $f(x+y) = 0$; that is, $x + y \in H_0$. Replacing x, y in (S) by $x + y$ and $x - y$, respectively, to have $f(x+y)^2 - f(x-y)^2 = f(2x) - f(2y) = 0$, shows that $(x - y) \in H_0$. (Also $tx, ty \in H_0$, $t \in \mathbb{R}$; that is, $-y \in H_0$ and so $x - y \in H_0$.) If $f(x-y) = 0$, then $x - y \in H_0$. Hence H_0 is a linear subspace (the case $f(x-y) = 0$ is similar), and let H_1 be the orthogonal complement of $H_0 = \{x \in H : \langle x, y\rangle = 0 \text{ for all } y \in H_0\}$.

Assert that the dimension of H_1 is one. Suppose e_1, e_2 are two orthonormal elements of H_1. Then $f(e_1) \neq 0$, $f(e_2) \neq 0$. Let $t_0 = -\frac{f(e_1)}{f(e_2)}$. Then $f(e_1) + t_0 f(e_2) = f(e_1) + f(t_0 e_2) = 0$ or $f(e_1)^2 = f(t_0 e_2)^2$. Now, with $x = e_1$, $y = t_0 e_2$, (S) yields $f(e_1 + t_0 e_2) f(e_1 - t_0 e_2) = 0$; that is, $e_1 + t_1 e_2 \in H_0$, where $t_0^2 = t_1^2$. But $e_1 + t_1 e_2 \in H_0$ implies $\langle e_1 + t_1 e_2, e_1\rangle = 0$, which cannot be. So, $\dim H_1 = 1$.

The continuity of f implies that H_0 is a closed linear subspace, so that every $x \in H$ can be written as $x = \langle x, e\rangle e + h_0$, where $h_0 \in H_0$ and $H_1 = \text{span}\{e\}$, with $||e|| = 1$.

Set $y = \langle x, e\rangle e$ in (S) to have $f(x)^2 = f(\langle x, e\rangle e)^2$; that is,

$$f(x) = \pm f(\langle x, e\rangle e) = \pm\langle x, e\rangle f(e) = e(x)g(x),$$

whose $e(x) = \pm 1$, $g(x) = \langle x, f(e)e\rangle$. Obviously, $g(x)$ satisfies (S). With $f(x) = e(x)g(x)$ in (S), we get

$$e(x+y)e(x-y)g(x+y)g(x-y) = g(x+y)g(x-y).$$

This shows that $e(x+y)e(x-y) = 1$; that is, $e(x) = e(y)$ for all $x, y \in H$. Thus $f(x) = \langle x, x_0\rangle$ with $x_0 = f(e)e$ or $x_0 = -f(e)e$.

The case $f(tx) = a(x)\sin b(x)t$ for all $x \in H, t \in \mathbb{R}$ can be argued similarly to produce $f(x) = a\sinh\langle x, x_0\rangle$ with a suitable $x_0 \in H$ (see [409]). See [409] also to prove that a "mixed solution" cannot occur; that is, either $f(tx) = a(x)t$ for all $x \in H$, $t \in \mathbb{R}$ or $f(tx) = ax\sin b(x)t$ for all $x \in H$, $t \in \mathbb{R}$. □

Corollary 3.45a. *Let M_n = set of all matrices of order n over $\mathbb{R}$ endowed with the usual topology of matrices and $f\ (\neq 0) : M_n \to \mathbb{R}$ be a continuous solution of* (S) *with $f(s^{-1}xs) = f(x)$ for $x \in M_n$ and $s \in M_n$ orthogonal. Then f has one of the forms following $f(x) = a \operatorname{tr} x$, $a \sin b(\operatorname{tr} x)$, or $a \sinh b(\operatorname{tr} x)$, where a, b are real and $\operatorname{tr} x$ = trace of the matrix x.*

Proof. If $x = (x_{ij})$ and $y = (y_{ij})$ are two matrices from M_n, and if we set

$$\langle x, y\rangle = \sum_{i,j=1}^{n} x_{ij} y_{ij},$$

then M_n becomes the Hilbert (Euclidean) real space. Since the functional f satisfies all conditions of Theorem 3.45, we find that f has one of the forms

$$f(x) = A(x), \quad f(x) = b \sin A(x), \quad f(x) = b \sinh A(x),$$

where $A(x) = \langle x, x_0\rangle$ and x_0 is some constant vector matrix from M_n. For every real number t and for every orthogonal matrix s, we have $f(s^{-1}txs) = f(tx)$, which implies $A(s^{-1}xs) = A(x)$. Indeed, this is obvious in the first case. In other cases, we have

$$\sin t A(s^{-1}xs) = \sin t A(x), \quad \sinh t A(s^{-1}xs) = \sinh t A(x).$$

If we take the derivative of these equations with respect to t and then set $t = 0$, we get the desired result. Thus the functional $A(x)$ is orthogonally invariant. This continuity of $A(x)$ and $A(x + y) = A(x) + A(y)$ imply $A(x) = a \operatorname{tr} x$ with some constant a ([575, Theorem 7]). Q.E.D.

3.4.5 More on Sine Functions on a Vector Space

Theorem 3.46. (Baker [87]). *Let $\underline{\overline{X}}$ be a real vector space, and suppose $f : \underline{\overline{X}} \to \mathbb{C}$ is a nonzero solution of* (S). *Then f is continuous along rays if and only if either* (i) *f is linear (that is, additive and real homogeneous) or* (ii) *there exists a linear map $A : \underline{\overline{X}} \to \mathbb{C}$ and a complex constant c such that*

$$f(x) = c \sin A(x).$$

Proof. By Theorem 3.44, if f is additive, then we have (i) since, for additive functions, continuity along rays is equivalent to real homogeneity.

Now suppose there exists $F : \underline{\overline{X}} \to \mathbb{C}$ and $k \in \mathbb{C}$ such that $F(x+y) = F(x)F(y)$ and

$$f(x) = k\{F(x) - F(-x)\}$$

for all $x \in \underline{\overline{X}}$. We will show that F is continuous along rays. Clearly we may assume $f \not\equiv 0$.

If $f(a) \neq 0$, then by [87] we have

$$F(ra) = \frac{f((r+1)a) - f((r-1)a)}{2f(a)} + \frac{f(ra)}{2k}$$

for all real r. Since f is continuous along rays, the mapping $r \to F(ra)$ is continuous. If $f(a) = 0$, choose $b \in \underline{X}$ such that $f(b) \neq 0$. Then

$$f(a+b)f(a-b) = f(a)^2 - f(b)^2 = -f(b)^2 \neq 0$$

and hence $f(a-b) \neq 0$. Thus the mappings $r \to F(rb)$ and $r \to F(r(a-b))$ are continuous. But $F(ra) = F(r(a-b))\,F(rb)$ for all real r, and hence $r \to F(ra)$ is a continuous mapping.

Thus F is continuous along rays and so there exists a linear function $A : X \to \mathbb{C}$ and a complex constant k such that

$$f(x) = k\{\exp\{A(x)\} - \exp\{-A(x)\}\}$$

for all $x \in X$. Thus (ii) follows by letting A by $-iA$ and $c = 2ik$. The converse is obvious. □

Corollary 3.46a. [87]. *Suppose $f : \mathbb{R}^n \to \mathbb{C}$ satisfies* (S) *for all $x, y, \in \mathbb{R}^n$. If there exists a measurable subset K of $\mathbb{R}^n$ of positive measure such that the restriction of f to K is measurable, then f is continuous and hence either*

(i) *there exists $\{a_i\}_{i=1}^n \subseteq \mathbb{C}$ such that $f(x) = \sum_{i=1}^n a_i x_i$ for all $x = (x_i) \in \mathbb{R}^n$ or*

(ii) *there exist $c \in \mathbb{C}$ and $\{a_i\}_{i=1}^n \subseteq \mathbb{C}$ such that*

$$f(x) = c \sin\left(\sum_{i=1}^n a_i x_i\right) \quad \text{for } x \in \mathbb{R}^n.$$

Note that the continuity assumption is weakened to measurability, and refer to Theorem 3.45.

There are other functional equations, namely

$$f(x+y) + f(x-y) = 2f(x)\left(1 - 2f\left(\frac{y}{2}\right)^2\right) \quad \text{for } x, y \in \mathbb{R}, \tag{3.67}$$

$$\begin{aligned} f(x-y) = {} & f(x)\left(1 - 2f\left(\frac{y}{2}\right)^2\right) \\ & - f(y)\left(1 - 2f\left(\frac{x}{2}\right)^2\right) \quad \text{for } x, y \in \mathbb{R}, \end{aligned} \tag{3.67a}$$

$$f(y)[f(x+y) + f(x-y)] = f(x)f(2y) \quad \text{for } x, y \in \mathbb{R}, \tag{3.68}$$

which also characterize the sine function [663]. We now investigate the relationship between them and (S). We begin with (3.67) and (3.67a) and prove the following theorem.

Theorem 3.47. (See also Parnami and Vasudeva [663]). *A nonzero function* $f : \mathbb{R} \to \mathbb{C}$ *satisfies* (3.67) *if and only if* f *satisfies* (3.67a). *Furthermore,* f *satisfies* (S) *and*

$$f(x+y) = f(x)\left[1 - 2f\left(\frac{y}{2}\right)^2\right] + f(y)\left[1 - 2f\left(\frac{x}{2}\right)^2\right], \tag{3.67b}$$

and f *is given by*

$$f(x) = c(E(x) - E(-x)) \quad \text{for } x \in \mathbb{R},$$

where E *is exponential* (E). *If* f *is continuous, then* $f(x) = c \sin bx,\ b, c \in \mathbb{C}$.

Proof. $f = 0$ is a solution of (S), (3.67), and (3.67a). We consider only the nonzero solution f.

Suppose f is a solution of (3.67). Set $y = 0$ in (3.67) to get $f(0) = 0$.

Now put $x = 0$ to have $f(y) + f(-y) = 0$; that is, f is odd.

Interchange x and y in (3.67) to obtain

$$f(x+y) - f(x-y) = 2f(y)\left[1 - 2f\left(\frac{x}{2}\right)^2\right], \quad x, y \in \mathbb{R}. \tag{3.69}$$

Adding (3.67) and (3.69), we have (3.67b). Subtracting (3.69) from (3.67), we obtain (3.67a).

Conversely, let f be a solution (nonzero) of (3.67a). With $y = x$, (3.67a) gives $f(0) = 0$. Now $x = 0$ in (3.67a) shows that f is odd. Changing y to $-y$ in (3.67a), we get (3.67b). Add (3.67a) and (3.67b) to have (3.67).

Let f be a solution of (3.67) ((3.67a)). Letting $y = x$ in (3.67), we have $f(2x) = 2f(x)\left(1 - 2f\left(\frac{x}{2}\right)^2\right),\ x \in \mathbb{R}$.

From this and (3.67a) and (3.67b), we get

$$\begin{aligned} f(x+y)^2 - f(x-y)^2 &= 4f(x)\left[1 - 2f\left(\frac{y}{2}\right)^2\right] f(y)\left[1 - 2f\left(\frac{x}{2}\right)^2\right] \\ &= f(2x)f(2y) \quad \text{for } x, y \in \mathbb{R}, \end{aligned}$$

which is (S).

From Theorem 3.44, it follows that

$$f(x) = A(x) \quad \text{or} \quad f(x) = c(E(x) - E(-x)), \quad x \in \mathbb{R},$$

where A is additive and E is exponential.

But $f(x) = A(x)$ in (3.67) gives $A(x)A(y)^2 = 0$; that is, $A = 0$ and so $f = 0$, which cannot be. So, $f(x) = c(E(x) - E(-x))$. If f is continuous, then $f(x) = c \sin bx,\ c, b \in \mathbb{C}$. This proves the theorem.

□

Remark 3.47a. [663]. This theorem shows that the equation (3.67) is preferable to the equation (S) for the characterization of the sine function. Obviously (S) does not imply (3.67) when $f \neq 0$.

Let us turn our attention to (S) and (3.68). It turns out that $H = \{x \in \mathbb{R} : f(x) = 0\}$ plays a vital role in this investigation.

Theorem 3.48. *Suppose* $f : \mathbb{R} \to \mathbb{C}$ *is a solution of* (S). *Then* f *satisfies* (3.68) *and* H *is a subgroup of* $\mathbb{R}$.

Proof. From (S), first replacing x by $\frac{x}{2} + y$, y by $\frac{x}{2}$, and then x by $\frac{x}{2}$ and y by $\frac{x}{2} - y$, we have

$$f(y)f(x+y) = f\left(\frac{x}{2} + y\right)^2 - f\left(\frac{x}{2}\right)^2 \quad \text{for } x, y \in \mathbb{R},$$
$$f(y)f(x-y) = f\left(\frac{x}{2}\right)^2 - f\left(\frac{x}{2} - y\right)^2 \quad \text{for } x, y \in \mathbb{R}.$$

Adding the two equations above and using (S), we have (3.68). It is easy to verify that H is a subgroup.

Remark 3.48a. [663]. The converse is not true; that is, that (3.68) does not imply (S) can be seen by the following example:

$$f(x) = \begin{cases} x, & \text{for } x \in \mathbb{Q}, \\ 0, & \text{for } x \text{ irrational.} \end{cases}$$

f is continuous at 0 and does not satisfy (S), but f is a solution of (3.68).

Indeed, for $x \in \mathbb{Q}$, $y \in$ irrational, $f(x+y)f(x-y) = 0$, but $f(x)^2 - f(y)^2 = x^2$. So, (S) is not satisfied.

As for (3.68), consider the following cases:

(i) If $y \in$ irrational, then both sides of (3.68) are zeros.
(ii) Let $y \in \mathbb{Q}$ for $x \in$ irrational, and $f(y)(f(x+y) + f(x-y)) = 0 = f(x)f(2y)$.
(iii)For $x \in \mathbb{Q}$, $f(y)(f(x+y) + f(x-y)) = y \cdot 2x = f(x)f(2y)$.

In all cases, (3.68) holds.

Now we show that $H = \{x \in \mathbb{R} : f(x) = 0\}$ plays a crucial role in proving the converse.

Theorem 3.48b. (See also [663]). *Suppose* $f : \mathbb{R} \to \mathbb{C}$ *satisfies* (3.68) *and* H *is a subgroup of* $\mathbb{R}$. *Then* f *is a solution of* (S).

Proof. Recall that f is odd. Note that whenever $x, y \in H$, then $2x, x \pm y \in H$. Several cases are to be considered.

Case (i). Let $x, y \in H$. Then $f(x+y)f(x-y) = 0 = f(x)^2 - f(y)^2$.

Case (ii). Let $x \in H$, $y \notin H$. Then $f(x) = 0 = f(2x)$, $f(y) \neq 0$. Equation (3.68) yields $f(y)(f(x+y) + f(x-y)) = 0$; that is, $f(x+y) + f(x-y) = 0$ and $f(x+y)f(x-y) = -f(x+y)^2 = -f(x-y)^2$.

Now $x + y \notin H$; otherwise $y \in H$. So, $\frac{x+y}{2} \notin H$. Similarly, $(x-y) \notin H$, and $\frac{x-y}{2} \notin H$. Further, $\frac{y}{2} \notin H$, and either $\frac{x}{2} \in H$ or $\frac{x}{2} \notin H$.

First, let $\frac{x}{2} \in H$. Now (3.68) gives

$$f\left(\frac{y}{2}\right)\left[f\left(\frac{x}{2}+\frac{y}{2}\right)+f\left(\frac{x}{2}-\frac{y}{2}\right)\right]=f\left(\frac{x}{2}\right)f(y)=0;$$

that is,

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=0.$$

Now let $\frac{x}{2} \notin H$. Now (3.68) gives

$$f\left(\frac{y}{2}\right)\left[f\left(\frac{y}{2}+\frac{x}{2}\right)+f\left(\frac{y}{2}-\frac{x}{2}\right)\right]=f\left(\frac{y}{2}\right)f(x)=0;$$

that is,

$$f\left(\frac{x+y}{2}\right)-f\left(\frac{x-y}{2}\right)=0 \quad \text{(use } f \text{ is odd)}.$$

From (3.68), we get

$$f\left(\frac{x-y}{2}\right)\left[f\left(\frac{x+y}{2}+\frac{x-y}{2}\right)+f\left(\frac{x+y}{2}-\frac{x-y}{2}\right)\right]$$
$$=f\left(\frac{x+y}{2}\right)f(x-y),$$

and since $f\left(\frac{x+y}{2}\right)$ is either $= f\left(\frac{x-y}{2}\right)$ or $= -f\left(\frac{x-y}{2}\right)$, we have, since $f(x) = 0$, either $f(y) = f(x-y)$ or $f(y) = -f(x-y)$.

In either case,

$$f(x+y)f(x-y)=-f(x-y)^2=f(x)^2-f(y)^2.$$

Thus, (S) holds.

Case (iii). Let $x, y \notin H$. Then $\frac{x}{2}, \frac{y}{2} \notin H$. Either $\frac{x+y}{2} \in H$ or $\frac{x+y}{2} \notin H$. Suppose $\frac{x+y}{2} \in H$. Then $(x+y) \in H$ and $\frac{x-y}{2} \notin H$ (otherwise, $x, y \in H$). Now (3.68) gives

$$f\left(\frac{x-y}{2}\right)\left[f\left(\frac{x+y}{2}+\frac{x-y}{2}\right)+f\left(\frac{x+y}{2}-\frac{x-y}{2}\right)\right]$$
$$=f\left(\frac{x+y}{2}\right)f(x-y)=0;$$

that is, $f(x)+f(y)=0$ and $f(x+y)f(x-y)=0=f(x)^2-f(y)^2$. So, (S) holds.

Suppose $\frac{x+y}{2} \notin H$. Then either $\frac{x-y}{2} \in H$ or $\frac{x-y}{2} \notin H$.

First, let $\frac{x-y}{2} \in H$. Then $(x-y) \in H$ and

$$f\left(\frac{x+y}{2}\right)\left[f\left(\frac{x-y}{2}+\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}-\frac{x+y}{2}\right)\right]$$
$$=f\left(\frac{x-y}{2}\right)f(x+y)=0;$$

that is, $f(x) - f(y) = 0$ (use f odd) and

$$f(x+y)f(x-y) = 0 = f(x)^2 - f(y)^2.$$

So, (S) holds.

Finally, let $\frac{x-y}{2} \notin H$. Then

$$f\left(\frac{x+y}{2}\right)\left[f\left(\frac{x-y}{2}+\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}-\frac{x+y}{2}\right)\right] = f\left(\frac{x-y}{2}\right)f(x+y)$$

and

$$f\left(\frac{x-y}{2}\right)\left[f\left(\frac{x+y}{2}+\frac{x-y}{2}\right)+f\left(\frac{x+y}{2}-\frac{x-y}{2}\right)\right] = f\left(\frac{x+y}{2}\right)f(x-y).$$

Multiplying the two equations above and noting that $f\left(\frac{x+y}{2}\right) \neq 0$, $f\left(\frac{x-y}{2}\right) \neq 0$, and f is odd, we get (S).

This proves the theorem. □

As a prelude to finding conditions for solutions of (3.68) to be solutions of (S) also, we have the following proposition.

Proposition 3.48c. [663]. *Let $f : \mathbb{R} \to \mathbb{C}$ be a solution of* (3.68) *but not of* (S). *Then H is a dense subset of $\mathbb{R}$.*

Now, the following result is self-evident.

Result 3.48d. [663]. *Let f be a nonzero continuous mapping from $\mathbb{R}$ into $\mathbb{C}$. Then every solution of* (3.68) *is a solution of* (S) *and vice versa. Moreover, f is of the form cx, $c \sin bx$, for $b, c \in \mathbb{C}$.*

From Theorem 3.47, it follows that every solution of (3.67) is a solution of (S). But the converse does not hold. Whereas a nonzero additive A is a solution of (S), A is not a solution of (3.67). The following theorem gives conditions for a solution of (S) also to be a solution of (3.67).

Theorem 3.49. (See also [663]). *Suppose $f : \mathbb{R} \to \mathbb{C}$ satisfies* (S). *Let $H = \{x \in \mathbb{R} : f(x) = 0\}$. If $H \neq 2H$, then there is a nonzero $c \in \mathbb{C}$ such that $F = cf$ satisfies* (3.67).

Proof. By hypothesis, there is an $\alpha \in H - 2H$. So, $\frac{\alpha}{2} \notin H$; otherwise $\alpha \in 2H$. Let $c = \frac{1}{f(\frac{\alpha}{2})}$ and define $F = cf$. Clearly, F is a solution of (S), $F(\alpha) = 0$, $F\left(\frac{\alpha}{2}\right) = 1$. Use (S) and ($S_1$) to show that F is a solution of (3.67).

Consider

$$\begin{aligned} F(x+y)+F(x-y) &= F(x+y)F\left(\frac{\alpha}{2}\right)+F(x-y)F\left(\frac{\alpha}{2}\right) \\ &= F\left(\frac{x+y}{2}+\frac{\alpha}{4}\right)^2 - F\left(\frac{x+y}{2}-\frac{\alpha}{4}\right)^2 \\ &\quad + F\left(\frac{x-y}{2}+\frac{\alpha}{4}\right)^2 - F\left(\frac{x-y}{2}-\frac{\alpha}{4}\right)^2 \\ &= F(x)F\left(y+\frac{\alpha}{2}\right)+F(x)F\left(-y+\frac{\alpha}{2}\right) \\ &= F(x)\left[F\left(y+\frac{\alpha}{2}\right)-F\left(y-\frac{\alpha}{2}\right)\right]. \end{aligned}$$

Now $y = \frac{\alpha}{2}$ yields $F\left(x+\frac{\alpha}{2}\right) + F\left(x-\frac{\alpha}{2}\right) = F(x)F(\alpha) = 0$, so that $F\left(x+\frac{\alpha}{2}\right) = -F\left(x-\frac{\alpha}{2}\right)$ for all $x \in \mathbb{R}$ and

$$F(x+y)+F(x-y) = 2F(x)F\left(y+\frac{\alpha}{2}\right) \quad \text{for } x, y \in \mathbb{R}. \tag{3.70}$$

But

$$\begin{aligned} 1-2F\left(\frac{y}{2}\right)^2 &= F\left(\frac{\alpha}{2}\right)^2 - F\left(\frac{y}{2}\right)^2 - F\left(\frac{y}{2}\right)^2 \\ &= F\left(\frac{\alpha+y}{2}\right)F\left(\frac{\alpha-y}{2}\right) - F\left(\frac{y}{2}\right)^2 \\ &= F\left(\frac{y}{2}+\frac{\alpha}{2}\right)^2 - F\left(\frac{y}{2}\right)^2 \\ &= F\left(y+\frac{\alpha}{2}\right)\cdot F\left(\frac{\alpha}{2}\right) = F\left(y+\frac{\alpha}{2}\right). \end{aligned} \tag{3.70a}$$

Thus, (3.70) and (3.70a) show that F is a solution of (3.67). Q.E.D.

Now we treat the functional equation

$$g(x-y) = g(x)g(y) + f(x)f(y) \quad \text{for } x, y \in \mathbb{R}, \tag{3.71}$$

where f and g are complex-valued functions defined on reals (cf. (3.5)) that are related to (3.67).

If $g = 0$ in (3.71), then $f = 0$. If $f = 0$, then $g = 0$ or 1. We consider only nonconstant solutions of (3.71).

Theorem 3.50. (See also Vaughan [814], Kurepa [580, 579], Aczél [12], Aczél and Dhombres [44], Parnami and Vasudeva [663], Klee [539]). *Suppose $f, g : \mathbb{R} \to \mathbb{C}$ are nonconstant solutions of* (3.71). *Then f satisfies* (3.67). *Further, $g(x) = 1 - 2f\left(\frac{x}{2}\right)^2$ and f and g also satisfy*

$$f(x+y)+f(x-y) = 2f(x)g(y), \tag{3.27}$$

$$g(x+y) = g(x)g(y) - f(x)f(y), \tag{3.72}$$

$$f(x+y) = f(x)g(y) + f(y)g(x) \quad \textit{for } x, y \in \mathbb{R}, \tag{3.73}$$

$$f(x-y) = f(x)g(y) - g(x)f(y) \quad \textit{for } x, y \in \mathbb{R}, \tag{3.73a}$$

f is a solution of (3.69)*, and g satisfies* (C)*. Moreover, f and g are given by*

$$f(x) = k(E(x) - E^*(x)) \quad \text{for } x \in \mathbb{R}, \tag{3.37a}$$

$$g(x) = \frac{E(x) + E^*(x)}{2} \qquad \text{for } x \in \mathbb{R}, \tag{3.37b}$$

where $E : \mathbb{R} \to \mathbb{C}^*$ is a homomorphism satisfying (E)*, $k^2 = \frac{1}{4}$.*

If one of f, g is continuous, then

$$g(x) = \cos bx, \quad f(x) = d \sin bx, \quad b, c \in \mathbb{C},$$

with $d^2 = 1$.

Proof. Refer also to Theorems 3.17 and 3.41.

Interchange x and y in (3.71) to get $g(x - y) = g(y - x)$; that is, g is even.

Replace y by $-y$ in (3.71) to have

$$g(x + y) = g(x)g(y) + f(x)f(-y) \quad \text{for } x, y \in \mathbb{R}. \tag{3.74}$$

Change x to $-x$ and y to $-y$ in (3.71), and use (3.71) to have

$$f(x)f(y) = f(-x)f(-y)$$

or, since $f \neq 0$,

$$f(x) = kf(-x) = k^2 f(x) \quad \text{and} \quad k^2 = 1.$$

When $k = 1$, f is even. Now (3.74) and (3.71) show that g is constant, which is not the case. So, f is odd. Now (3.74) becomes (3.72).

Adding (3.71) and (3.72), we get that g is cosine (C). $y = 0$ in (3.71) yields $g(0) = 1$ since f is odd.

Now, $y = x$ in (3.71) gives $f(x)^2 + g(x)^2 = 1$, and (3.72) gives

$$g(2x) = g(x)^2 - f(x)^2 = 1 - 2f(x)^2$$

or

$$g(x) = 1 - 2f\left(\frac{x}{2}\right)^2 \quad \text{for } x \in \mathbb{R}.$$

Replace y by $y + z$ in (3.71), and use (3.71) and (3.72) and associativity to have

$$\begin{aligned} g(x - y - z) &= g(x)g(y + z) + f(x)f(y + z), \quad \text{for } x, y, z \in \mathbb{R} \\ &= g(x)[g(y)g(z) - f(y)f(z)] + f(x)f(y + z) \\ &= g(x - y)g(z) + f(x - y)f(z) \\ &= [g(x)g(y) + f(x)f(y)]g(z) + f(x - y)f(z) \end{aligned}$$

or

$$f(x)[f(y+z) - f(y)g(z)] = f(z)[f(x-y) + g(x)f(y)];$$

that is,

$$f(y+z) - f(y)g(z) = f(z)h(y)$$

for some h (obtained by specializing x with $f(x) \neq 0$).

Now, interchange z and y to have (use $f \neq 0$)

$$f(y)g(z) + f(z)h(y) = f(z)g(y) + f(y)h(z)$$

or

$$f(y)(h(z) - g(z)) = f(z)(h(y) - g(y))$$

or

$$h(y) = g(y) + cf(y), \quad \text{for } y \in \mathbb{R},$$

and

$$f(y+z) = f(y)g(z) + f(z)(g(y) + cf(y)).$$

Change y to $-y$ and z to $-z$, and use g even and f odd to get

$$-f(y+z) = -f(y)g(z) - f(z)g(y) + cf(y)f(z),$$

which by addition yields $c = 0$, and so (3.73), (3.73a), (3.27), (3.67), and (3.69) hold.

Since g satisfies (C) by Theorem 3.21, we get (3.37b) for some homomorphism $E : \mathbb{R} \to \mathbb{C}^*$. Using this g in (3.71), we have

$$\frac{E(x-y) + E^*(x-y)}{2} = \frac{E(x) + E^*(x)}{2} \cdot \frac{E(y) + E^*(y)}{2} + f(x)f(y);$$

that is,

$$f(x)f(y) = \frac{E(x) - E^*(x)}{2} \cdot \frac{E(y) - E^*(y)}{2},$$

which yields (3.37a).

It is now easy to obtain the continuous solution. This proves the theorem. □

Now we treat a sort of converse.

Theorem 3.51a. [663]. *Suppose a nonzero* $f : \mathbb{R} \to \mathbb{R}$ *satisfies* (3.69) *or* (3.67). *Let* $g(x) = 1 - 2f\left(\frac{x}{2}\right)^2$. *Then* g *and* f *satisfy* (3.71).

Proof. We will first treat the case where f satisfies (3.69).

Set $y = 0$ in (3.69) to have $f(0)\left(1 - 2f\left(\frac{x}{2}\right)^2\right) = 0$. If $1 - 2f\left(\frac{x}{2}\right)^2 = 0$, then from (3.69) it follows that $f =$ constant, which is not the case. So, $f(0) = 0$.

Then $x = 0$ in (3.69) shows that f is odd, and $g(x) = 1 - 2f\left(\frac{x}{2}\right)^2$ gives that g is even and $g(0) = 1$.

In (3.69), first replace x by $x+z$ and then replace y by $y+z$ to have

$$f(x+y+z) - f(x-y+z) = 2f(y)g(x+z),$$
$$f(x+y+z) - f(x-y-z) = 2f(y+z)g(x),$$

which by subtraction yields

$$\begin{aligned} f(x-y+z) - f(x-y-z) &= 2f(y+z)g(x) - 2f(y)g(x+z) \\ &= 2f(z)g(x-y) \quad \text{(using (3.69))}. \end{aligned}$$

Thus,

$$f(y)g(x+z) = g(x)f(y+z) - f(z)g(x-y) \quad \text{for } z, x, y \in \mathbb{R}. \tag{3.75}$$

Putting $x = 0$ in (3.75), we have

$$f(y+z) = f(z)g(y) + f(y)g(z), \tag{3.73}$$

which is (3.73). Then we obtain (3.73a). From (3.75) and (3.73), we obtain

$$f(y)[g(x+z) - g(x)g(z)] = -f(z)[g(x-y) - g(x)g(y)].$$

Specializing in z, we get

$$g(x-y) = g(x)g(y) + f(y)k(x),\ x, y \in \mathbb{R}, \tag{3.71a}$$

for some function k.

Interchanging x and y in (3.71a) yields $f(y)k(x) = f(x)k(y)$ or $k(x) = cf(x)$, and (3.71a) becomes

$$g(x-y) = g(x)g(y) + cf(x)f(y). \tag{3.71b}$$

First put $y = x$ in (3.71b) and then $y = -x$ in (3.71b) to get $1 = g(x)^2 + cf(x)^2$ and $g(2x) = g(x)^2 - cf(x)^2$. But $g(2x) = 1 - 2f(x)^2 = (1 - cf(x)^2) - cf(x)^2$. So, $c = 1$ and (3.71b) becomes (3.71).

Suppose f satisfies (3.67). From Theorem 3.47, it follows that f is a solution of (3.71).

This proves the theorem. □

Theorem 3.51b. (See also [663].) *Let $g : \mathbb{R} \to \mathbb{R}$ be a nonzero solution of* (C). *Then there is an $f : \mathbb{R} \to \mathbb{R}$ such that f and g satisfy* (3.71).

Proof. From Theorem 3.21, we see that

$$g(x) = \frac{E(x) + E^*(x)}{2} \quad \text{for } x \in \mathbb{R},$$

where E is a homomorphism from $\mathbb{R}$ to $\mathbb{R}$ satisfying (E).

Define $f(x) = \frac{E(x)+E^*(x)}{2}$ for $x \in \mathbb{R}$. Then

$$\begin{aligned} g(x-y) - g(x)g(y) &= \frac{E(x-y)+E^*(x-y)}{2} - \frac{E(x)+E^*(x)}{2} \cdot \frac{E(y)+E^*(y)}{2} \\ &= \frac{1}{4}[E(x)E^*(y) + E^*(x)E(y) - E(x)E(y) - E(x)E^*(y)] \\ &= \frac{E(x)-E^*(x)}{2} \cdot \frac{E(y)-E^*(y)}{2} \\ &= f(x)f(y). \end{aligned}$$

This completes the proof of this theorem. □

Now, we treat the functional equation connected with the sine functional equation (S) and prove the following theorem.

Theorem 3.52. (See also Corovei [172].) *Let f and g be nonzero real-valued functions of real variables satisfying the equation*

$$f(x)g(y-x) = f\left(\frac{y}{2}\right) g\left(\frac{y}{2}\right) - f\left(x - \frac{y}{2}\right) g\left(x - \frac{y}{2}\right), \quad x, y \in \mathbb{R}. \tag{3.76}$$

Then g is a solution of (S)*, and f and g are given by*

$$g(x) = A(x), \qquad f(x) = c + dA(x), \tag{3.77a}$$

$$g(x) = b(E(x) - E(-x)), \quad f(x) = c[E(x) - E(-x)] + dE(-x), \tag{3.77b}$$

for $x \in \mathbb{R}$, where A satisfies (A)*, E satisfies* (E)*, and b, c, d are constants.*

Proof. Replace x by $x + y$ and y by $2x$ in (3.76) to have

$$f(x+y)g(x-y) = f(x)g(x) - f(y)g(y) \quad \text{for } x, y \in \mathbb{R}. \tag{3.78}$$

Interchanging x and y in (3.78), we have

$$f(x+y)g(y-x) = -f(x+y)g(x-y) \quad \text{or} \quad f(u)g(v) = -f(u)g(-v),$$

and since $f \neq 0$, g is odd.

First set $y = -x$ in (3.78) to have

$$f(0)g(2x) = g(x)(f(x) + f(-x)) \quad \text{for } x \in \mathbb{R}. \tag{3.78a}$$

Now, change y to $-y$ in (3.78) to have

$$f(x-y)g(x+y) = f(x)g(x) + f(-y)g(y), \quad x, y \in \mathbb{R}. \tag{3.78b}$$

Subtract (3.78) from (3.78b) and use (3.78a) to have

$$f(x-y)g(x+y) - f(x+y)g(x-y) = g(y)(f(-y) + f(y));$$

that is,

$$f(v)g(u) - f(u)g(v) = f(0)g(u-v) \quad \text{for } u, v \in \mathbb{R}. \tag{3.79}$$

Case I. Suppose $f(0) = 0$.

Then $f(v)g(u) = f(u)g(v)$ or $f(u) = cg(u)$ for some constant c since $g \neq 0$. This f in (3.78) shows that g is a solution of (S).

Case II. Suppose $f(0) \neq 0$.

From (3.78a), (3.78) and (3.78b),

$$
\begin{aligned}
&f(0)^2 g(x+y)g(x-y)\\
&= g\left(\frac{x+y}{2}\right)\left[f\left(\frac{x+y}{2}\right) + f\left(-\frac{x+y}{2}\right)\right]\\
&\quad \cdot g\left(\frac{x-y}{2}\right)\left[f\left(\frac{x-y}{2}\right) + f\left(-\frac{x-y}{2}\right)\right]\\
&= \left[f\left(\frac{x+y}{2}\right)g\left(\frac{x-y}{2}\right) + f\left(-\frac{x+y}{2}\right)g\left(\frac{x-y}{2}\right)\right]\\
&\quad \cdot \left[f\left(\frac{x-y}{2}\right)g\left(\frac{x+y}{2}\right) + f\left(-\frac{x-y}{2}\right)g\left(\frac{x+y}{2}\right)\right]\\
&= \left[f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) - f\left(\frac{y}{2}\right)g\left(\frac{y}{2}\right) + f\left(-\frac{y}{2}\right)g\left(-\frac{y}{2}\right) - f\left(-\frac{x}{2}\right)g\left(-\frac{x}{2}\right)\right]\\
&\quad \cdot \left[f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) - f\left(-\frac{y}{2}\right)g\left(-\frac{y}{2}\right) + f\left(\frac{y}{2}\right)g\left(\frac{y}{2}\right) + f\left(-\frac{x}{2}\right)g\left(\frac{x}{2}\right)\right]\\
&= -f\left(\frac{y}{2}\right)g\left(\frac{y}{2}\right)^2\left[f\left(\frac{y}{2}\right) + f\left(-\frac{y}{2}\right)\right] - f\left(-\frac{y}{2}\right)g\left(\frac{y}{2}\right)^2\left[f\left(\frac{y}{2}\right) + f\left(-\frac{y}{2}\right)\right]\\
&\quad + f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right)^2\left[f\left(\frac{x}{2}\right) + f\left(-\frac{x}{2}\right)\right] + f\left(-\frac{x}{2}\right)g\left(\frac{x}{2}\right)^2\left[f\left(\frac{x}{2}\right) + f\left(-\frac{x}{2}\right)\right]\\
&= -f(0)g(y)g\left(\frac{y}{2}\right)\left[f\left(\frac{y}{2}\right) + f\left(-\frac{y}{2}\right)\right] + f(0)g(x)g\left(\frac{x}{2}\right)\left[f\left(\frac{x}{2}\right) + f\left(-\frac{x}{2}\right)\right]\\
&= -f(0)^2 g(y)^2 + f(0)^2 g(x)^2;
\end{aligned}
$$

thus g satisfies (S).

Since g is a solution of (S), by Theorem 3.44, $g(x) = A(x)$ or $g(x) = b[E(x) - E(-x)]$ for $x \in \mathbb{R}$, where A satisfies (A), E satisfies (E), and b is a constant.

When $g(x) = A(x)$, (3.79) yields

$$f(0)[A(u) - A(v)] = A(u)f(v) - A(v)f(u)$$

or

$$A(u)(f(v) - f(0)) = A(v)(f(u) - f(0));$$

that is,

$$f(u) = c + dA(u), \quad \text{for } u \in \mathbb{R}, \text{ since } A \neq 0.$$

Corresponding to $g(x) = b(E(x) - E(-x))$, (3.79) gives

$$f(0)\left[\frac{E(u)}{E(v)} - \frac{E(v)}{E(u)}\right] = f(v)[E(u) - E(-u)] - f(u)[E(v) - E(-v)];$$

that is,

$$f(0)[E(2u) - E(2v)] = f(v)[E(2u) - 1]E(v) - f(u)[E(2v) - 1]E(u)$$

or

$$[E(2u) - 1][f(v)E(v) - f(0)] = (E(2v) - 1)[f(u)E(u) - f(0)]$$

or

$$f(v)E(v) - f(0) = c(E(2v) - 1),$$

meaning

$$f(v) = c[E(v) - E(-v)] + dE(-v) \quad \text{for } v \in \mathbb{R}.$$

This completes the proof of this theorem. □

Corollary 3.52a. [172]. *Nonzero continuous solutions are given by*

$$\begin{aligned} &g(x) = ax, && f(x) = bx + c;\\ &f(x) = c\sin ax + d\cos ax, && g(x) = b\sin ax;\\ &g(x) = b\sinh ax, && f(x) = c\sinh ax + d\cosh ax, \quad x \in \mathbb{R}. \end{aligned}$$

3.4.6 Sine Solution

If one is interested in obtaining only the sine solution or in eliminating the linear solution cx and hyperbolic solution $\sinh cx$, one can consider either (Van Vleck [813])

$$f(x - y + d) - f(x + y + d) = 2f(x)f(y), \quad x, y \in \mathbb{R}, \tag{3.80}$$

or (Kannappan [428])

$$h(x + y + d)h(x - y + d) = h(x)^2 - h(y)^2, \quad x, y \in \mathbb{R}. \tag{3.81}$$

First we treat (3.80) in Theorem 3.53 and then consider (3.81) in Theorem 3.53a.

Theorem 3.53. (See also [813].) *Suppose $f(\neq 0) : \mathbb{R} \to \mathbb{C}$ satisfies* (3.80)*, where d is a nonzero real constant. Then f is a periodic function with period $4d$, $\varphi(x) = f(x + d)$ is a solution of* (C)*, and $f(x) = \frac{E(x)-E(-x)}{2E(d)}$, where E satisfies* (E) *with $E(d)^2 = -1$. Moreover, if f is a nonzero continuous solution of* (3.80)*, then f is given by $f(x) = (-1)^n \sin\frac{(2n+1)\pi x}{2d}$ for $x \in \mathbb{R}$, $n \in \mathbb{Z}$.*

Proof. Let $y = 0$ in (3.80) to have $f(0) = 0$. Replacing y with $-y$ in (3.80) shows that f is odd (that is, $f(-y) = -f(y)$) for $y \in \mathbb{R}$.

Put $x = -d$ in (3.80) to get $f(-d) = -1 = -f(d)$. Set $y = d$ in (3.80) to have $f(x) = -f(x + 2d) = f(x + 4d)$; that is, f has period $4d$ and $f(2d) = 0$. Replacing x by $x + d$ and y by $y + d$ in (3.80), we get

$$f(x - y + d) + f(x + y + d) = 2f(x + d)f(y + d);$$

that is, $\varphi(x) = f(x+d)$ satisfies (C) and has period $4d$. Since $f(2d) = -f(0) = 0$, $\varphi(d) = 0$. From Theorem 3.21, we see that

$$\varphi(x) = \frac{E(x) + E(-x)}{2} \quad \text{with } E(d)^2 = -1,$$

where E is a solution of (E). Hence

$$\begin{aligned} f(x) = \varphi(x-d) &= \frac{E(x)E(-d) + E(-x)E(d)}{2} \\ &= \frac{E(x) - E(-x)}{2E(d)} \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

Suppose f is continuous. Then φ is continuous and $\varphi(x) = \cos cx$, for some constant c, with period $4d$; that is, $\cos cx = \cos c(x+4d)$ and $4cd = 2n\pi$ or $c = \frac{n\pi}{2d}$. Hence

$$\begin{aligned} f(x) &= \cos c(x-d) = \cos \frac{n\pi}{2d}(x-d) \\ &= \cos\left(\frac{n\pi x}{2d} - \frac{n\pi}{2}\right). \end{aligned}$$

Since $f(2d) = 0$, $\cos \frac{n\pi}{2} = 0$ or n is odd. So,

$$\begin{aligned} f(x) &= \cos\left(\frac{(2n+1)\pi x}{2d} - \frac{(2n+1)\pi}{2}\right) \\ &= \sin\left[\frac{(2n+1)\pi x}{2d} - n\pi\right] \\ &= \cos n\pi \, \sin\left(\frac{(2n+1)\pi x}{2d}\right). \end{aligned}$$

This completes the proof of this theorem. □

Now we take up (3.81).

Theorem 3.53a. [428]. *Let h be a nonzero complex-valued function of a real variable satisfying* (3.81) *for $x, y \in \mathbb{R}$, where $d(\neq 0)$ is a real constant. Then h is periodic with period $2d$, $g(x) = h(x+d)$ is a solution of* (S) *with period $2d$ and only this, and $h(x) = \frac{b}{E(d)}(E(x) - E(-x))$, where E is a solution of* (E) *with $E(d)^2 = 1$. Further, if h is continuous, then $h(x) = b\cos(n\pi)\sin\frac{n\pi x}{d}$ for $x \in \mathbb{R}$.*

Proof. Let $x = 0$, $y = 0$ in (3.81) to get $h(d) = 0$. Replace y by $-y$ in (3.81) and use (3.81) to have $h(y)^2 = h(-y)^2$ for all $y \in \mathbb{R}$. So, $h(-d) = 0$. $x = -d$ in (3.81) gives $h(y)h(-y) = -h(y)^2$.

So, if $h(y) \neq 0$, then $h(-y) = -h(y)$. But if $h(y) = 0$, since $h(y)^2 = h(-y)^2$, $h(-y) = 0$; that is, h is an odd function.

Letting $y = d$ in (3.81) and using $f(d) = 0$, we get

$$h(x+2d)h(x) = h(x)^2.$$

Hence $h(x+2d) = h(x)$ whenever $h(x) \neq 0$.

Suppose $h(x_0) = 0$. Replace x by x_0 and y by $x_0 + 2d$ in (3.81) to get $h(x_0 + 2d) = 0$. Thus

$$h(x + 2d) = h(x) \text{ for all } x \in \mathbb{R};$$

that is, h is periodic with period $2d$, $h(2d) = h(0) = 0$.

Now change x to $x + d$ and y to $y + d$ in (3.81) to have

$$h(x + y + d)h(x - y + d) = h(x + d)^2 - h(y + d)^2;$$

that is, g satisfies (S), where $g(x) = h(x + d)$ and g has period $2d$. This eliminates the additive solution, and from Theorem 3.44, we see that $g(x) = b(E(x) - E(-x))$, where E is a solution of (E). Since $g(d) = h(2d) = 0$, we have $E(d) - E(-d) = 0$ or $E(d)^2 = 1$. Then $h(x) = g(x - d) = b\frac{E(x)-E(-x)}{E(d)}$ for $x \in \mathbb{R}$.

As for the continuous solution,

$$g(x) = b \sin cx = b \sin c(x + 2d),$$

so that $cd = n\pi$ and

$$\begin{aligned} h(x) = g(x - d) &= b \sin \frac{n\pi}{d}(x - d) \\ &= b \cos n\pi \sin \frac{n\pi x}{d} \quad \text{for } x \in \mathbb{R},\ n \in \mathbb{Z}. \end{aligned}$$

□

3.4.7 Characterization of the Sine

As in the case of the cosine, the sine function can also be characterized with the help of a functional equation in a single variable. It is natural to expect to assume some additional conditions to achieve this.

Theorem 3.54. (Dubikajtis [231] (cf. Theorem 3.37)). *$\varphi(x) = \sin x$ is the only solution satisfying the following conditions: $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies*

(i) *φ is a solution of the equation*

$$2\varphi(x)^2 + \varphi\left(\frac{\pi}{2} - 2x\right) = 1 \quad \text{for } x \in \mathbb{R}, \tag{3.82}$$

(ii) *φ is odd (i.e., $\varphi(-x) = -\varphi(x)$) for $x \in \mathbb{R}$,*
(iii) *φ is positive for $x \in]0, \frac{\pi}{2}[$, and*
(iv) *φ is continuous in a neigbourhood of 0.*

Proof. Claim

(v) φ is periodic with period 2π.

Replace x by $-x$ in (3.82) and use (ii) to imply

$$\varphi\left(\frac{\pi}{2}-2x\right)=\varphi\left(\frac{\pi}{2}+2x\right);$$

that is,

$$\varphi\left(\frac{\pi}{2}-x\right)=\varphi\left(\frac{\pi}{2}+x\right).$$

Change x to $x-\frac{\pi}{2}$ to get

$$\varphi(x)=\varphi(\pi-x),$$

which by replacing x by $x+\pi$ yields

$$\varphi(x)=-\varphi(-x)=-\varphi(\pi+x)=\varphi(2\pi+x).$$

So, φ has period 2π.

Some remarks are in order.

Remark 1. Condition (ii) cannot be replaced by (v), as can be seen by the example $\varphi(x)=\frac{1}{2}$.

Remark 2. Condition (iii) is necessary, as can be seen by the example $\varphi(x)=(-1)^n\sin[(2n+1)x]$, where $n\in\mathbb{Z}_+$. But condition (iii) can be replaced by if $0\le x\le\frac{\pi}{2}$, $0<\varphi(x)<\frac{1}{\sqrt{2}}$.

Remark 3. Condition (iv) can be replaced by the continuity of φ in $]0,\epsilon[$ (use condition (ii)).

To continue the proof, the sign of $\varphi(x)$ is uniquely determined by (i), (ii), (iii), and the periodicity of φ.

Indeed, (ii) shows $\varphi(0)=0$ and (i) yields $\varphi\left(\frac{\pi}{2}\right)=1$. As x varies in $]0,\frac{\pi}{2}[$, $\pi-x$ varies in $]\frac{\pi}{2},\pi[$. Then $\varphi(x)=\varphi(\pi-x)$ shows that $\varphi(x)\ge 0$ in $]0,\pi[$ and (ii) implies $\varphi(x)\le 0$ in $]-\pi,0[$. Now, the periodicity implies that φ has a unique sign at every point on $\mathbb{R}$.

Claim now that φ is determined uniquely at all points of the form $x=\frac{k\pi}{2^{n-1}}$ for $k\in\mathbb{Z}$ and $n\in\mathbb{Z}_+$.

The proof is by induction on n. For $n=0$, consider $\varphi\left(\frac{k\pi}{2^{-1}}\right)=\varphi(2k\pi)=\varphi(0)=0$.

Suppose the value of $\varphi\left(\frac{k\pi}{2^{n-1}}\right)$ is determined for all k. From (3.82) there follows

$$\varphi\left(\frac{k\pi}{2^n}\right)=\pm\sqrt{\frac{1}{2}\left\{1-\varphi\left(\frac{2^{n-2}-k\pi}{2^{n-1}}\pi\right)\right\}},$$

and the sign is determined uniquely. The proof of continuity of f on $\mathbb{R}$ from (iv) is similar to the proof in Theorem 3.38. Thus, there is only one f satisfying (i) to (iv), and $\varphi(x)=\sin x$ satisfies all these conditions. □

3.4.8 General Trigonometric Functional Equations—The Addition and Subtraction Formulas

From (3.1) and (3.5) arise the functional equations

$$f(x+y) = f(x)g(y) + f(y)g(x), \tag{3.73}$$

$$f(x-y) = f(x)g(y) - f(y)g(x), \tag{3.73a}$$

$$g(x+y) = g(x)g(y) - f(x)f(y), \tag{3.72}$$

$$g(x-y) = g(x)g(y) + f(x)f(y), \quad \text{for } x, y \in \mathbb{G}, \tag{3.71}$$

where $f, g : G \to F$, G is a group, and F is a field.

Note that $f(x) = \sin x$ and $g(x) = \cos x$ satisfy all these equations on $\mathbb{R}$. These functional equations have been investigated together and individually by several authors (see, among others, [12, 33, 1, 40, 426, 162, 580, 488, 816, 308, 306, 733, 729, 282, 156, 830, 831]). It was observed that (3.71) alone is enough to characterize both sine and cosine, which none of the remaining equations are able to do.

Functional equations connected to hyperbolic functions include

$$g(x+y) = g(x)g(y) + f(x)f(y), \tag{3.72a}$$

$$g(x-y) = g(x)g(y) - f(x)f(y), \quad \text{for } x, y \in \mathbb{R}, \tag{3.71a}$$

where $f, g : \mathbb{R} \to \mathbb{C}$. Obviously, hyperbolic functions $g(x) = \cosh x$, $f(x) = \sinh x$ are solutions of (3.73a) and (3.71a) and (3.73) and (3.72a). It can be shown [816, 830, 831] that the general solutions of (3.71a) and (3.73a) and (3.72) and (3.72a), respectively, are given by

$$g(x) = \frac{E_1(x) + E_1(-x)}{2}, \qquad f(x) = \frac{E_1(x) - E_1(-x)}{2},$$

and

$$g(x) = \frac{E_1(x) + E_2(-x)}{2}, \qquad f(x) = \frac{E_1(x) - E_2(-x)}{2},$$

where E_i $(i = 1, 2)$ are exponential functions satisfying (E).

First we will consider the functional equation

$$g(x-y) = g(x)g(y) + f(x)f(y) \quad \text{for } x, y \in \mathbb{G}, \tag{3.71}$$

where $f, g : G \to \mathbb{C}$ and G is a group.

Proof. Suppose $g = 0$. Then $f = 0$.

Suppose $f = 0$. Then $g(x-y) = g(x)g(y) = g(y-x)$. That is, g is even and $g(x+y) = g(x)g(y)$, so that $g(x+y) = g(x-y)$, meaning $g(x) =$ constant, say c. So, $c = c^2$ or $c = 0$ or $c = 1$. If $f(x) =$ constant, then so is g. We will consider only nontrivial (nonconstant) solutions.

Interchanging of x and y in (3.71) yields

$$g(x-y) = g(y-z);$$

that is, g is even. Now we will show that f is odd.

Replace y by $-y$ in (3.71) to get

$$g(x+y) = g(x)g(y) + f(x)f(-y) \quad \text{for } x, y \in G. \tag{3.82}$$

Change x to $-x$ in (3.71) to have

$$g(x+y) = g(x)g(y) + f(-x)f(y) \quad \text{for } x, y \in G. \tag{3.82a}$$

Thus

$$f(x)f(-y) = f(-x)f(y);$$

that is, $f(x)f(y) = f(-x)f(-y)$ and $f(x)^2 = f(-x)^2$. Hence, $f(-x) = \pm f(x)$ for all $x \in G$.

If $f(x_0) = 0$, then $f(-x_0) = 0 = -f(x_0)$. Suppose $f(x_0) \neq 0$ and $f(-x_0) = f(x_0)$. Then $f(x)f(x_0) = f(-x)f(-x_0)$, implying $f(x) = f(-x)$, for all x, since $f \neq 0$. That is, f is even and (3.71) and (3.72) show that g is a constant, which is not the case.

Hence f is odd and (3.82) becomes

$$g(x+y) = g(x)g(y) - f(x)f(y) \quad \text{for } x, y \in G, \tag{3.72}$$

and g satisfies (C) (add (3.71) and (3.72)).

Further, applying associativity and (3.72), we get

$$\begin{aligned} g(x+y+z) &= [g(x)g(y) - f(x)f(y)]g(z) - f(x+y)f(z) \\ &= g(x)[g(y)g(z) - f(y)f(z)] - f(x)f(y+z); \end{aligned}$$

that is,

$$[f(x+y) - f(y)g(x)]f(z) = [f(y+z) - f(y)g(z)]f(x)$$

or

$$f(x+y) - f(y)g(x) = h(y)f(x), \tag{3.83}$$

where

$$h(y) = \frac{1}{f(z_0)}[f(y+z_0) - f(y)g(z_0)]$$

with $f(z_0) \neq 0$.

First, setting $x = -y$ in (3.83), we have $f(y)(h(y) - g(y)) = 0$ for all y, and then interchange x and y in (3.83) to get $f(x)(h(y) - g(y)) = f(y)[h(x) - g(x)]$. Then $0 = f(x)f(y)(h(y) - g(y)) = f(y)^2(h(x) - g(x))$; that is, $h(x) = g(x)$, since $f \neq 0$. Hence

$$f(x+y) = f(x)g(y) + f(y)g(x) \quad \text{for } x, y \in G. \tag{3.73}$$

Replace y by $-y$ in (3.73) to obtain

$$f(x-y) = f(x)g(y) - f(y)g(x) \quad \text{for } x, y \in G. \tag{3.73a}$$

Suppose g satisfies (K). Then, by Theorem 3.21,

$$g(x) = \frac{E(x) + E^*(x)}{2} \quad \text{for } x \in G, \tag{3.84}$$

where $E : G \to \mathbb{C}^*$ is a homomorphism, satisfying (E). This value of g in (3.71) results in

$$f(x)f(y) = \frac{E(x) - E^*(x)}{2} \cdot \frac{E(y) - E^*(y)}{2} \quad \text{for } x, y \in G,$$

and since $f \neq 0$,

$$f(x) = k(E(x) - E^*(x)) \quad \text{for } x \in G, \tag{3.85}$$

with $k^2 = \frac{1}{4}$. Thus we have proved the following theorem.

Theorem 3.55. *Suppose $f, g : G \to \mathbb{C}$ satisfy* (3.71) *with g satisfying* (K). *Nonconstant f, g satisfying* (3.71) *are given by* (3.84) *and* (3.85). *Further, f and g satisfy* (3.71), (3.72), (3.73), *and* (3.73a).

Corollary 3.55a. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be a nonconstant solution of* (3.71) *with g continuous. Then f is also continuous and $g(x) = \cos bx$ and $f(x) = k \sin bx$ with $k^2 = \frac{1}{4}$, where b is a complex constant.*

A Generalization

Now we consider the functional equation (treated in Chung, Kannappan, and Ng [162])

$$f(x+y) = f(x)g(y) + g(x)f(y) + h(x)h(y) \quad \text{for } x, y \in G, \tag{3.86}$$

where $f, g, h : G \to \mathbb{C}$ and G is a group, which is a generalization of many trigonometric identities, for example (3.73), (3.72), and (3.72a). The familiar $\cos(x+y)$ and $\sin(x+y)$ admit such expansions as in (3.86).

Result 3.56. [162]. *Let $f \neq 0$ and $g, h : G \to \mathbb{C}$ be mappings. If they satisfy the equation* (3.86), *then they have one of the forms*

$$f = \sum_{i=1}^{3} a_i E_i, \qquad g = \sum_{i=1}^{3} b_i E_i, \qquad h = \sum_{i=1}^{3} c_i E_i, \tag{3.86a}$$

with

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix};$$

or

$$
\begin{aligned}
f &= a_1E_1 + a_2E + a_3EA,\\
g &= b_1E_1 + b_2E + b_3EA,\\
h &= c_1E_1 + c_2E + c_3EA,
\end{aligned}
\tag{3.86b}
$$

with

$$
\begin{bmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3 \end{bmatrix}
\begin{bmatrix} b_1 & b_2 & b_3\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3 \end{bmatrix}
=
\begin{bmatrix} a_1 & 0 & 0\\ 0 & a_2 & a_3\\ 0 & a_3 & 0 \end{bmatrix};
$$

or

$$
\begin{aligned}
f &= a_1EA_1 + a_2E + a_3EA + a_4EA^2,\\
g &= b_1EA_1 + b_2E + b_3EA + b_4EA^2,\\
h &= c_1EA_1 + c_2E + c_3EA + c_4EA^2,
\end{aligned}
\tag{3.86c}
$$

with

$$
\begin{bmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\\ a_4 & b_4 & c_4 \end{bmatrix}
\begin{bmatrix} b_1 & b_2 & b_3 & b_4\\ a_1 & a_2 & a_3 & a_4\\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}
=
\begin{bmatrix} 0 & a_1 & 0 & 0\\ a_1 & a_2 & a_3 & a_4\\ 0 & a_3 & 2a_4 & 0\\ 0 & a_4 & 0 & 0 \end{bmatrix},
$$

where E, E_i $(i = 1, 2, 3)$ *are exponentials satisfying* (E)*, and* A, A_1 *satisfy* (A)*.*

The converse holds.

As special cases of (3.86), we can obtain the general solution of the functional equations (3.73) and (3.72a) on G. The equation (3.73) was solved by Abel [1] using differentiation. Hilbert's fifth problem raised the question of solving Abel's equation under weaker assumptions (Abel's equations will be dealt with in Chapter 10) (see [33]). In [162], the general solution of (3.73) was determined.

Corollary 3.56a. [162]. *Let* f $(\neq 0)$ *and* $g : G \to \mathbb{C}$ *be mappings. If they satisfy the functional equation*

$$
f(x + y) = f(x)g(y) + g(x)g(y), \tag{3.73}
$$

then they are of the form

$$
\begin{aligned}
f &= a_1E_1 + a_2E + a_3EA,\\
g &= b_1E_1 + b_2E + b_3EA,
\end{aligned}
$$

with the constraint

$$
\begin{bmatrix} a_1 & b_1\\ a_2 & b_2\\ a_3 & b_3 \end{bmatrix}
\begin{bmatrix} b_1 & b_2 & b_3\\ a_1 & a_2 & a_3 \end{bmatrix}
=
\begin{bmatrix} a_1 & 0 & 0\\ 0 & a_2 & a_3\\ 0 & a_3 & 0 \end{bmatrix}.
$$

Conversely, f and g given above with the constraints above satisfy (3.73) (*cf. Remark* 3.56b).

The proof follows from Result 3.56.

Remark 3.56b. The general solutions of (3.73) are given by

$$f = a_1E_1 - a_1E, \quad g = \frac{1}{2}E_1 + \frac{1}{2}E; \quad f = EA,\ g - E. \tag{3.87}$$

In Corollary 3.56a, the functional equation is of interest only when $f \neq 0$; i.e., only the case $(a_1, a_2, a_3) \neq 0$ is of interest. By the rank consideration in the constraints, we get

$$\operatorname{rank} \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & a_3 \\ 0 & a_3 & 0 \end{bmatrix} \leq 2.$$

This implies $a_3 = 0$ or $a_1 = 0$. Having assumed $(a_1, a_2, a_3) \neq 0$, this implies $b_3 = 0$ or $b_1 = 0$, respectively. Thus f $(\neq 0)$ and g are either in $\operatorname{span}\{E_1, E\}$ or in $\operatorname{span}\{E, EA\}$.

(1) Suppose $a_3 = 0$. Then $b_3 = 0$. Since $a_i = 0$ implies $b_i = 0$ for $i = 1, 2$, we may assume $a_1, a_2 \neq 0$ by adjusting E_1, E. Then the form of f and g can be replaced by

$$f = a_1E_1 - a_1E \quad \text{and} \quad g = \frac{1}{2}E_1 + \frac{1}{2}E.$$

(2) Suppose $a_3 \neq 0$. Then $a_1 = 0 = b_1$. We may take $a_3 = 1$ by adjusting A. With these, (3.81) can be replaced by

$$f = EA \quad \text{and} \quad g = E.$$

Corollary 3.56c. [162]. *Let $f, g : G \to \mathbb{C}$ be mappings. If they satisfy the functional equation*

$$g(x + y) = f(x)f(y) + g(x)g(y) \quad \text{for } x, y \in G, \tag{3.72a}$$

then $f(x + y) = f(x)g(y) + g(x)f(y) + 2\alpha f(x)f(y)$ for some constant α, and f and g are of the form

$$g = a_1E_1 + a_2E + a_3EA,$$
$$f = c_1E_1 + c_2E + c_3EA,$$

with the constraint

$$\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \\ a_3 & c_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & a_3 \\ 0 & a_3 & 0 \end{bmatrix},$$

where E, E_1 satisfy (E) *and A satisfies* (A).

Conversely, if f and h are given as above, where the coefficients satisfy the constraints above, then they form a solution of (3.72a), (*cf. Remark* 3.56a).

Remark 3.56d. The general solutions of (3.72a) are given by

$$\begin{gathered} f(x) = c_1(E_1 - E), \quad g(x) = a_1E_1 + (1 - a_1)E; \\ f(x) = \pm iEA, \ g(x) = E + EA. \end{gathered} \tag{3.88}$$

Consider the constraint in Corollary 3.52c. Since the rank of the matrices on the left is at most 2, we get

$$\operatorname{rank} \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & a_3 \\ 0 & a_3 & 0 \end{bmatrix} \leq 2.$$

This implies either $a_3 = 0$ or $a_1 = 0$.

(1) If $a_3 = 0$, then $c_3 = 0$, and so $f, h \in \operatorname{span}\{E_1, E\}$. When $a_i = 0$, we must also have $c_i = 0$. We may therefore assume $a_1, a_2 \neq 0$ by adjusting E_1, E if necessary. With this assumption, f and g can be replaced by

$$f = a_1E_1 + (1 - a_1)E \quad \text{and} \quad g = c_1E_1 - c_1E$$

with $a_1^2 + c_1^2 = a_1$ ($a_1 \neq 0, 1$ optional).

(2) If $a_3 \neq 0$, then $a_1 = c_1 = 0$. We can take $a_3 = 1$ by adjusting A. With this, f and g can be replaced by

$$f = E + EA \quad \text{and} \quad g = \pm iEA.$$

Result 3.57. (Aczél [12]). *The nonzero continuous solutions $f, g : \mathbb{R} \to \mathbb{R}$ of* (3.72) *are given by*

$$\begin{aligned} g(x) &= \frac{1}{1 - b^2} \exp cx, & f(x) &= \frac{1}{1 - b^2} b \exp cx, \quad (b \neq \pm 1); \\ g(x) &= (1 + bx) \exp cx, & f(x) &= \pm bx \exp cx; \\ g(x) &= \exp cx(\cos bx + d \sin bx), & f(x) &= \sqrt{1 + d^2} \exp cx \sin bx; \\ g(x) &= \exp cx(\cosh bx + d \sinh bx), & f(x) &= \sqrt{d^2 - 1} \exp ex \sinh bx; \end{aligned}$$

and that of (3.73a) *is given by*

$$\begin{aligned} g(x) &= 1 + dx, & f(x) &= dx; \\ g(x) &= \cos dx + b \sin dx, & f(x) &= b \sin dx; \\ g(x) &= \cosh dx + b \sinh dx, & f(x) &= b \sinh dx, \end{aligned}$$

where b, c, d are constants.

Many, including Steinitz and Neder (see [12, pp. 136]), treated the functional equation

$$f(x+y) = f(x)f\left(\frac{\pi}{2}+y\right) + f\left(\frac{\pi}{2}+x\right)f(y), \quad \text{for } x, y \in \mathbb{R}, \tag{3.89}$$

under continuity of $f : \mathbb{R} \to \mathbb{C}$. We determine the solutions of (3.89) using (3.73).

Theorem 3.58. *Suppose* $f(\neq 0) : \mathbb{R} \to \mathbb{C}$ *satisfies* (3.89). *Then* f *is given by*

$$f(x) = \frac{1}{2}E_2\left(x - \frac{\pi}{2}\right) \quad \text{or} \quad f(x) = a(E_1(x) - E_2(x)), \tag{3.89a}$$

with $2a = E_1\left(-\frac{\pi}{2}\right)$, $E_1\left(-\frac{\pi}{2}\right)E_2\left(\frac{\pi}{2}\right) = -1$.

Further, if f *is* (*measurable*) *continuous, then* f *has the form*

$$f(x) = \frac{1}{2}\exp c\left(x - \frac{\pi}{2}\right) \quad \text{or} \quad f(x) = a(\exp cx - \exp dx), \tag{3.89b}$$

with $2a = \exp(-c\frac{\pi}{2})$, $\exp(c-\alpha)\frac{\pi}{2} = -1$.

Proof. Define $g(x) = f\left(\frac{\pi}{2}+x\right)$ for $x \in \mathbb{R}$. Then (3.89) becomes (3.73). We have to determine the solution corresponding to (1) of (3.87) (note that a cannot be zero),

$$f(x) = a(E_1(x) - E_2(x)), \quad f\left(\frac{\pi}{2}+x\right) = \frac{1}{2}E_1(x) + \frac{1}{2}E_2(x),$$

where E_1, E_2 satisfy (E). Thus

$$\frac{1}{2}E_1(x) + \frac{1}{2}E_2(x) = a\left[E_1(x)E_1\left(\frac{\pi}{2}\right) - E_2(x)E_2\left(\frac{\pi}{2}\right)\right]$$

or

$$E_1(x)\left[\frac{1}{2} - aE_1\left(\frac{\pi}{2}\right)\right] = -E_2(x)\left[\frac{1}{2} + aE_2\left(\frac{\pi}{2}\right)\right]. \tag{3.90}$$

Suppose $\frac{1}{2} + aE_2\left(\frac{\pi}{2}\right) = 0$ in (3.90). Then either $E_1(x) = 0$ or $\frac{1}{2} - aE_1\left(\frac{\pi}{2}\right) = 0$.

Hence, when $E_1(x) = 0$, we have $f(x) = -aE_2(x)$ with $a = -\frac{1}{2}E_2\left(-\frac{\pi}{2}\right)$, which is (3.89a) (first part). If $\frac{1}{2} - aE_1\left(\frac{\pi}{2}\right) = 0$, then

$$a = \frac{1}{2}E_1\left(-\frac{\pi}{2}\right) \quad \text{and} \quad E_1\left(\frac{\pi}{2}\right) + E_2\left(\frac{\pi}{2}\right) = 0,$$

$f(x) = a(E_1(x) - E_2(x))$, which is the second part of (3.89a). If $\frac{1}{2} + aE_2\left(\frac{\pi}{2}\right) \neq 0$, then $E_2(x) = cE_1(x)$, and we get (3.89a) (first part).

Corresponding to (2) of (3.87), we get

$$\begin{aligned} f(x) &= E(x)A(x), \\ f\left(\frac{\pi}{2}+x\right) &= E(x) \\ &= E(x)E\left(\frac{\pi}{2}\right)\left(A(x) + A\left(\frac{\pi}{2}\right)\right); \end{aligned}$$

that is, since $E \neq 0$, $A(x) + A\left(\frac{\pi}{2}\right) = E\left(-\frac{\pi}{2}\right)$, meaning A is a constant and so $A = 0$, which cannot be. This case is not possible.

The continuous solution can be obtained either from (3.89) or from Remark 3.56b. □

3.4.9 Vibration of String Equation (VS)

Finally, in this section, we treat the equation (VS), mentioned in the introduction, considered by d'Alembert, namely

$$\text{(VS)} \qquad f(x+y) - f(x-y) = h(x)k(y) \quad \text{for } x, y \in \mathbb{R},$$

where $f, h, k : \mathbb{R} \to \mathbb{C}$ (or $\mathbb{R}$). (See also Sinopoulos [746].)

First we consider the trivial solutions. If f = constant, then either $h = 0$, k arbitrary or h arbitrary, $k = 0$.

(3.91) When $k =$ constant, $y = 0$ gives $h = 0$, $f =$ constant.

Suppose $h(x) = d$. Then (VS) becomes

$$f(x+y) - f(x-y) = dk(y), \quad d \neq 0. \tag{3.92}$$

Set $x = y$ in (3.92) to get $f(2y) - f(0) = dk(y)$.

Setting $x + y = u$ and $x - y = v$ in (3.92), we get

$$f(u) - f(v) = dk\left(\frac{u-v}{2}\right) = f(u-v) - f(0)$$

and

$$f(v) + f(-v) = 2f(0)$$

(interchange u and v or put $u = 0$). Interchange x and y in (3.92) and add the resultant to (3.92) to obtain

$$2f(x+y) - 2f(0) = dk(x) + dk(y);$$

that is (from (PA), Result 1.56),

$$f(x) = A(x) + b,\ dk(x) = 2A(x),$$

where A is additive (it could have used that $f(u) - f(0)$ is additive). So, the solution is

$$f(x) = A(x) + b, \quad h(x) = d, \quad k(x) = \frac{2}{d}A(x), \quad d \neq 0. \tag{3.91a}$$

The continuous solution is

$$f(x) = cx + b, \quad h(x) = d, \quad k(x) = \frac{2}{d}cx. \tag{3.91b}$$

We will now consider the nontrivial solutions of (VS). Put $x = 0$ in (VS) to have

$$f(y) - f(-y) = h(0)k(y) \quad \text{for } y \in \mathbb{R}. \tag{3.93}$$

We now consider the cases according to whether $h(0) = 0$ or $h(0) \neq 0$.

Case 1. Suppose $h(0) = 0$. Equation (3.93) shows that f is even. Interchange x and y in (VS) to get

$$f(x+y) - f(x-y) = h(y)k(x), \quad \text{for } x, y \in \mathbb{R},$$

or

$$h(x)k(y) = h(y)k(x);$$

that is,

$$k(x) = c\,h(x) \quad \text{for } x \in \mathbb{R}.$$

Note that c cannot be zero. Otherwise $k = 0$ and f is a constant.

Then (VS) becomes

$$f(x+y) - f(x-y) = c\,h(x)h(y) \quad \text{for } x, y \in \mathbb{R}. \tag{3.94}$$

Replace x by $\frac{x}{2}$ and y by $\frac{x}{2}$ to have

$$f(x) = f(0) + c\,h\left(\frac{x}{2}\right)^2 = b + c\,h\left(\frac{x}{2}\right)^2,$$

so that (3.94) becomes

$$h\left(\frac{x+y}{2}\right)^2 - h\left(\frac{x-y}{2}\right)^2 = h(x)h(y)$$

or

$$h(u+v)h(u-v) = h(u)^2 - h(v)^2 \tag{S}$$

(set $x = u + v$, $y = u - v$), which is the sine equation (S) and the general solution of (S) as given by Theorem 3.44,

$$h(x) = A(x) \quad \text{or} \quad h(x) = d(E(x) - E(-x)) \quad \text{for } x \in \mathbb{R}, \tag{3.91c}$$

where $E : \mathbb{R} \to \mathbb{C}^*$ is exponential and A is additive. Then

$$k(x) = cA(x) \quad \text{or} \quad k(x) = cd(E(x) - E(-x)) \quad \text{for } x \in \mathbb{R}, \tag{3.91d}$$

$$f(x) = b + \frac{c}{4}A(x)^2 \quad \text{or} \quad f(x) = b + cd^2\left(E\left(\frac{x}{2}\right) - E\left(\frac{-x}{2}\right)\right)^2. \tag{3.91e}$$

If f is measurable or continuous and $f : \mathbb{R} \to \mathbb{C}$, then

$$(3.91d_1) \qquad h(x) = dx \quad \text{or} \quad d \sin ax, \quad a, d \in c,$$

$$(3.91d_2) \qquad k(x) = cdx \quad \text{or} \quad cd \sin ax,$$

$$(3.91d_3) \qquad f(x) = b + \frac{c}{4}d^2x^2 \quad \text{or} \quad b + 4cd^2 \sin^2 \frac{ax}{2},$$

where $a, b, c, d \in \mathbb{C}$.

If $f, k, h : \mathbb{R} \to \mathbb{R}$ are continuous and satisfy (VS), then (Theorem 3.43)

$$(3.91d_4) \qquad \begin{cases} h(x) = dx & \text{or } d \sin ax & \text{or } a \sin h(ax), \\ k(x) = cdx & \text{or } cd \sin ax & \text{or } cd \sin h(ax), \\ f(x) = b + \dfrac{cd^2}{4}x^2 & \text{or } b - 4cd^2 \sin^2 \dfrac{ax}{2} & \text{or } b + cd^2 \sin h^2 \dfrac{ax}{2} \\ \quad = b + \dfrac{cd^2}{4}x^2 & \text{or } b - 2cd^2 + 2cd^2 \cos ax & \text{or } b + cd^2 \sin h^2 \dfrac{ax}{2}, \end{cases}$$

where $a, b, c, d \in \mathbb{R}$.

Case 2. Let $h(0) \neq 0$. Then, from (3.93) we see that k is odd. Replacing x by $\frac{x}{2}$ and y by $\frac{x}{2}$ in (VS), we have

$$f(x) = f(0) + h\left(\frac{x}{2}\right) k\left(\frac{x}{2}\right) = b + h\left(\frac{x}{2}\right) k\left(\frac{x}{2}\right)$$

and

$$(3.95) \qquad h\left(\frac{x+y}{2}\right) k\left(\frac{x+y}{2}\right) - h\left(\frac{x-y}{2}\right) k\left(\frac{x-y}{2}\right) = h(x)k(y)$$

or

$$(3.95a) \qquad h(u)k(u) - h(v)k(v) = h(u+v)k(u-v) \quad \text{for } u, v \in \mathbb{R}.$$

Replace v by $-v$ in (3.95a) and subtract (3.95a) from the resultant to obtain

$$\begin{aligned} k(u+v)h(u-v) - k(u-v)h(u+v) &= k(v)(h(v) + h(-v)) \\ &= h(0)k(2v) \quad (\text{let } u = v, \text{ use } k(0) = 0), \end{aligned}$$

which can be put in the form

$$(3.96) \qquad k(x)h(y) - k(y)h(x) = h(0)k(x-y) \quad \text{for } x, y \in \mathbb{R}.$$

Replace y by $-y$ in (3.96) and add the resultant to have

$$h(0)(k(x+y) + k(x-y)) = k(x)(h(y) + h(-y)),$$

which is the same as (3.27) (take $2\varphi(y) = \frac{1}{h(0)}(h(y) + h(-y))$) (we do not need φ). Using Theorem 3.41, we get $k = 0$, which cannot be, or k is Jensen, and since k is odd,

(3.91f) $$k(x) = A(x) \quad \text{or} \quad k(x) = c(E(x) - E(-x)),$$

for $x \in \mathbb{R}$, where E satisfies (E) and A satisfies (A) (see (3.27)).

Suppose $k(x) = A(x)$ and (3.96) yields

$$A(x)(h(y) - c_1) = A(y)(h(x) - c_1), \quad c_1 = h(0),$$

and since A is not zero, we get

(3.91f$_1$) $$h(x) = c_1 + c_2 A(x),$$

and then from (3.95) we have

(3.91f$_2$) $$f(x) = b + \frac{c_1}{2}A(x) + \frac{c_2}{4}A(x)^2, \quad x \in \mathbb{R}.$$

If f is measurable or continuous,

(3.91g) $$k(x) = dx, \quad h(x) = c_1 + c_2 dx, \quad f(x) = b + \frac{c_1 d}{2}x + \frac{c_2}{4}d^2x^2,$$

for $x \in \mathbb{R}$, where d, c_1, c_2, b are constants.

Now we consider the case $k(x) = c(E(x) - E(-x))$. As before, substituting for k in (3.96), we obtain

$$(E(x) - E(-x))h(y) - (E(y) - E(-y))h(x) = c_1(E(x - y) - E(-x + y))$$

or

$$\left(E(x) - \frac{1}{E(x)}\right)h(y) - \left(E(y) - \frac{1}{E(y)}\right)h(x) = c_1\left(\frac{E(x)}{E(y)} - \frac{E(y)}{E(x)}\right);$$

that is,

$$(E(x)^2 - 1)(E(y)h(y) - c_1) = (E(y)^2 - 1)(E(x)h(x) - c_1),$$

meaning

(3.91f$_3$) $$h(x) = c_3 E(x) + (c_1 - c_3)E(-x) \quad \text{for } x \in \mathbb{R}.$$

(Note that if $E(y)^2 = 1$ for all y, then $E(y) = E(-y)$, and then $k = 0$, which is not the case.)

Now (3.95) yields

(3.91f$_4$) $$f(x) = b + cc_1 - 2cc_3 + cc_3 E(x) - c(c_1 - c_3)E(-x).$$

Suppose $f, h, k : \mathbb{R} \to \mathbb{C}$ are continuous and satisfy (VS). Then

(3.97) $$\begin{cases} k(x) = c(\exp dx - \exp(-dx)), \\ h(x) = c_3 \exp dx + (c_1 - c_3)\exp(-dx), \\ f(x) = b + cc_1 - 2cc_3 + cc_3 \exp dx - c(c_1 - c_3)\exp(-dx). \end{cases}$$

If $f, h, k : \mathbb{R} \to \mathbb{R}$ are continuous and satisfy (VS), then

$$k(x) = c \sin dx \quad \text{or} \quad c \sin hdx. \tag{3.97a}$$

Substituting $k(x) = c \sin dx$ in (3.96) gives

$$\begin{aligned} h(y)c \sin dx - h(x)c \sin dy &= cc_1 \sin d(x - y) \\ &= cc_1(\sin dx \cos dy - \cos dx \sin dy) \end{aligned}$$

or

$$\sin dy(h(x) - c_1 \cos dx) = \sin dx(h(y) - c_1 \cos dy),$$

yielding

$$h(x) = c_1 \cos dx + c_4 \sin dx. \tag{3.97b}$$

Using (3.97a) and (3.97b) in (VS), we get

$$\begin{aligned} f(x + y) - f(x - y) &= (c_1 \cos dx + c_4 \sin dx)c \sin dy \\ &= cc_1 \cos dx \sin dy + cc_4 \sin dx \sin dy \\ &= \frac{cc_1}{2}[\sin d(x + y) - \sin d(x - y)] \\ &\quad + \frac{cc_4}{2}[\cos d(x - y) - \cos d(x + y)], \end{aligned}$$

yielding (compare Result 3.1a and Result 3.2a)

$$f(x) = \frac{cc_1}{2} \sin dx - \frac{cc_4}{2} \cos dx + \frac{cc_4}{2} + b. \tag{3.97c}$$

$k(x) = c \sin hdx$ results in

$$h(x) = c_1 \cos h(dx) + c_4 \sin h(dx), \tag{3.97d}$$

$$f(x) = \frac{cc_1}{2} \sin h(dx) - \frac{cc_4}{2} \cos h(dx) + \frac{cc_4}{2} + b. \tag{3.97e}$$

Thus we have proved the following theorem.

Theorem 3.59. *Suppose* $f, h, k : \mathbb{R} \to \mathbb{C}$ *satisfy* (VS). *Then* f, h, k *are given by* (3.91) *and* (3.91a)*;* (3.91c), (3.91d), *and* (3.91e)*;* (3.91f), ($3.91f_1$), *and* ($3.91f_2$)*; and* (3.91f), ($3.91f_3$), *and* ($3.91f_4$). *Further, if* f, h, k *are continuous, then* k, h, f *are given by* ($3.91d_1$), ($3.91d_2$), ($3.91d_3$), ($3.91d_4$), (3.91g), *and* (3.97).

If $f, h, k : \mathbb{R} \to \mathbb{R}$ *are continuous and satisfy* (VS), *then* f, h, k *are given by* (3.91b), ($3.91d_4$), *and* (3.91g)*;* (3.97a), (3.97b), *and* (3.97c)*; and* (3.97d) *and* (3.97e).

3.4.10 Wilson's Equations

Finally, we consider the functional equations

$$S(x-y) = S(x)C(y) - S(y)C(x), \tag{3.98}$$

$$C(x-y) = C(x)C(y) - k^2 S(x)S(y), \tag{3.98a}$$

where $S, C : G \to \mathbb{C}$, treated by Wilson in [580, 729, 12], and present a simple, direct proof of the following theorem.

Theorem 3.60. *Suppose $S, C : G \to \mathbb{C}$, where G is a group, satisfy* (3.98) *and* (3.98a) *with C satisfying* (K)*, and k is a complex constant. Then $S = 0$, $C = 0$; $S = 0$, C an even homomorphism; when $S \neq 0$ and $C \neq 0$, S is odd and C is even; when $k \neq 0$, we have*

$$C(x) = \frac{E(x) + E(-x)}{2}, \quad S(x) = \pm\frac{1}{2k}(E(x) - E(-x)), \tag{3.98b}$$

where E is exponential, satisfying (E)*; and when $k = 0$ and G is 2-divisible, then*

$$C(x) = 1, \quad S(x) = A(x), \tag{3.98c}$$

where A is additive, satisfying (A).

Proof. It is easy to see that $S = 0$, $C = 0$ is a solution. Suppose $S = \text{constant} = C \neq 0$. Then (3.98) shows that $C(x) = \text{constant}$ and then $C = 0$, which cannot be. Suppose $S = 0$ and $C \neq 0$. Then (3.98a) shows that $C(x-y) = C(x)C(y)$. Interchanging x and y shows that C is even and that $C(x+y) = C(x)C(y)$; that is, C is a homomorphism. So, we will consider only $S \neq 0$ and $C \neq 0$. First we prove that C satisfies (C).

An interchange of x and y in (3.98) yields that S is odd and $S(0) = 0$ and in (3.98a) shows that C is even.

Replace y by $-y$ in (3.98a) to get

$$C(x+y) = C(x)C(y) + k^2 S(x)S(y),$$

and adding this with (3.98a) gives

$$C(x+y) + C(x-y) = 2C(x)C(y),$$

which is (C). Then, by Theorem 3.21, there exists an exponential $E : G \to \mathbb{C}^*$ such that

$$C(x) = \frac{E(x) + E(-x)}{2}.$$

Substituting this $C(x)$ in (3.98a) gives

$$k^2 S(x)S(y) = \frac{1}{4}(E(x) - E(-x))(E(y) - E(-y))$$

or

$$S(x) = d(E(x) - E(-x))$$

(when $k \neq 0$) since $S \neq 0$. Putting this $S(x)$ in the equation above, we get $d = \pm\frac{1}{2k}$ when $k \neq 0$ and $S(x) = \pm\frac{1}{2k}(E(x) - E(-x))$, giving (3.98b).

Alternatively, replace y with $-y$ in (3.98) to get

$$S(x + y) = S(x)C(y) + S(y)C(x). \tag{3.73}$$

Define $E : G \to \mathbb{C}$ by

$$E(x) = C(x) + kS(x) \quad \text{for } k \neq 0.$$

Now

$$\begin{aligned} E(x + y) &= C(x + y) + kS(x + y) \\ &= C(x)C(y) + k^2 S(x)S(y) + k(S(x)C(y) + S(y)C(x)) \\ &= (C(x) + kS(x))(C(y) + kS(y)) \\ &= E(x)E(y) \end{aligned}$$

shows that E is exponential. Then

$$E(-x) = C(x) - kS(x),$$

giving again (3.98b).

When $k = 0$, we get $C(x - y) = C(x)C(y) = C(y - x)$; that is, C is even, $C(x + y) = C(x - y)$, $C(x) = \text{constant} = d$ (here we use that G is 2-divisible), and $d^2 = d$, yielding $d = 1$. Hence $C(x) = 1$. Then (3.73) shows that S is additive, yielding (3.98c). This completes the proof of this theorem. □

3.4.11 Analytic Solutions

In [701], the author says that to his surprise he found

$$|\sin(x + iy)| = |\sin x + \sin iy| \quad \text{for } x, y \in \mathbb{R}.$$

It is also true that

$$|\sinh(x + iy)| = |\sinh x + \sinh iy| \quad \text{for } x, y \in \mathbb{R}.$$

From these arise the functional equation

$$|f(x + iy)| = |f(x) + f(iy)| \quad \text{for } x, y \in \mathbb{R}, \tag{3.99}$$

and the following result holds.

Theorem 3.61. (H. Haruki [349, 350]). *The only functions* $f : \mathbb{C} \to \mathbb{C}$ *regular* (*analytic*) *for* $|z| < \gamma$ *satisfying* (3.99) *are*

$$f(z) = az, \quad f(z) = a \sin bz, \quad \text{or} \quad f(z) = a \sinh bz, \tag{3.99a}$$

where a *is a complex constant and* b *is real.*

Proof. Suppose $f : \mathbb{C} \to \mathbb{C}$ satisfying the sine equation (S) is an entire function and $f(z)$ is real on $\mathbb{R}$.

Then f is odd (i.e., $f(-z) = -f(z)$, $z \in \mathbb{C}$) and $\overline{f(s+it)} = f(s-it)$ for $s, t \in \mathbb{R}$.

Now,

$$\begin{aligned} |f(s+it)|^2 &= f(s+it)\overline{f(s+it)} = f(s+it)f(s-it) \\ \text{by (S)} \quad &= f(s)^2 - f(it)^2 = f(s)^2 + f(it)f(-it) \\ &= f(s)^2 + f(it)\overline{f(it)} = |f(s)|^2 + |f(it)|^2 \end{aligned}$$

leads to the study of the Pythagorean functional equation (Hille [373], H. Haruki [348, 349])

$$|f(x+iy)|^2 = |f(x)|^2 + |f(iy)|^2, \quad x, y \in \mathbb{R}. \tag{3.99b}$$

Equations similar to (3.99) are (3.99b), (3.99c), and (3.99d):

$$|f(x+iy)|^2 = |f(x)|^2 + |f(iy)|^2 - 1, \tag{3.99c}$$

$$|f(x+iy)| = |f(x) + f(iy)|, \quad \text{for } x, y, \mathbb{R}. \tag{3.99d}$$

Result 3.61a. [373, 348]. *Suppose* $f : \mathbb{C} \to \mathbb{C}$ *is an entire function. If* (i) f *satisfies* (3.99b)*, then* f *has the form* (3.99a)*;* (ii) *if* f *is a solution of* (3.99c)*, then* f *is given by* $f(z) = \exp i\theta \cos bz$ *or* $\exp i\theta \cosh bz$*; and* (iii) *if* f *is a solution of* (3.99d)*, then* f *is given by* $f(z) = az^2$ *or* $a \sin^2 bz$ *or* $a \sinh^2 bz$ *for* $z \in \mathbb{C}$.

Some Generalizations

Suppose $f, g, h : \mathbb{C} \to \mathbb{C}$ are entire functions satisfying the functional equation

$$|f(x+iy)| = |g(x)| + |h(iy)| \quad \text{for } x, y \in \mathbb{R}. \tag{3.99e}$$

It is easy to check that f satisfies the equation

$$|f(x+iy)| + |f(0)| = |f(x)| + |f(iy)| \quad \text{for } x, y \in \mathbb{R},$$

the solutions of which are given by [373, 146, 349]

$$f(z) = a(z-b)^2 \quad f(z) = (ae^{\lambda z} + be^{-\lambda z})^2,$$

where $a, b, \lambda \in \mathbb{C}$, $\lambda^2 \in \mathbb{R}$. Thus, determining the solutions of (3.99e) boils down to solving the equations

$$|g(x)| + |h(iy)| = |a(z-b)^2| \tag{3.99e$_1$}$$

and

$$|g(x)| + |h(iy)| = |ae^{\lambda z} + be^{-\lambda z}|^2, \tag{3.99e$_2$}$$

where $z = x + iy,\ x, y \in \mathbb{R}$.

Let Ω denote the set of all entire functions such that their power series expansions have real coefficients. The following results hold.

Result 3.61b. (H. Haruki [355]). *The only system of entire solutions g and h of equation* (3.99e$_1$) *are*

$$g(z) = a(z-s_0)^2 e^{ip(z)}, \quad h(z) = a(z-it_0)^2 e^{q(z)+ir(z)},$$

where $b = s_0 + it_0$ and $p, q, r \in \Omega$ and q and r are odd and even, respectively.

Result 3.61c. [355]. *The only entire solutions of* (3.99e$_2$) *in the case $\lambda \in \mathbb{R}$ are*

$$\begin{cases} g(z) = a^2 e^{2\lambda z + ip(z)}, \\ h(z) = 0, \end{cases}$$

when $b = 0$, and

$$\begin{cases} g(z) = \dfrac{J_1(z)}{I_1(z)} e^{ip(z)} (ae^{\lambda(z+it_0)} + be^{-\lambda(z+it_0)})^2, \\ h(z) = \dfrac{J_2(z)}{I_2(z)} e^{q(z)+ir(z)} (ae^{\lambda(s_0+z)} + be^{-\lambda(s_0+z)})^2, \end{cases}$$

when $a \neq 0,\ b \neq 0$, where $p, q, r \in \Omega,\ q$ and r are odd and even, respectively, $s_0 = (1/2\lambda)\log|b/a|,\ t_0 = \varphi/2\lambda,\ \varphi \in \arg(-b/a)$, and I_1, J_1, I_2, J_2 are given by the formulas

$$I_1(z) = z^m \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right) e^{\frac{z}{a_j}},$$

$$J_1(z) = z^m \prod_{j=1}^{\infty} \left(1 - \frac{z}{b_j}\right) e^{\frac{z}{b_j}}, \quad b_j \in \{a_j, \overline{a}_j\},$$

$$I_2(z) = z^n \prod_{j=1}^{\infty} \left(1 - \frac{z}{c_j}\right) e^{\frac{z}{c_j}},$$

$$J_2(z) = z^n \prod_{j=1}^{\infty} \left(1 - \frac{z}{d_j}\right) e^{\frac{z}{d_j}}, \quad d_j \in \{e_j, -\overline{c}_j\}.$$

Theorem 3.62. *$P(z)$ and $Q(z)$ are integral functions satisfying*

$$P(z)^2 - Q(z)^2 = 1, \quad \text{for } z \in \mathbb{C}, \tag{3.100}$$

if and only if there exists an analytic function $f(z)$ with the property that

$$P(z) = \cosh f(z), \quad Q(z) = \sinh f(z), \quad z \in \mathbb{C}. \tag{3.100a}$$

Proof. (See also Ganapathy [298].) Note that (3.100a) includes the familiar solutions $(\cosh z, \sinh z)$.

Let (3.100) hold. Then

$$P(z)^2 - Q(z)^2 = [P(z) + Q(z)][P(z) - Q(z)] = 1.$$

Hence $P(z) + Q(z)$ and $P(z) - Q(z)$ are integral functions without zeros. Hence there exists an integral function $f(z)$ such that

$$P(z) + Q(z) = \exp f(z) \quad \text{and} \quad P(z) - Q(z) = \exp(-f(z)).$$

Thus (3.100a) holds. This completes the proof of the theorem. □

Theorem 3.62a. *If the solutions* (3.100a) *of* (3.100) *are periodic, then the same is true of $f'(z)$. Conversely, if $g(z)$ is an integral function with $g'(z)$ periodic, then there is a constant μ such that $\mu g(z)$ put in* (3.100a) *for $f(z)$ gives a periodic solution of* (3.100).

Proof. Suppose the solutions (3.100a) of (3.100) are periodic with period λ. Then we have

$$\cosh f(z + \lambda) - \cosh f(z) = 0;$$

that is,

$$\exp f(z + \lambda) + \exp(-f(z + \lambda)) - \exp f(z) - \exp(-f(z)) = 0,$$

meaning

$$[\exp\{f(z + \lambda) - f(z)\} - 1][\exp f(z) - \exp(-f(z + \lambda))] = 0,$$

from which it follows immediately that, for every complex z, either

$$f(z + \lambda) - f(z) = 2k\pi i$$

or

$$f(z + \lambda) + f(z) = 2\ell\pi i,$$

where k and ℓ are integers. Since f is an integral function, we can conclude that either the former is true at all z or the latter holds at all z. If the former holds for all z, then we have

$$f'(z + \lambda) - f'(z) = 0,$$

from which we see that $f'(z)$ is periodic with period λ.

In the latter case, we get

$$f'(z+\lambda)+f'(z)=0 \quad \text{or} \quad f'(z+2\lambda)=f'(z),$$

with the result that $f'(z)$ is periodic with period 2λ. In any case, we see that $f'(z)$ is periodic.

Now let $g(z)$ be an integral function with $g'(z)$ periodic, say with period λ. Then

$$g(z+\lambda)-g(z)=c \text{ constant}.$$

If $c=0$, taking $\mu=1$, $f(z)=g(z)$ in (3.100a) gives a periodic solution of (3.100). If $c\neq 0$, take $\mu=\frac{2k\pi i}{c}$, where k is an integer ($\neq 0$). Then $f(z)=\mu g(z)$ in (3.100a) gives a periodic solution of (3.100), which can be easily verified from the identities

$$\cosh C-\cosh D=2\sinh\frac{C+D}{2}\cdot\sinh\frac{C-D}{2}$$

and

$$\sinh C-\sinh D=2\cosh\frac{C+D}{2}\cdot\sinh\frac{C-D}{2}.$$

This completes the proof of this theorem. □

Theorem 3.62b. *If the solutions in* (3.100a) *of* (3.100) *are of finite order, then* $f(z)$ *in* (3.100a) *is a polynomial in* z. *If, further,* $P(z)$ *and* $Q(z)$ *are periodic solutions, then* $f(z)$ *is linear in* z *so that* $P(z)=\cosh(az+b)$ *and* $Q(z)=\sinh(az+b)$.

Proof. If $P(z)$ and $Q(z)$ are of finite order, then so is $P(z)+Q(z)$. Hence, from $P(z)\pm Q(z)=\exp(\pm f(z))$, we see that $f(z)$ is a polynomial. If, further, $P(z)$ and $Q(z)$ are periodic solutions, from Theorem 3.62a we have that $f'(z)$ is periodic. Hence $f'(z)$ is a constant and so $f(z)$ is linear in z. Thus the proof of this theorem is complete. □

Theorem 3.62c. *There exist no integral functions* $P(z)$ *and* $Q(z)$ *satisfying the equation*

$$P(z)^n-Q(z)^n=1, \tag{3.100b}$$

where n *is any integer* ≥ 3, *except when both* $P(z)$ *and* $Q(z)$ *are constants.*

Proof. Let $w=\exp\frac{\pi i}{n}$ and $1,\ \alpha,\ \alpha^2,\ldots,\ \alpha^{n-1}$ be the nth roots of unity. Then

$$\begin{aligned}P(z)^n-Q(z)^n&=[P(z)+wQ(z)][P(z)+\alpha wQ(z)]\ldots[P(z)+\alpha^{n-1}wQ(z)]\\&=1.\end{aligned}$$

If $P(z)$ and $Q(z)$ are integral functions satisfying the conditions above, then

$$P(z)+wQ(z),\ldots,P(z)+\alpha^{n-1}wQ(z)$$

are integral functions having no zeros. Thus the meromorphic function $\frac{P(z)}{Q(z)}$ cannot take the n values $-w,-\alpha w,\ldots,-\alpha^{n-1}w$.

But, by Picard's Theorem, $\frac{P(z)}{Q(z)}$ can omit at most two values, unless $\frac{P(z)}{Q(z)}$ is constant. Thus, for $n \geq 3$, (3.100b) has no integral solutions. If $\frac{P(z)}{Q(z)}$ is a constant, from (3.100b) it is evident that $P(z)$ and $Q(z)$ are constants. This completes the proof of this theorem. □

3.4.12 Equation (3.72) on Analytic Functions

Now we determine the continuous linear functionals F and K on $H(S)$, the algebra of analytic functionals on an open set $S \subseteq \mathbb{C}$ satisfying the functional equation

$$F(fg) = F(f)F(g) - K(f)K(g) \quad \text{for } f, g \in H(S). \tag{3.101}$$

Theorem 3.63. (Kannappan and Nandakumar [488]). *Let F and K be nontrivial linear functionals on $H(S)$ satisfying the functional equation* (3.101)*. Then one of the following is true:*

(i) *If $K(1) \neq 0$, there exist constants k_1, k_2, homomorphism N, and $z_1 \in S$ such that*

$$F = k_1 N, \quad K = k_2 N;$$

i.e.,

$$K(f) = k_2 f(z_1), \quad F(f) = k_1 f(z_1), \quad \text{for } f \in H(S).$$

(ii) *If $K(1) = 0$, $F(1) = 1$, and $K(I) \neq 0$, then K and F satisfy the functional equation*

$$K(fg) = K(f)F(g) + K(g)F(f) + cK(f)K(g), \quad f, g \in H(S), \tag{3.101a}$$

and there exist a $z_1 \in S$ and a constant c with $c^2 = 4$ such that

$$K(f) = K(I) f'(z_1), \quad F(f) = f(z_1) - \frac{K(I)}{2} f'(z_1), \quad \text{for } f \in H(S),$$

(iii) *there exist $z_1, z_2 \in S$, $z_1 \neq z_2$, and a constant $c \neq \pm 2$ with $1 + \frac{4-c^2}{(z_1-z_2)^2} K(I)^2 = 0$ such that*

$$K(f) = \frac{K(I)}{z_1 - z_2}[f(z_1) - f(z_2)],$$

$$F(f) = \frac{1}{2}[f(z_1) + f(z_2)] + \frac{c}{2}\frac{K(I)}{z_1 - z_2}[f(z_1) - f(z_2)], \quad \text{for } f \in H(S).$$

Proof. In the following, we need the Result 3.63a in Rubel [714], Zaleman [838], and Nandakumar [643], and some auxiliary results.

Result 3.63a. *If K and F are two linear functionals satisfying the functional equation*

$$K(fg) = K(f)F(g) + F(f)K(g) \quad \text{on } H(S), \tag{3.101b}$$

then one of the following is true:

(a) *There exists* $z_1 \in S$ *such that*

$$K(f) = K(1)f(z_1), \quad F(f) = \frac{1}{2}f(z_1), \quad f \in H(S);$$

(b) *there exists* $z_1 \in S$ *such that*

$$K(f) = K(I)f'(z_1), \quad F(f) = f(z_1), \quad f \in H(S);$$

(c) *or there exist* $z_1, z_2 \in S$ *with* $z_1 \neq z_2$ *such that*

$$K(f) = \frac{K(I)}{z_1 - z_2}[f(z_1) - f(z_2)], \; F(f) = \frac{1}{2}[f(z_1) + f(z_2)], \quad f \in H(S).$$

From now on, we assume that the linear functionals K and F satisfy (3.101).

Lemma 3.63b. *Let* F *and* K *be nontrivial linear functionals on* $H(S)$ *satisfying the functional equation* (3.101) *and let* $K(I) \neq 0$. *Then* F *and* K *are constant multiples of a homomorphism; i.e., there exist constants* k_1 *and* k_2 *and a homomorphism* N *such that* $F = k_1 N$ *and* $K = k_2 N$ *with* $k_1 = k_1^2 - k_2^2$.

Proof. Setting $f = g = 1$ in (3.101), we get

$$F(1) = F(1)^2 - K(1)^2.$$

Suppose $K(1) \neq 0$. Then $F(1)$ is neither 0 nor 1. Hence, taking $g = 1$ in (3.101), we obtain

$$F(f) = F(f)F(1) - K(f)K(1)$$

so that

$$F(f) = kK(f),$$

where $k = \frac{K(1)}{F(1)-1} \neq 0$. Substituting this relation in (3.101), we obtain

$$kK(fg) = k^2K(f)K(g) - K(f)K(g) = (k^2 - 1)K(f)K(g)$$

or $N = \frac{k^2-1}{k}K$ is a homomorphism. Note that $k \neq 0, \pm 1$; otherwise K is a trivial linear functional. Thus K and F are constant multiples of a homomorphism.

Lemma 3.63c. *Let* F *and* K *be nontrivial linear functionals on* $H(S)$ *satisfying the functional equation* (3.101). *If* $K(1) = 0$, *then* $F(1) \neq 0$.

Proof. Suppose $F(1) = 0$. Then

$$F(f) = F(1 \cdot f) = F(1)F(f) - K(1)K(f) = 0 \quad \text{for all } f \in H(S),$$

which contradicts the hypothesis that F is nontrivial. Hence $F(1) \neq 0$.

Lemma 3.63d. *Let* F *and* K *be linear functionals on* $H(S)$ *satisfying the functional equation* (3.101). *If* $K(1) = 0$, $F(1) = 1$, *and* $K(I) = 0$, *then* F *is a homomorphism and* K *is a trivial linear functional.*

Proof. Let $F(I) = z_0$. We claim that $z_0 \in S$. Suppose not. Then $\frac{1}{I-z_0} \in H(S)$. Then, since $K(1) = 0$, $F(1) = 1$, and $K(I) = 0$, we have by linearity of F and K that

$$\begin{aligned} 1 &= F\left((I - z_0)\frac{1}{I - z_0}\right) = F(I - z_0)F\left(\frac{1}{I - z_0}\right) - K(I - z_0)K\left(\frac{1}{I - z_0}\right) \\ &= (F(I) - F(z_0))F\left(\frac{1}{I - z_0}\right) - (K(I) - K(z_0))K\left(\frac{1}{I - z_0}\right) = 0, \end{aligned}$$

which is a contradiction. Thus $z_0 \in S$.

Since $z_0 \in S$, we have $\frac{f-f(z_0)}{I-z_0} \in H(S)$. Hence

$$\begin{aligned} F(f) - f(z_0) &= F(f - f(z_0)) \\ &= F\left((I - z_0)\frac{f - f(z_0)}{(I - z_0)}\right) \\ &= F(I - z_0)F\left(\frac{f - f(z_0)}{I - z_0}\right) - K(I - z_0)K\left(\frac{f - f(z_0)}{I - z_0}\right) \\ &= (F(I) - F(z_0))F\left(\frac{f - f(z_0)}{I - z_0}\right) \\ &\quad - (K(I) - K(z_0))K\left(\frac{f - f(z_0)}{I - z_0}\right) = 0. \end{aligned}$$

Thus $F(f) = f(z_0)$, which is a homomorphism. This in turn implies $K = 0$.

Lemma 3.63e. *Let F and K be nontrivial linear functionals on $H(S)$ satisfying the functional equation* (3.101)*. If $K(1) = 0$, $F(1) = 1$, and $K(I) \neq 0$, then F and K satisfy the equation*

$$K(fg) = K(f)F(g) + K(g)F(f) + cK(f)K(g), \quad f, g \in H(S), \tag{3.101a}$$

where c is a constant.

Proof. From $F(fgh) = F(fhg)$, and applying (3.101) a couple of times, we get

$$\begin{aligned} F(fgh) &= F(fg)F(h) - K(fg)K(h) \\ &= [F(f)F(g) - K(f)K(g)]F(h) - K(fg)K(h) \end{aligned}$$

and

$$\begin{aligned} F(fhg) &= F(fh)F(g) - K(fh)K(g) \\ &= [F(f)F(h) - K(f)K(h)]F(g) - K(fh)K(g). \end{aligned}$$

Comparing the two equations above, we obtain

$$[K(fg) - K(f)F(g)]K(h) = [K(fh) - K(f)F(h)]K(g).$$

Setting $h = I$ and using $K(I) \neq 0$, we have

$$K(fg) = K(f)F(g) + N(f)K(g), \tag{3.102}$$

where $N(f) = K(I)^{-1}[K(If) - K(I)K(f)]$. Interchanging f and g in the preceding equation, we get

$$K(f)(N(g) - F(g)) = K(g)(N(f) - F(f)). \tag{3.102a}$$

Replacing g by I in (3.102a) and solving for $N(f)$ using $K(I) \neq 0$, we obtain $N(f) = F(f) + cK(f)$, where $c = K(I)^{-1}[N(I) - F(I)]$. Using this in (3.102), we have

$$K(fg) = K(f)F(g) + K(g)F(f) + cK(f)K(g). \tag{3.101a}$$

Lemma 3.63f. [714]. *If F is a homomorphism on $H(S)$, then there exists a $z_0 \in S$ such that $F(f) = f(z_0)$ for all $f \in H(S)$, hence the result.*

Proof of the Theorem

Proof. (For another proof, see [488].)

Case 1. $K(1) \neq 0$. By Lemma 3.63b, there exists a homomorphism N such that $F = k_1 N$ and $K = k_2 N$. Since by Result 3.63a there exists a $z_1 \in S$ such that $N(f) = f(z_1)$ for all $f \in H(S)$, the result (i) follows.

If $K(1) = 0$, then $F(1) = F(1)^2 - K(1)^2$, which implies that $F(1) = 0$ or $F(1) = 1$. But, by Lemma 3.63c, $F(1) \neq 0$. Thus we have the following two cases when $K(1) = 0$.

Case 2. $K(1) = 0$, $F(1) = 1$, and $K(I) = 0$. From Lemma 3.63d, this is not possible since this leads to $K = 0$.

Case 3. $K(1) = 0$, $F(1) = 1$, and $K(I) \neq 0$. By Lemma 3.63e, we have

$$K(fg) = K(f)F(g) + K(g)F(f) + cK(f)K(g), \quad f, g \in H(S), \tag{3.101a}$$

where c is a constant. Rewriting the above, we obtain

$$K(fg) = K(f)[F(g) + \frac{c}{2}K(g)] + K(g)\left[F(f) + \frac{c}{2}K(f)\right],$$

which is of the form (3.101b). Since $K(1) \neq 0$, solution (a) of Result 3.63a is not possible. Now we apply solution (b) of Result 3.63a. Thus there exists a $z_1 \in S$ such that

$$K(f) = K(I)f'(z_1), \quad F(f) + \frac{c}{2}K(f) = f(z_1), \quad \text{for } f \in H(S);$$

that is,

$$K(f) = K(I)f'(z_1), \quad F(f) = f(z_1) - \frac{c}{2}K(I)f'(z_1), \quad \text{for } f \in H(S),$$

which is (ii) of the theorem. This K and F satisfy (3.101b), provided $c^2 = 4$. Note that $K(1) = 0$ and $F(1) = 1$ are satisfied and the solutions are

$$K(f) = K(I)f'(z_1), \quad F(f) = f(z_1) \pm K(I)f'(z_1), \quad \text{for } f \in H(S).$$

Finally, corresponding to solution (c), there exists $z_1, z_2 \in S$ with $z_1 \neq z_2$ such that

$$K(f) = \frac{K(I)}{z_1 - z_2}[f(z_1) - f(z_2)],$$
$$F(f) + \frac{c}{2}K(f) = \frac{1}{2}[f(z_1) + f(z_2)], \quad \text{for } f \in H(S);$$

that is,

$$K(f) = \frac{K(I)}{z_1 - z_2}[f(z_1) - f(z_2)],$$
$$F(f) = \frac{1}{2}[f(z_1) + f(z_2)] - \frac{c}{2}\frac{K(I)}{z_1 - z_2}[f(z_1) - f(z_2)], \quad \text{for } f \in H(S),$$

which is the second part of (iii) of the theorem. These F and K satisfy (3.101b) when

$$1 + \frac{4 - c^2}{(z_1 - z_2)^2}K(I)^2 = 0, \quad \text{where } c \neq \pm 2.$$

If $c = 0$, $K(I)^2 = -\frac{(z_1 - z_2)^2}{4}$ and the solution takes the form

$$K(f) = \pm\frac{i}{2}[f(z_1) - f(z_2)], \quad F(f) = \frac{1}{2}[f(z_1) + f(z_2)].$$

This completes the proof of the theorem. □

Remarks

1. The solution of the equation $F(fg) = F(f)F(g) + K(f)K(g)$ can be obtained from (3.101b) by replacing K by iK.
2. The solution of $K(fg) = K(f)F(g) - F(f)K(g)$ is $K = 0$ and F arbitrary (interchange f and g).
3. We have shown that the linear functionals satisfying (3.101b) are linear combinations of evaluation functionals, which implies that these functionals are continuous on $H(S)$ equipped with the topology of uniform convergence on compact subsets of S. Hence we ask whether we obtain additional solutions if the requirement of linearity of the functionals is weakened to additivity.

3.4.13 Some Generalizations

In this section, we treat many generalizations. We start off with the general functional equation

$$f(x+y)+g(x-y)=h(x)k(y) \quad \text{for } x,y\in G, \tag{3.103}$$

where $f,g,h,k:G\to\mathbb{C}$ and G is a 2-divisible Abelian group.

First we will show that the study of (3.103) is equivalent to the study of

$$F(x+y)N(x-y)=H(x)+K(y) \quad \text{for } x,y\in G, \tag{3.103a}$$

where $F, N, H, K : G\to\mathbb{C}$.

Putting $x+y=2u$ and $x-y=2v$ first in (3.103) results in

$$h(u+v)k(u-v)=f(2u)+g(2v),$$

which is (3.103a), and second in (3.103a) it goes over to

$$H(u+v)+K(u-v)=F(2u)N(2v),$$

which is (3.103).

So, we will consider only (3.103) (see Wilson [831]). Special cases of (3.103) have been discussed elsewhere in this book and in the literature. Some of the more familiar special cases include Jensen's equation (J) ($g=h=f, k=2$), the Pexider-Cauchy equations ($g=0, h=k=f$), the cosine equation (C) ($g=f, h=f, k=2f$), the generalized cosine equation (3.27) ($g=f, h=f, k=2\varphi$), and the vibration of strings equation (VS) ($g=-f$), among others.

As a matter of fact, we determine the most general solutions of (3.103) in the following theorem connected to (3.27) and (VS) and the usual Cauchy equations (A) and (E).

Theorem 3.64. (See also [831].) *Suppose* $f,g,h,k:G\to\mathbb{C}$ (*with* G, *a 2-divisible Abelian group*) *fulfill equation* (3.103). *Then they are given by:*

(i) $h=0$ *or* $k=0$, $f=c, g=-c$.

(ii) **Case (i).** $h=$ *constant* $c\ (\neq 0)$, $k\neq 0$.

$$f(x)=A(x)+c_1,\ g(x)=-A(x)+c_2,\ k(x)=\frac{2}{c}A(x)+c_3,$$
$$\text{with } c_1+c_2=cc_3.$$

Case (ii). $k=c\ (\neq 0)$, $h\neq 0$.

$$f(x)=A(x),\quad g(x)=A(x)+cc_1,\quad h(x)=\frac{2}{c}A(x)+c_1.$$

(iii) **Case (i).** $f = 0,\ g \neq 0,\ h, k \neq 0.$

$$g(x) = bcE(x), \quad h(x) = bE(x), \quad k(x) = cE(-x).$$

Case (ii). $g = 0,\ f, h, k \neq 0.$

$$f(x) = bcE(x), \quad h(x) = bE(x), \quad k(x) = cE(x).$$

(iv) f, g, h, k *are nonconstant.*

Case (i). $h(0) = 0$ *or* $k(0) = 0$ *reduces to* (VS) *and*

(a) $f(x) = b + \frac{c}{4}A(x)^2 = -g(-x),\ h(x) = cA(x),\ k(x) = A(x),$
$f(x) = b + cd^2(E(x) + E(-x) - 2) = -g(-x),$
$h(x) = ck(x) = cd(E(x) - E(-x)),$
$f(x) = b + \frac{c_1}{2}A(x) + \frac{c_2}{4}A(x)^2 = -g(-x),\ h(x) = A(x),$
$k(x) = c_1 + c_2A(x),$
$f(x) = d + cc_3E(x) - c(c_1 - c_3)E(-x) = -g(-x),$
$h(x) = c(E(x) - E(-x)),$
$k(x) = c_3E(x) + (c_1 - c_3)E(-x);$

(b) $f(x) = b + \frac{c}{4}A(x)^2 = -g(x),\ h(x) = A(x),\ k(x) = cA(x),$
$f(x) = b + cd^2(E(x) + E(-x) - 2) = -g(x),$
$k(x) = ch(x) = cd(E(x) - E(-x)),$
$f(x) = b + \frac{c_1}{2}A(x) + \frac{c_2}{4}A(x)^2 = -g(x),\ h(x) = c_1 + c_2A(x),$
$k(x) = A(x),$
$f(x) = d + cc_3E(x) - c(c_1 - c_3)E(-x) = -g(x),$
$h(x) = c_3E(x) + (c_1 - c_3)E(-x),$
$k(x) = c(E(x) - E(-x)).$

Case (ii). $h(0) \neq 0,\ k(0) \neq 0.$

(a) $f(x) = \frac{1}{4}c_2A(x)^2 + c_4A(x) + c_5,\ h(x) = A(x) + b,$
$g(x) = -\frac{1}{4}c_2A(x)^2 + c_6A(x) + c_7,\ k(x) = c_2A(x) + c_3,$
with $2c_4 = c_3 + bc_2,\ 2c_6 = c_5 - bc_2,\ c_5 + c_7 = bc_3;$

(b) $f(x) = c_{10}(E(x) + E(-x)) + c_{11}(E(x) - E(-x)) + c_5,$
$g(x) = c_{13}(E(x) + E(-x)) + c_{12}(E(x) - E(-x)) + c_7,$
$h(x) = \frac{d}{2}(E(x) + E(-x)) + b(E(x) - E(-x)),$
$k(x) = c_8(E(x) + E(-x)) + c_9(E(x) - E(-x)),$
with $c_{10} = \frac{dc_8}{2} + bc_4,\ c_{11} = \frac{dc_9}{2} + bc_8,\ c_{13} = \frac{kc_8}{2} - bc_9,$
$c_{12} = bc_8 - \frac{kc_9}{2},\ c_5 + c_7 = 0,$ *where* A *is additive satisfies* (A), E *satisfies* (E), *and* b, c, d, *and* c_1 *to* c_{13} *are constants.*

Proof. We have to consider several cases.

Solution (i) is obvious. Suppose $h = c \neq 0,\ k \neq 0$. Equation (3.103) becomes

$$f(x + y) + g(x - y) = ck(y) \quad \text{for } x, y \in G.$$

With $x = y$, we get $f(2y) + g(0) = ck(y)$, so that we have

$$f(x + y) + g(x - y) = f(2y) + g(0) \quad \text{for } x, y \in G.$$

Set $x+y=u,\ x-y=v$ to obtain

$$f(u)+g(v)=f(u-v)+g(0),$$
$$f(0)+g(v)=f(-v)+g(0),$$

and

$$f(u)+f(-v)=f(u-v)+f(0),$$

which with $A(x)=f(x)-f(0)$ yields $A(u-v)=A(u)+A(-v)$, which is additive. This solution results in (ii) case (i) (and could have used the Pexider equation, too).

As for (ii) case (ii), assume $k=c\neq 0,\ h\neq 0$. Then

$$f(x+y)+g(x-y)=ch(x);$$

that is, $f(x+y)+g(x-y)=f(2x)+g(0)$ (set $y=x$ in the equation above).

As before, the equation reduces to

$$f(u)+g(v)=f(u+v)+g(0),$$

which is a Pexider equation, and

$$f(u)=A(u),\quad g(u)=A(u)+g(0),\quad ch(x)=2A(x)+g(0).$$

This gives the solution (ii) case (ii).

Let us now consider the case $f=0, g, h, k\neq 0$. Then (3.103) takes the form

$$g(x-y)=h(x)k(y)\quad \text{for } x,y\in G.$$

This is a Pexider equation. With $x=0$ and $y=0$, separately we have

$$g(-y)=h(0)k(y),\quad g(x)=h(x)k(0),$$

so that

$$g(x-y)=g(x)g(-y)\frac{1}{h(0)k(0)}$$

($h(0)=0$ or $k(0)=0$ imply $g=0$, which is not the case). Now $g(x)=dE(x)$, where E satisfies (E). This yields the solution (iii) case (i). Similarly, $g=0,\ f,h,k\neq 0$ gives the solution case (ii) of (iii).

Finally, we treat the case where f,g,h,k are not constants. We will show how (VS) and (3.27) arise naturally to determine the solution (iv).

First we consider the case $h(0)=0$ or $k(0)=0$. Suppose $h(0)=0$. Then $x=0$ in (3.103) gives

$$f(y)+g(-y)=0$$

and

$$f(y+x)-f(y-x)=h(x)k(y),$$

which is (VS). By Theorem 3.59, this gives the solution (iv) case (i) (a).

When $k(0) = 0,\ y = 0$ in (3.103) shows that

$$f(x) + g(x) = 0$$

and

$$f(x + y) - f(x - y) = h(x)k(y) \quad \text{for } x, y \in G.$$

This is (VS), and Theorem 3.59 yields the solution (iv) case (i) (b).

Lastly, we consider the subcase (ii), $h(0) \neq 0,\ k(0) \neq 0$. Set $y = x$ and $y = -x$ separately in (3.103) to get

$$f(2x) = h(x)k(x) + c_5, \tag{3.103b}$$

$$g(2x) = h(x)k(-x) + c_7, \quad \text{for } x \in G. \tag{3.103c}$$

From (3.103), (3.103b), and (3.103c), letting $x + y = 2u, x - y = 2v$ (using that G is 2-divisible), we have

$$\begin{aligned} h(u + v)k(u - v) &= f(2u) + g(2v) \\ &= h(u)k(u) + h(v)k(-v) + c_5 + c_7 \quad \text{for } u, v \in G. \end{aligned} \tag{3.104}$$

Let $v = u$ and $v = -u$ in (3.104) separately to have

$$h(2u)k(0) = h(u)(k(u) + k(-u)) + c_5 + c_7, \tag{3.105}$$

$$k(2u)h(0) = k(u)(h(u) + h(-u)) + c_5 + c_7, \quad \text{for } u \in G. \tag{3.105a}$$

Interchange u and v in (3.104) to obtain

$$h(u + v)k(v - u) = h(v)k(v) + h(u)k(-u) + c_5 + c_7 \quad \text{for } u, v \in G. \tag{3.104a}$$

First add (3.104) and (3.104a) and use (3.105) and (3.105a) to get

$$\begin{aligned} h(u + v)2\varphi(u - v)k(0) &= h(u)2\varphi(u) + h(v)2\varphi(v) - 2c_5 - 2c_7 \\ &= h(2u)k(0) + h(2v)k(0), \end{aligned}$$

where

$$2\varphi(x)k(0) = k(x) + k(-x) \quad \text{for } x \in G. \tag{3.106}$$

Setting $u + v = x$ and $u - v = y$, using $k(0) \neq 0$, we have

$$h(x + y) + h(x - y) = 2h(x)\varphi(y) \quad \text{for } x, y \in G,$$

which is (3.27). From Theorem 3.41, it follows that the nonzero solutions are

$$\varphi \equiv 1,\ h \text{ is Jensen (J); that is, } h(x) = A(x) + b, \tag{3.107}$$

$$\begin{cases} \varphi(x) = \dfrac{(k(x) + k(-x))}{2k(0)} = \dfrac{E(x) + E(-x)}{2}, \\ h(x) = \dfrac{d}{2}(E(x) + E(-x)) + b(E(x) - E(-x)), \end{cases} \tag{3.107a}$$

where A satisfies (A) and E satisfies (E).

First let us consider the case (3.107a) $h(x) = A(x) + b$ and obtain the solution (iv) case (ii) (a).

Subtracting (3.104a) from (3.104) and utilizing (3.106) with $\varphi = 1$, we get

$$A(u+v)(k(u-v)-k(0)) = A(u)(k(u)-k(0)) - A(v)(k(v)-k(0))$$

(cf. (3.95a)). Replace u by $u+v$ in the equation above to obtain

$$\alpha(u) + 2A(v)(k(u)-k(0)) = \alpha(u+v) - \alpha(v),$$

where $\alpha(x) = A(x)(k(x)-k(0))$ for $x \in G$.

Interchanging u and v yields

$$A(v)(k(u)-k(0)) = A(u)(k(v)-k(0));$$

that is,

$$k(u) - k(0) = b_1 A(u) \quad \text{for } u \in G,$$

where b_1 is constant.

These values of h and k in (3.104) and (3.104a) give f and g and produce the solution (iv) case (ii) (a).

Finally, we tackle (iv) case (ii) (b). Now (3.107a) holds.

Subtracting (3.104a) from (3.104), we get

$$h(u+v)\alpha(u-v) = h(u)\alpha(u) - h(v)\alpha(v), \quad u, v \in G, \tag{3.108}$$

where $\alpha(u) = k(u) - k(-u)$ for $u \in G$ is odd (cf. (3.95a) and (3.95)).

Change v to $-v$ in (3.108) and use that α is odd to obtain

$$\begin{aligned} h(u+v)\alpha(u-v) - h(u-v)\alpha(u+v) &= -\alpha(v)(h(v)+h(-v)) \\ &= -h(0)\alpha(2v) \quad (\text{put } u = v \text{ on the left side}), \end{aligned}$$

which by letting $u+v = x$ and $u-v = y$ becomes

$$h(0)\alpha(x-y) = -h(x)\alpha(y) + \alpha(x)h(y) \quad \text{for } x, y \in G,$$

which is (3.73a). Change y to $-y$ and add to have

$$h(0)(\alpha(x+y) + \alpha(x-y)) = \alpha(x)(h(y)+h(-y)),$$

which is (3.27) with $2\varphi(y) = \frac{h(y)+h(-y)}{h(0)}$.

We know from (3.107a) that $\varphi(y) = E(y) + E(-y)$, so we need only $\alpha(x)$ from Theorem 3.21:

$$\alpha(x) = k(x) - k(-x) = \frac{d_1}{2}(E(x) + E(-x)) + b_1(E(x) - E(-x)).$$

From $\alpha(x)$ and (3.106) with (3.107a), we get $k(x)$. Then (3.103b) and (3.103c) determine f and g. This solution results in (iv) case (ii) (b). This completes the proof of this theorem. □

3.4.14 Levi-Civitá Functional Equation, Convolution Type Functional Equations, and Generalization of Cauchy-Pexider Type and d'Alembert Equations

The object now is to discuss the most general functional equations

$$f(x+y)=\sum_{i=1}^{n} h_i(x)k_i(x) \quad \text{(Levi-Civitá equation (Székelyhidi [794]))} \tag{3.109}$$

and

$$f(x+y)+g(x-y)=\sum_{i=1}^{n} h_i(x)k_i(y) \quad \text{for } x, y \in G, \tag{3.109a}$$

$$f(x+y)g(x-y)=\sum h_i(x)k_i(y) \quad \text{for } x, y \in G, \tag{3.109b}$$

where f, g, h_i, k_i ($i = 1$ to n) are complex-valued functions defined on a group G. Equations of the form (3.109), (3.109a), and (3.109b) have been studied by many authors in [44, 9, 182, 630, 631, 551, 594, 596, 717, 796, 653, 295, 790, 747, 787, 761, 794, 818], etc. The equation (3.109) is a common generalization of Cauchy-Pexider (additive, exponential) equations, whereas equation (3.109a) includes the above and more—Jensen, cosine, sine, (VS), (3.102), (Q), square norm equations, etc.

The functional equation (3.109), known as the Levi-Civitá equation (Székelyhidi [787]), has been studied by many authors under various assumptions, such as differentiability, measurability, etc. The earliest general results concerning (3.109) are due to Vincze [818]. He introduced the method of determinants to solve special cases of (3.109). In McKiernan [630], it was shown that (3.109) is closely related to the matrix equation (E) and can be solved by simultaneous diagonalizations of the matrix $E(x)$, and the solutions can be expressed in terms of additive and exponential functions where the additive functions satisfy additional functional equations. Also McKiernan [630, 631] obtained the general solution of (3.109) under some conditions, and in Sinopoulos [747] it is shown that the solution is an exponential polynomial.

In Székelyhidi [797], it was shown that (3.109a) has significant applications in signal processing. Notice that the left side of (3.109a) is the general solution to the wave equation, while the right side is the inner product of vectors $h(x) = (h_1(x), \ldots, h_n(x))$, $k(x) = (k_1(x), \ldots, k_n(x))$. We will consider this inner product form a little later.

In Rukhin [717], the author obtains conditions for the solutions of (3.109a) to be matrix elements of a finite-dimensional representation of the group. Whereas in Székelyhidi [790] the author tackles the problem of determining the general solution of (3.109a) raised by Aczél in [9], Székelyhidi and Aczél obtain the solution of (3.109a) by reducing it to matrix equation (E) and (C) as exponential polynomials (see inner product version). Here are some interesting results.

Let $M_n(\mathbb{C})$ denote all $n \times n$ matrices with complex entries.

Result 3.65. [747, 630, 631]. *Let G be a topological Abelian group and $f, h_i, k_i : G \to \mathbb{C}$ $(i = 1, 2, \ldots, n)$ be continuous solutions of* (3.109)*. Then f is an exponential polynomial.*

Result 3.65a. [787]. *Let G be an Abelian group, n a positive integer, and $f, k_i, h_i : G \to \mathbb{C}$ be functions $(i = 1, 2, \ldots, n)$ where both the sets $(k_1, k_2, \ldots, k_n)$ and $(h_1, h_2, \ldots, h_n)$ are linearly independent. The functions f, k_i, h_i $(i = 1, 2, \ldots, n)$ form a nondegenerate solution of* (3.109) *if and only if there exist positive integers $k, n_1, n_2, \ldots, n_k$ with $n_1 + n_2 + \ldots + n_k = n$, different exponentials $E_1, E_2, \ldots, E_k$, and for every $j = 1, 2, \ldots, k$ there exist linearly independent sets $(A_{j1}, A_{j2}, \ldots, A_{jn_j-1})$ of additive functions and polynomials P_j, Q_{ij}, R_{ij} $(i = 1, 2, \ldots, n)$ in $n_j - 1$ complex variables and of degree at most $n_j - 1$ such that we have for all x in G*

$$f(x) = \sum_{j=1}^{n} P_j(A_{j1}(x), A_{j2}(x), \ldots, A_{jn_j-1}(x))E_j(x),$$

$$k_i(x) = \sum_{j=1}^{n} Q_{ij}(A_{j1}(x), A_{j2}(x), \ldots, A_{jn_j-1}(x))E_j(x),$$

$$h_i(x) = \sum_{j=1}^{n} R_{ij}(A_{j1}(x), A_{j2}(x), \ldots, A_{jn_j-1}(x))E_j(x).$$

Result 3.66. [790]. *Let G be a topological group and $f : G \to M_n(\mathbb{C})$ be a continuous solution of* (C) *with $f(0) = I$, and $f(x_0)^2 - I$ is invertible for some $x_0 \in G$. Then, there is an $E : G \to M_n(\mathbb{C})$ satisfying* (E) *such that*

$$f(x) = \frac{1}{2}[E(x) + E(-x)]$$

holds for all $x \in G$ (see Kannappan [424]).

The proof follows the method in [424].

Result 3.66a. [790]. *Let G be a 2-divisible topological Abelian group and $f : G \to M_n(\mathbb{C})$ be a continuous function satisfying* (C). *If $f(0) = I$, then every component of f is an exponential polynomial.*

Result 3.67. [790]. *Let G be a 2-divisible, locally compact Abelian group and $f, g, h_i, k_i : G \to \mathbb{C}$ be measurable solutions of* (3.109a)*. Then f and g are exponential polynomials.*

3.4.15 Operator-Valued Solution of (C)

As mentioned earlier, the d'Alembert functional equation (C) has a long history, studied under different conditions, with a rich literature. Also, the operator-valued

versions have been studied by several authors, including [181, 621, 421, 282, 796]. Let $L(H)$ denote the set of all bounded linear operators in a Hilbert space H and $N(H)$ all normal operators on H.

A mapping $f : G \to L(H)$ satisfying the equation (C) with $f(0) = I$ is called a *cosine operator function*, and that of $f : G \to N(H)$ satisfying (C) is called a *normal cosine operator function* where G is (an Abelian) a group.

Result 3.68. (Stäckel [763]). *Let G be an Abelian group and H be a Hilbert space. Suppose $f : G \to N(H)$ is a normal cosine operator satisfying* (C)*. Then there is a homomorphism* (*exponential*) $E : G \to N(H)$ (*multiplicative semigroup*) *such that*

$$f(x) = \frac{1}{2}(E(x) + E(-x)) \tag{3.110}$$

holds for all $x \in G$ (*see* [424])*; that is, f is an even part of a homomorphism of G into the multiplicative semigroup of $N(H)$.*

Cosine operator functions have also been studied on non-Abelian groups, where usually the commutativity of G is replaced by the condition (K) (see [424]).

Result 3.68a. [763]. *Let G be a group and H a Hilbert space. Then any operator $f : G \to N(H)$ satisfying* (C) *with* (K) *has the form* (3.110)*.*

Corollary 3.68b. *Let G be a group, H_0 an inner product space, H its completion, and $f : G \to N(H_0)$ a solution of* (C) *with* (K)*. Then f has the form* (3.110)*, where $E : G \to N(H)$ is a homomorphism.*

In applications, we frequently encountered many specific cases of equation (3.109a); for example, Jensen, quadratic, d'Alembert, or cosine, and Wilson (see [12, 40, 398, 424, 830, 831]). The general solutions of these equations are usually obtained by solving them using procedures tailored to the particular equation; that is, there is no universal scheme to cover many of them at the same time. To extract the solution to these special cases from [717, 790] is not simple. The frequent occurrences of these and similar equations warrants the listing of general solutions to special equations of lower order n. With this in mind, we now treat the functional equation

$$f(x + y) + f(x - y) = f(x)g(y) + g(x)f(y) \tag{3.111}$$

in Result 3.69a. In the process, we solve some key special cases and encounter some conditional Cauchy equations, which is an area that has received much attention lately (see [218, 422]).

3.4.16 Solution of Equation (3.111)

Result 3.69. *Suppose $E : G \to \mathbb{C}^*$ is exponential. Let $f : G \to \mathbb{C}$ be a mapping satisfying the functional equation*

$$f(x + y) + f(x - y) = (E(x) + E(-x))f(y) + f(x)(E(y) + E(-y)) \tag{3.112}$$

for $x, y \in G$ with

(K) $$f(x+y+z) = f(x+z+y) \quad \textit{for all } x, y, z \in G.$$

Then f must be of the form

$$f(x) = A(x)(E(x) + E(-x)) \quad \textit{when } E(x) \neq E(-x)$$

or

$$f(x) = E_0(x)B(x,x) \quad \textit{when } E(x) = E(-x) = E_0(x) \quad (\textit{say})$$

and $A : G \to \mathbb{C}$ is additive and $B : G \times G \to \mathbb{C}$ is biadditive.

The converse is also true.

Result 3.69a. *Suppose f $(\neq 0)$, $g : G \to \mathbb{C}$ are mappings satisfying the functional equation* (3.111) *with* (K)*. Then they have one of the following forms:*

$$\begin{cases} f(x) = A(x)(E(x) - E(-x)), & \textit{when } E(x) \neq E(-x), \\ g(x) = E(x) + E(-x), \end{cases}$$

$$\begin{cases} f(x) = E_0(x)B(x,x), & \textit{when } E(x) = E(-x) = E_0(x) \quad (\textit{say}), \\ g(x) = 2E_0(x), \end{cases}$$

$$\begin{cases} f(x) = \frac{1}{2k}[E_1(x) + E_1(-x) - E_2(x) - E_2(-x)], & k \neq 0, \\ g(x) = \frac{1}{2}[E_1(x) + E_1(-x) + E_2(x) + E_2(-x)], & E_1 \neq E_2, \end{cases}$$

where A is additive, $E_2 : G \to \mathbb{C}$, $E, E_1 : G \to \mathbb{C}^*$ are exponential, and $B : G \times G \to \mathbb{C}$ is biadditive. Conversely, for nonzero A and B, f and g given above satisfy (3.111) *with nonzero f.*

3.4.17 Inner Product Version of (3.109)

First of all, (3.109) can be rewritten as

$$f(x+y) = \langle h(x), \overline{k(y)} \rangle \quad \text{for } x, y \in G,$$

where $h(x) = (h_1(x), \ldots, h_n(x))$, $k(x) = (k_1(x), \ldots, k_n(x))$, and $\langle\ ,\ \rangle$ is the standard inner product in $\mathbb{C}^n$, where f, h, k are complex-valued functions on G.

In O'Connor [653], the author determined the continuous solutions of the functional equation

(3.109c) $$f(x-y) = \sum_{k=1}^{n} a_k(x)\overline{a_k(y)} \quad \text{for } x, y \in G,$$

where $f, a_k : G \to \mathbb{C}$ as

$$f(x) = \sum_{k=1}^{n} b_k \mathcal{X}_k(x),$$

where $\mathcal{X}_k$ are continuous characters of G and b_k are constants. This result was obtained by means of the Bochner representation theorem for positive definite functions. In Gajda [295], the author found the forms of $a_k s$ also. Equation (3.109c) can be put in the form

$$f(x-y) = \langle \alpha(x), \alpha(y) \rangle \quad \text{for } x, y \in G, \tag{3.109d}$$

where $\alpha(x) = (a_1(x), \ldots, a_n(x))$ is a mapping $\alpha : G \to \mathbb{C}^n$. If $\mathbb{C}^n$ is replaced by a Hilbert space H over $\mathbb{C}$, solutions of (3.109d) can be described in terms of a unitary representation of G.

Recall that a homomorphism $\varphi : G \to \mathcal{U}(H)$ (all unitary operators on H) is called a *unitary representation* of a topological group G.

A unitary representation U is said to be *continuous* provided, that for every $s \in H$, the transformation $\beta : G \to H$ given by $\beta(x) = U(x)s$, for $x \in G$, is continuous.

An element $s_0 \in H$ is called a *cyclic vector* of a representation U provided

$$c\ell \operatorname{span}\{U(x)s : x \in G\} = H,$$

where $c\ell$ stands for the closure operator in H and $\operatorname{span}\{S\}$ means the linear space generated by a subset S of H.

Result 3.70. [295]. *Let G be a topological group and H be a Hilbert space over $\mathbb{C}$. Suppose that $f : G \to \mathbb{C}$ and $\alpha : G \to H$ are continuous functions. Then f and α satisfy* (3.109d) *if and only if there exists a continuous unitary representation U of the group G in the space $H_0 := c\ell \operatorname{span}\{\alpha(G)\}$ with the cyclic vector $s_0 = \alpha(0)$ such that*

$$\begin{aligned} \alpha(x) &= U(x)s_0, \qquad x \in G, \\ f(x) &= \langle U(x)s_0, s_0 \rangle, \quad x \in G. \end{aligned}$$

Let $M_{mn}(\mathbb{C})$ denote the set of all $m \times n$ matrices over $\mathbb{C}$ and $\delta_{i\,j}$, the Kronecker delta given by

$$\delta_{i\,j} = \begin{cases} 0, & \text{for } i \neq j, \\ 1, & \text{for } i = j. \end{cases}$$

Result 3.71a. [295]. *Let G be an Abelian topological group. Then continuous functions $f, \alpha_j : G \to \mathbb{C}$ $(j = 1, \ldots, n)$ satisfy* (3.109d) *if and only if there exist continuous characters $\mathcal{X}_1, \ldots, \mathcal{X}_m$ of the group G $(m \leq n)$, complex numbers $b_1, \ldots, b_m$, and a matrix $\beta \in M_{mn}(\mathbb{C})$ such that*

$$\sum_{j=1}^{n} \beta_{i\,j} \overline{\beta_{k\,j}} = \delta_{ik} \quad i, k = 1, \ldots, m,$$

$$f(x) = \sum_{i=1}^{m} |b_i|^2 \mathcal{X}_i(x), \quad x \in G,$$

$$\alpha_j(x) = \sum_{i=1}^{m} b_i \beta_{ji}(x), \qquad \text{for } x \in G,\ j = 1, \ldots, n.$$

3.4.18 A Functional Equation of d'Alembert's Type

We will now consider d'Alembert's equation in the general form (inner product version) [295]

$$f(x+y) + f(x-y) = 2\langle \alpha(x), \alpha(y) \rangle \quad \text{for } x, y \in G, \tag{3.113}$$

where $f : G \to \mathbb{C}$, $\alpha : G \to H$, G a group, and H a Hilbert space over $\mathbb{C}$.

Theorem 3.71. (See also [295].) *Let G be a topological group* (*not necessarily Abelian*) *and H a Hilbert space over $\mathbb{C}$. Suppose $f : G \to \mathbb{C}$ and $\alpha : G \to H$ are continuous solutions of equation* (3.113) *with f satisfying* (K) *and there exists a $z_0 \in G$ such that $\alpha(z_0) = 0$. Then*

$$f(x) = \langle \alpha(x), \alpha(0) \rangle \quad \text{for } x \in G, \tag{3.113a}$$

and there exists a continuous unitary representation U of G in $H_0 = c\ell\ \text{span}\{\alpha(G)\}$ with the cyclic vector $s_0 = \alpha(0)$ such that

$$\alpha(x) = \frac{1}{2}(U(x)s_0 + U(-x)s_0), \quad x \in G, \tag{3.113b}$$

and so

$$f(x) = \frac{1}{2}\langle U(x)s_0 + U(-x)s_0, s_0 \rangle, \quad x \in G. \tag{3.113c}$$

Conversely, if $f : G \to \mathbb{C}$ and $\alpha : G \to H$ are given by (3.113b) *and* (3.113c) *with some unitary representation U and cyclic vector s_0, then equation* (3.113) *holds.*

Proof. The converse is easy to check. Assume that (3.113) holds. Set $y = 0$ (identity in G) in (3.113) to get

$$f(x) = \langle \alpha(x), \alpha(0) \rangle \quad \text{for } x \in G. \tag{3.113a}$$

Note that no continuity is used.

Changing y to $-y$ in (3.113), we have

$$\langle \alpha(x), \alpha(y) \rangle = \langle \alpha(x), \alpha(-y) \rangle \quad \text{for } x, y \in G,$$

which implies that $\alpha(-y) = \alpha(y)$ for $y \in G$. So, α is even and then, from (3.113a), $f(x) = f(-x)$, $x \in G$.

Now, set $x = 0$ in (3.113) to obtain

$$f(y) = \langle \alpha(0), \alpha(y)\rangle = \overline{\langle \alpha(y), \alpha(0)\rangle} = \overline{f(y)}.$$

Condition (K) on f implies that (i) α satisfies (K) (use (3.113)) and (ii) $f(x+y) = f(y+x)$ for $x, y \in G$, which also follows from taking conjugates on both sides of (3.113),

$$\langle \alpha(x), \alpha(y)\rangle = \overline{\langle \alpha(x), \alpha(y)\rangle} = \langle \alpha(y), \alpha(x)\rangle,$$

and use (3.113) again.

Setting $y = z_0$ in (3.113) and using $\alpha(z_0) = 0$, we have

$$\begin{aligned} &f(x+z_0) + f(x-z_0) = 0 \\ &\text{or} \quad f(x+2z_0) + f(x) = 0 \\ &\text{or} \quad f(x) + f(x-2z_0) = 0 \quad \text{for all } x \in G. \end{aligned} \tag{3.114}$$

From (3.113), (3.114), and (K), for $x, y \in G$ we obtain

$$\begin{aligned} &\langle \alpha(x), \alpha(y)\rangle + \langle \alpha(x+z_0), \alpha(y+z_0)\rangle \\ &\quad = \frac{1}{2}[f(x+y) + f(x-y) + f(x+z_0+y+z_0) + f(x+z_0-z_0-y)] \\ &\quad = \frac{1}{2}[f(x+y) + f(x-y) - f(x+y) + f(x-y)] = f(x-y). \end{aligned} \tag{3.115}$$

Define $g : G \to \mathbb{C}$, $\beta : G \to H$ by

$$g(x) = f(x) + if(x+z_0) \quad \text{for } x \in G, \tag{3.116}$$

$$\beta(x) = \alpha(x) + i\alpha(x+z_0) \quad \text{for } x \in G. \tag{3.116a}$$

Note that $g(x+y) = g(y+x)$ and $\beta(x+y) = \beta(y+x)$ for $x, y \in G$, and g and β satisfy (K).

Using (3.115) and (3.114), for $x, y \in G$,

$$\begin{aligned} g(x-y) &= f(x-y) + if(x-y+z_0) \\ &= \langle \alpha(x), \alpha(y) =\rangle + \langle \alpha(x+z_0), \alpha(y+z_0)\rangle \\ &\quad + i\langle \alpha(x+z_0), \alpha(y)\rangle + \langle \alpha(x+2z_0), \alpha(y+z_0)\rangle \\ &= \langle \alpha(x) + i\alpha(x+z_0), \alpha(y) + i\alpha(y+z_0)\rangle \\ &= \langle \beta(x), \beta(y)\rangle, \end{aligned} \tag{3.116}$$

which is the same as (3.109d). By Result 3.70, there is a continuous, unitary representation U of G in $H_0 = c\ell\ \text{span}\{\beta(G)\}$ with cyclic vector $s_0 = \beta(0) = \alpha(0)$ such that

$$\beta(x) = U(x)s_0, \tag{3.116b}$$

$$g(x) = \langle U(x)s_0, s_0\rangle, \quad \text{for } x \in G. \tag{3.116c}$$

In view of (3.113), (3.114), and the (K) condition on f,

$$\begin{aligned} 2\langle \alpha(x+z_0)+\alpha(x-z_0), \alpha(y)\rangle &= f(x+z_0+y)+f(x+z_0-y) \\ &\quad + f(x-z_0+y)+f(x-z_0-y) \\ &= f(x+y+z_0)+f(x+y-z_0)+f(x-y+z_0) \\ &\quad + f(x-y-z_0) \\ &= 0 \quad \text{for all } x, y \in G. \end{aligned}$$

Thus

$$\alpha(x+z_0)+\alpha(x-z_0)=0 \quad \text{for all } x \in G. \tag{3.117}$$

Now, turning to the definition of β in (3.116a), using (3.117) we see that

$$\beta(x)+\beta(-x)=\alpha(x)+\alpha(-x)=2\alpha(x) \quad \text{for } x \in G,$$

$\operatorname{span}\{\alpha(G)\} = \operatorname{span}\{\beta(G)\}$, and so $s_0 \in c\ell \operatorname{span}\{\alpha(G)\}$. Now, (3.113b) follows from (3.116b), and (3.113c) follows from (3.113a) and (3.113b). This completes the proof of this theorem. □

Finally in this chapter, we can deduce from Theorem 3.71 the following result.

Result 3.72. [295]. *Let G be an Abelian topological group and let $f, \alpha_j : G \to \mathbb{C}$ $(j = 1, \ldots, n)$ be continuous functions. Moreover, suppose that there exists a $z_0 \in G$ such that*

$$\alpha_j(z_0) = 0 \quad \textit{for } j = 1, \ldots, n.$$

If the functions f and α_j $(j = 1, \ldots, n)$ satisfy the equation

$$f(x+y)+f(x-y)=2\sum_{j=1}^{n}\alpha_j(x)\overline{\alpha_j(y)}, \quad x, y \in G, \tag{3.118}$$

then there exist continuous characters $\mathcal{X}_1, \ldots, \mathcal{X}_m$ of the group G $(m \le n)$, complex numbers $b_1, \ldots, b_n$, and a matrix $\beta_{i\,j} \in M_{mn}(\mathbb{C})$ such that

$$\sum_{j=1}^{n}\beta_{i\,j}\beta_{k\,j}=\delta_{i\,k}, \qquad i, k = 1, \ldots, m;$$

$$f(x)=\sum_{i=1}^{m}|b_i|^2 \operatorname{Re}\mathcal{X}_i(x) \qquad x \in G;$$

$$\alpha_j(x)=\sum_{i=1}^{m} b_i\beta_{i\,j}\operatorname{Re}\mathcal{X}_i(x), \quad x \in G, \ j = 1, \ldots, n.$$

3.5 Survey—Summary of Stetkaer

We conclude this chapter with extracts from the beautiful, fascinating article (from a talk at the Functional Equation meeting held in Opava in June 2004 (ISFE42)) by Stetkaer [750].

3.5.1 Abstract

I will describe the solutions $f : G \to \mathbb{C}$ of classical functional equations such as those of d'Alembert, Wilson, and Jensen and the quadratic functional equation on a group G. Most of the previous studies assume that G is Abelian or at least that the solution f satisfies Kannappan's condition

$$\text{(K)} \qquad f(xyz) = f(xzy) \quad \text{for all } x, y, z \in G.$$

The emphasis will be on what can be said about the solutions if G belongs to certain classes of non-Abelian groups such as the nilpotent groups and the semisimple Lie groups and Kannappan's condition is no longer enforced.

Throughout this part, let G be a group with neutral element e. For $x, y \in G$, $[x, y] = xyx^{-1}y^{-1}$ denotes the commutator of x and y and $G' = [G, G]$ denotes the commutator subgroup of G generated by all commutators (see Remark 3.22).

Note that G' is a normal subgroup of G and that G/G' is Abelian.

Of course, if G is Abelian, then $G' = \{e\}$. At the other extreme, if G is a semisimple connected Lie group like $SL(2, \mathbb{R})$, then $G' = G$.

Given a subgroup H of G, we consider the coset space $G/H = \{xH \mid x \in G\}$. Let X be any set. A function $f : G \to X$ is a function on G/H if it is constant on each coset xH; i.e., if $f(xk) = f(x)$ for all $x \in G$ and $k \in H$. We will not distinguish between the function f on G and the corresponding function on G/H.

Definition. Let $f : G \to X$, where X is a set. We say that f satisfies *Kannappan's condition* if it is a function on G/G', or equivalently if (K) holds.

If G is Abelian, then any function on it satisfies Kannappan's condition.

If f is a function on G, we let f^* denote the function $f^*(x) = f(x^{-1})$, $x \in G$.

For any topological space X, we let $C(X)$, $C_b(X)$ and $C_c(X)$. denote the algebra of complex-valued continuous functions on X, the subalgebra of bounded functions, and the subalgebra of compactly supported functions.

G is a locally compact Hausdorff topological group with identity element e. The group operation will be written multiplicatively unless the group is Abelian; in that case, we use $+$. $C(G)$ denotes the algebra of all continuous complex-valued functions on G equipped with the topology of uniform convergence on compacta.

3.5.2 The Abelian Solution and Related Extensions of It

Due to the fact that the classical functional equations for many years have been studied intensively on Abelian groups, it is interesting that something new has emerged in the last five years. Sinopoulos [775] considered the following versions of d'Alembert's equation, Jensen's equation, and the quadratic equation, respectively:

$$\begin{aligned} g(x+y) + g(x+\sigma y) &= 2g(x)g(y), && x, y \in S, \\ g(x+y) + g(x+\sigma y) &= 2g(x), && x, y \in S, \\ g(x+y) + g(x+\sigma y) &= 2g(x) + 2g(y), && x, y \in S, \end{aligned}$$

where the underlying space S is a commutative semigroup (not necessarily a group) and σ is an endomorphism of S such that $\sigma(\sigma x) = x$ for all $x \in S$ (see also Sinopoulos [745]). The range of g is respectively a quadratically closed (commutative) field of characteristic different from 2, a 2-cancellative Abelian group, and an Abelian group uniquely divisible by 2. The solution formulas of Stetkaer [775] are the same as in Sinopoulos [743], so the new feature is that the solution formulas extend from groups to semigroups and from $\mathbb{C}$ to more general structures.

Many functional equations, such as d'Alembert's functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y), \quad x, y \in \mathbb{R}, \tag{C}$$

have been extensively studied on Abelian groups such as $G = \mathbb{R}$. Here $f : \mathbb{R} \to \mathbb{C}$ is a function that we want to know more about.

To set the stage for the further discussion, let me mention how the situation is on the real line: The continuous nonzero solutions of (C) are the functions of the form

$$f_\lambda(x) = \frac{e^{i\lambda x} + e^{-i\lambda x}}{2}, \quad x \in \mathbb{R},$$

where $\lambda \in \mathbb{C}$; i.e., the complex cosine functions (see Theorem 3.9).

The functional equation (C) begs to be generalized to any group G by the prescription

$$f(xy) + f(xy^{-1}) = 2f(x)f(y), \quad x, y \in G, \tag{CE}$$

where $f : G \to \mathbb{C}$ is a function that we want to know more about. We use the name *d'Alembert's functional equation* for the functional equation (CE).

We shall first consider the case where G is Abelian. If G is Abelian or just if f satisfies Kannappan's condition (K), then Kannappan in his seminal paper [424] from 1968 found that the nonzero solutions of (CE) are the functions of the form

$$f_\gamma = \frac{\gamma + \gamma^*}{2}, \tag{3.37}$$

where γ ranges over the homomorphisms $\gamma : G \to \mathbb{C}^*$. In the example of the continuous solutions on $G = \mathbb{R}$, we have $\gamma(x) = e^{i\lambda x}$ for $x \in \mathbb{R}$. The character γ is essentially unique: If $\gamma_1 : G \to \mathbb{C}^*$ is a homomorphism and

$$f_\gamma = \frac{\gamma + \gamma^*}{2} = \frac{\gamma_1 + \gamma_1^*}{2},$$

then either $\gamma_1 = \gamma$ or $\gamma_1 = \gamma^*$ (see Lemma 3.16).

Stetkaer [775, 770] has studied vector- and matrix-valued solutions of Wilson's functional equation on Abelian groups G such that $G = 2G$ (= 2-divisible groups). Let $\mathcal{F}$ denote an algebraically closed field of characteristic 0. He determines in [775] the general solution of

$$F(x+y) + F(x-y) = a(y)F(x), \quad x, y \in G,$$

where a is a 3×3 matrix-valued function and F a vector-valued function with linearly independent components, both with entries from $\mathcal{F}$. Using that, he can write all solutions $f : G \to \mathcal{F}$ of the scalar equation

$$f(x+y)+f(x-y) = g_1(x)h_1(y)+g_2(x)h_2(y)+g_3(x)h_3(y), \quad x, y \in G,$$

on a long list.

In [770] Stetkaer determines the general solution of $f : G \to \mathcal{F}$ of

$$f(x+y)-f(x-y) = \sum_{i=1}^{2} g_i(x)h_i(y), \quad x, y \in G.$$

The new feature is the minus on the left-hand side.

Let G be a locally compact, Hausdorff topological group. We let K be a compact Hausdorff topological transformation group of G acting by automorphisms on G. We let $k \cdot x$ denote the action by $k \in K$ on $x \in G$. The normalized Haar measure on K is denoted by dk. An important special case is $K = \mathbb{Z}_2 = \{\pm 1\}$, where

$$\int_K f(k)dk = \frac{f(1)+f(-1)}{2}.$$

The most recent result is by Shin'ya [734, Corollary 3.12], but important earlier works are due to Kannappan [424] and Chojnacki [156].

Theorem. (Kannappan [424], Czerwik [186], Sinopoulos [744]). *Let G be Abelian. If $\phi \in C(G)$ is a nonzero solution of*

$$\int_K \phi(x+k\cdot y)dk = \phi(x)\phi(y) \quad \textit{for all } x, y \in G,$$

then there exists a continuous homomorphism $\chi : G \to \mathbb{C}^*$ such that

$$\phi(x) = \int_K \chi(k\cdot x)dk \quad \textit{for all } x \in G.$$

If ϕ is bounded, then χ may be taken as a unitary character (i.e., a continuous homomorphism of G into the unit circle).

Let G now be a locally compact, Hausdorff topological group that is not necessarily Abelian, and let K be a compact subgroup of G. In classical harmonic analysis, you encounter not just functions on G/K but even functions $f : G \to \mathbb{C}$ that are K-bi-invariant; i.e., such that $f(k_1xk_2) = f(x)$ for all $k_1, k_2 \in K$ and $x \in G$, or equivalently are functions on $K\backslash G/K$. We say that (G, K) is a *Gelfand pair* if the convolution algebra $C_c(K\backslash G/K)$ is commutative. This is a weaker requirement than G being Abelian.

We note that the continuous (algebra) homomorphisms Φ of $C_c(G)$ onto $\mathbb{C}$ are the mappings of the form

$$\Phi : f \mapsto \int_G f(x)\phi(x)dx,$$

where $\phi : G \to \mathbb{C}$ ranges over the continuous (nonunitary) characters of G (i.e., over the continuous functions $\phi \neq 0$) such that ϕ satisfies (M). So this is an equivalent characterization of the continuous characters.

Let K be a compact subgroup of G. The continuous (algebra) homomorphisms $\Phi : C_c(K\backslash G/K) \to \mathbb{C}$ are the mappings of the form

$$\Phi : f \mapsto \int_G f(x)\phi(x)dx,$$

where $\phi : G \to \mathbb{C}$ ranges over the *spherical functions* (i.e., over the continuous functions $\phi \neq 0$) such that

$$\int_K \phi(xky)dk = \phi(x)\phi(y) \quad \text{for all } x, y \in G.$$

Here dk denotes the normalized Haar measure on the compact group K. So characters must be replaced by the more general notion of spherical functions, and the underlying space will be $K\backslash G/K$ rather than G.

This philosophy has been taken up by M. Akkouchi and E. Elqorachi and other Moroccan mathematicians. Kannappan's condition (K), which replaces the condition about G being Abelian, is in this setting substituted by

$$\text{(K}') \qquad \int_K \int_K f(xkyhz)dkdh = \int_K \int_K f(ykxhz)dkdh \quad \text{for all } x, y, z \in G.$$

As a sample, we mention two (corollaries of) results from [252, Theorem 2.1 and Theorem 2.2] that are in complete analogy to Kannappan's earlier result. It should be mentioned that the results of [252] are much more general.

Theorem. (Elqorachi and Akkouchi [257]). *Let $g \in C_b(G)$ be a nonzero solution of the generalized d'Alembert functional equation*

$$\int_K g(xky)dk + \int_K g(xky^{-1})dk = 2g(x)g(y), \quad \textit{for all } x, y \in G, \tag{3.119}$$

satisfying the generalized Kannappan condition (K′).

(i) *Then there exists $\phi \in C_b(G)$ satisfying* (3.119) *such that*

$$g = \frac{\phi + \phi^*}{2}.$$

(ii) *If*

$$g = \frac{\phi + \phi^*}{2} = \frac{\phi_1 + \phi_1^*}{2},$$

where $\phi, \phi_1 \in C_b(G)$ satisfy (3.119)*, then either $\phi_1 = \phi$ or $\phi_1 = \phi^*$.*

Proof. The proof goes along the same lines as Kannappan's original proof from [424].

Kannappan's condition is clearly satisfied if G is Abelian. A similar result holds in the present framework.

Result. [257]. *Let (G, K) be a Gelfand pair. Then any solution $g \in C_b(G)$ of the generalized d'Alembert functional equation*

$$\int_K g(xky)dk + \int_K g(xky^{-1})dk = 2g(x)g(y), \quad \text{for all } x, y \in G,$$

satisfies the generalized Kannappan condition (K′).

In a recent paper [130], B. Bouikhalene and E. Elqorachi examine the functional equation

$$\int_G \int_K f(xtk \cdot y)\overline{\chi}(k)dkd\mu(t) = g(x)h(y), \quad x, y \in G, \tag{3.120}$$

where μ is a Gelfand measure on G, dk denotes the normalized Haar measure of K, and χ is a character of K, and where $f \in C_b(G)$ satisfies the Kannappan type condition

$$\int_G \int_G f(zsxty)d\mu(s)d\mu(t) = \int_G \int_G f(zsytx)d\mu(s)d\mu(t) \tag{K″}$$

for all $x, y, z \in G$. The equation encompasses as special cases the functional equations

$$\int_G \int_K f(xtk \cdot y)dkd\mu(t) = f(x)f(y), \quad x, y \in G,$$

$$\int_K f(xk \cdot y)\overline{\chi}(k)dk = g(x)h(y), \quad x, y \in G,$$

$$\int_K f(xk \cdot y)\overline{\chi}(k)dk = g(x)f(y), \quad x, y \in G,$$

$$\int_G f(xty)d\mu(t) \pm \int_G f(xt\sigma(y))d\mu(t) = 2g(x)h(y), \quad x, y \in G,$$

where σ is a continuous involution of G.

A new feature of [130] is that K acts as a morphism on G, not necessarily as an automorphism in equation (3.120).

In [769], Stetkaer considers

$$\int_K f(x(k \cdot y))\overline{\chi(k)}dk = \sum_{\ell=1}^{N} g_\ell(x)h_\ell(y), \quad x, y \in G, \tag{3.121}$$

in which $f, g_1, \ldots, g_N, h_1, \ldots, h_N \in C(G)$ are unknown functions. Here $\chi : K \to \{z \in \mathbb{C} \mid |z| = 1\}$ is a continuous homomorphism of the group K into the unit circle. So G, K, χ, and N are given in advance.

Stetkaer [769, Theorem 4.8] shows that each solution of (3.121) can be expressed in terms of an $N \times N$ matrix-valued K-spherical function (i.e., a continuous map $\Phi \in C(G, M(N, \mathbb{C}))$ such that

$$\int_K \Phi(x(k \cdot y))dk = \Phi(x)\Phi(y), \quad \text{for all } x, y \in G, \text{ and } \Phi(e) = I.$$

The paper studies the matrix-valued K-spherical functions associated with the solutions of equation (3.121) and applies this knowledge to the solutions of various functional equations of the form (3.121).

3.5.3 d'Alembert's Functional Equation

As mentioned above, d'Alembert's functional equation on a group G is the functional equation (CE), where $g : G \to \mathbb{C}$ is a function that we want to know more about. From Kannappan's paper [424], we know its solutions if G is Abelian or if just g satisfies Kannappan's condition (K). If G is not Abelian, the solution set is not known in general.

If $\gamma : G \to \mathbb{C}^*$ is a character, then the function $g = (\gamma + \gamma^*)/2$ is a nonzero solution of d'Alembert's functional equation on G. We will call solutions of this special form and the zero solution *classical solutions.* The constant function $g = 1$ is an example of a classical solution on any group, and it is on some groups the only classical solution apart from zero; for example, on simple groups and connected semisimple Lie groups because $G = [G, G]$ for such groups.

Example 1. The function $g(x) = \frac{1}{2}\mathrm{tr}(x)$, $x \in SL(2, \mathbb{C})$ (i.e., the normalized trace) is a solution of d'Alembert's functional equation on the group $G = SL(2, \mathbb{C})$, as an elementary computation reveals. Apart from the (trivial) classical solution $g(x) = 1$ that exists on any group, I know of no other nonzero solutions of d'Alembert's functional equation on $SL(2, \mathbb{C})$.

Example 2. Ng [40, Remark 5] gives another example of a nonclassical solution g of d'Alembert's functional equation. The group is the group of unit quaternions $\{a+bi+cj+dk \mid a^2+b^2+c^2+d^2 = 1\}$, and the function is $g(a+bi+cj+dk) = a$.

Actually, the two examples are related. Viewing the group of unit quaternions as $SU(2) \subseteq SL(2, \mathbb{C})$, it turns out that g is the restriction of the normalized trace function $\frac{1}{2}\mathrm{tr}$ from Example 1 to $SU(2)$. Restricting still further to the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, we get Corovei's [173] example of a nonclassical solution of d'Alembert's functional equation. It is the only such one on Q_8.

The groups $SL(2, \mathbb{C})$ and $SU(2)$ in Examples 1 and 2 above are connected semisimple Lie groups, so they are far from being Abelian. Indeed, $[G, G] = G$, so the commutator subgroup is as large as it can be. For nilpotent groups, the situation is the converse because the commutators by definition become successively smaller and after a finite number of steps reduce to $\{e\}$. An example is the Heisenberg group $G = H_3(\mathbb{R})$, defined by

$$H_3(\mathbb{R}) = \left\{ (x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\},$$

where $[G, G] = Z(G)$ (the centre of G), so $[G, [G, G]] = \{e\}$. The Heisenberg group is an example of a connected nilpotent Lie group (see examples in the section "Solutions of (CE) on a non-Abelian group" not of the form (3.37)).

Result. (de Place Friis [292]). *Let G be a connected nilpotent Lie group and K a quadratically closed field with characteristic different from* 2. *Any nonzero solution $g : G \to K$ of d'Alembert's functional equation* (CE) *or even of the more general functional equation*

$$g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y), \quad x, y \in G, \tag{3.122}$$

is classical; i.e., there exists a homomorphism $\gamma : G \to \mathbb{C}^*$ such that (3.37) *holds.*

If G is not a connected nilpotent Lie group, then new phenomena occur, even for step 2 nilpotent groups (i.e., for groups G such that $[G, [G, G]] = \{e\}$). We have already observed this for $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ (the quaternion group). Here g_0, defined by $g_0(\pm 1) = \pm 1$, $g_0(\pm i) = g_o(\pm j) = g_0(\pm k) = 0$, is a nonclassical solution of d'Alembert's functional equation (actually the only one).

Proposition. [771, 266]. *On any step* 2 *nilpotent group, the solutions of d'Alembert's long functional equation* (3.122) *are the same as the solutions of d'Alembert's functional equation.*

3.5.4 Wilson's Functional Equation

By *Wilson's functional equation* on the group G we mean the functional equation

$$f(xy) + f(xy^{-1}) = 2f(x)g(y), \quad x, y \in G \tag{3.27}$$

(see Theorems 3.11 and 3.41), where $f, g : G \to \mathbb{C}$ are to be determined. For $g = f$, Wilson's functional equation becomes (C), but there are more relations between the two functional equations.

Theorem. [292]. *Let G be a connected nilpotent Lie group. Let the pair $f, g : G \to \mathbb{C}$ be a solution of Wilson's functional equation such that $f \neq 0$. Assume finally that g is not identically* 1. *Then there exist a homomorphism $\gamma : G \to \mathbb{C}^*$ and constants $a, b \in \mathbb{C}$ such that*

$$f = a\frac{\gamma + \gamma^*}{2} + b\frac{\gamma - \gamma^*}{2}, \quad g = \frac{\gamma + \gamma^*}{2}.$$

Conversely, if f and g have this form, then the pair f, g is a solution of Wilson's functional equation.

So the formulas for the solutions are the same as on an Abelian group except possibly for the case of $g = 1$. The special case of $g = 1$ reduces Wilson's functional equation to Jensen's functional equation.

Result. [776]. *Let G be a step 2 nilpotent group. Let the pair $f, g : G \to \mathbb{C}$ be a solution of Wilson's functional equation* (3.27)*, in which g has the form $g = (\gamma + \gamma^*)/2$ for some homomorphism $\gamma : G \to \mathbb{C}^*$. Then:*

(a) *If $\gamma \neq \gamma^*$, then f has the form*

$$f = a\frac{\gamma + \gamma^*}{2} + b\frac{\gamma - \gamma^*}{2}$$

for some constants $a, b \in \mathbb{C}$.

(b) *If $\gamma = \gamma^*$, then f has the form $f = F\gamma$, where $F : G \to \mathbb{C}$ is a solution of Jensen's functional equation on G, i.e., $F(xy) + F(xy^{-1}) = 2F(x)$ for all $x, y \in G$.*

If $\gamma = \gamma^*$, then $\gamma = 1$ on groups that are generated by their squares, which is the case for connected Lie groups.

If g is nonclassical, we obtain the following result, in which g_0 denotes the solution of d'Alembert's functional equation on the quaternion group Q_8 given by $g_0(\pm 1) = \pm 1,\ g_0(\pm i) = g_0(\pm j) = g_0(\pm k) = 0$.

3.5.5 A Variant of Wilson's Functional Equation

In this section, we discuss the functional equation

$$f(xy) + f(y^{-1}x) = 2f(x)g(y), \quad x, y \in G, \tag{3.123}$$

where $f, g : G \to \mathbb{C}$ are functions on G for which we want information. The functional equation (3.123) is very similar in appearance to Wilson's functional equation (3.27) but differs from it in the second term, where the new equation has $f(y^{-1}x)$, while Wilson's equation has $f(xy^{-1})$. The two versions of course agree if G is Abelian.

The Heisenberg group with integer entries shows that equation (3.123) and Wilson's equation (3.27) may have different solutions (see [768, Example IX.2]). Actually, neither solution set is contained in the other.

Result. [768]. *Let the pair $(f, g) : G \to \mathbb{C}$ be a solution of equation* (3.123) *such that $f \neq 0$. Define $m_g : G \to \mathbb{C}$ by $m_g(x) = 2g(x)^2 - g(x^2),\ x \in G$. Then*

(a) *$f(xyx^{-1}) = m_g(x)f(y)$ for all $x, y \in G$.*

(b) *m_g is a homomorphism of G into the multiplicative group $\{\pm 1\}$.*

The homomorphism m_g is in many instances identically 1*. Then $m_g \equiv 1$ if one of the following conditions holds:*

(c) *G is Abelian.*

(d) *G is generated by its squares; i.e., $G = \langle \text{squares} \rangle$. This is the case if G is a connected Lie group.*
(e) *G is a finite group of odd order.*
(f) *G is a connected topological group and g is continuous.*

Then either

(1) *$\{f, g\} = \{cg, g\}$, where $c \in \mathbb{C}^*$ and g is a nonzero solution of d'Alembert's functional equation that does not satisfy Kannappan's condition, or*
(2) *there exists a homomorphism $\chi : G \to \mathbb{C}^*$ such that $g = (\chi + \chi^*)/2$. This fixes the homomorphism χ except for the interchange of χ and χ^*.*
   (i) *If $\chi \neq \chi^*$, then*
$$f = a\frac{\chi + \chi^*}{2} + b\frac{\chi - \chi^*}{2}$$
   for some constants $a, b \in \mathbb{C}$.
   (ii) *If $\chi = \chi^*$, then $f = (\alpha + A)\chi$, where $\alpha \in \mathbb{C}$ and $A : G \to \mathbb{C}$ is additive.*

Conversely, any pair $\{f, g\}$ of functions of the forms described above satisfies (3.123).

So we get the complete solution of the variant (3.123) of Wilson's functional equation if only we can solve d'Alembert's functional equation on G.

3.5.6 Other Equations

The trigonometric addition formulas

$$f(xy) = f(x)g(y) + f(y)g(x), \quad x, y \in G, \tag{3.73}$$
$$g(xy) = g(x)g(y) + f(x)f(y), \quad x, y \in G, \tag{3.72a}$$

were solved in [162] during the discussion of the cosine-sine functional equation

$$f(xy) = f(x)g(y) + f(y)g(x) + h(x)h(y), \quad x, y \in G. \tag{3.86}$$

Poulsen and Stetkaer [673] continued these investigations by finding explicit formulas for the solutions of the following versions of the addition and subtraction formulas for sine and cosine:

$$f(x\sigma(y)) = f(x)g(y) - g(x)f(y), \quad x, y \in G,$$
$$f(x\sigma(y)) = f(x)g(y) + g(x)f(y), \quad x, y \in G,$$
$$g(x\sigma(y)) = g(x)g(y) + f(x)f(y), \quad x, y \in G,$$

where $\sigma : G \to G$ is a homomorphism such that $\sigma \circ \sigma = I_G$, where I_G denotes the identity map. The classical addition and subtraction formulas on an Abelian group G correspond to $\sigma = \pm I_G$.

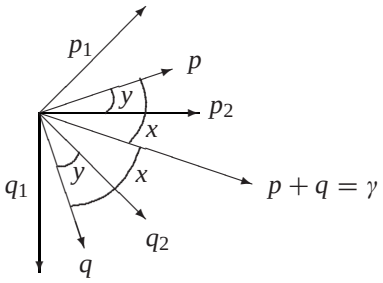
We end this section with an example (application) where the cosine equation arises naturally.

Derivation of (C)

Let $2y$ be the angle between two unit vectors, and denote by $2f(y)$ the length of their resultant. Let us consider two such pairs p_1, p_2 and q_1, q_2 with resultants p and q. Let $2x$ be the angle between them (see the figure below). Then

$$|p| = |p_1 + p_2| = |q| = |q_1 + q_2| = 2f(y),$$
$$|\gamma| = |p + q| = |p_1 + p_2 + q_1 + q_2| = 2f(x)2f(y).$$

(By the proportionality of the length, $2f(x)$ is multiplied by the common length $2f(y)$.)



But

$$\begin{aligned}|\gamma| &= |p_1 + p_2 + q_1 + q_2| \\ &= |p_1 + q_1 + p_2 + q_2| \\ &= 2f(x + y) + 2f(x - y).\end{aligned}$$

Thus (C) holds.

4

Quadratic Functional Equations

Quadratic functional equations, bilinear forms equivalent to the quadratic equation, and some generalizations are treated in this chapter.

Among the normed linear spaces (n.l.s.), inner product spaces (i.p.s.) play an important role. The interesting question when an n.l.s. is an i.p.s. led to several characterizations of i.p.s. starting with Fréchet [291], Jordan and von Neumann [398], etc. Functional equations are instrumental in many characterizations. One of the objectives of the next chapter is to bring out the involvement of functional equations in various characterizations of i.p.s. The basic algebraic (norm) condition that makes the n.l.s. an i.p.s. is the *parallelogram identity*, also known as the *Jordan–von Neumann identity* (*or the Appolonius law* or *norm equation*),

$$\text{(JvN)} \qquad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for } x, y \in V,$$

where V is an n.l.s. This translates into a functional equation well known as the *quadratic functional equation*,

$$\text{(Q)} \qquad q(x+y) + q(x-y) = 2q(x) + 2q(y) \quad \text{for } x, y \in V,$$

where V is a linear space.

This chapter is devoted to the study of the quadratic functional equation (Q) and its generalizations, which form a basis for the next chapter, on i.p.s. We will show in Chapter 5 how (Q) is used in various characterizations of i.p.s.

Recall that a mapping $B : G \times G \to F$ (G a group and $F = \mathbb{R}$ or $\mathbb{C}$) is called *biadditive* if B is additive in each variable; that is, if $B(\cdot, y)$ and $B(x, \cdot)$ are additive in $(\cdot)$.

4.1 General Solution and Properties

Our first result is the following theorem giving the general solution of the quadratic functional equation (Q).

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Theorem 4.1. *The general solution $q : G \to F$ (G a group and $F = \mathbb{R}$ or $\mathbb{C}$) satisfying* (Q) *and the condition*

$$q(x+y+z) = q(x+z+y), \quad \text{for all } x, y, z \in G, \tag{K}$$

is given by

$$q(x) = B(x,x) \quad \text{for } x \in G, \tag{4.1}$$

where $B : G \times G \to F$ is a symmetric, biadditive function. This function B is unique and is given by

$$\begin{aligned} B(x,y) &= \frac{1}{4}[q(x+y) - q(x-y)] \\ &= \frac{1}{2}[q(x+y) - q(x) - q(y)] \quad \text{for } x, y \in G. \end{aligned} \tag{4.2}$$

Proof. It is easy to verify that (4.1) with symmetric B and (K) satisfies (Q).

Conversely, let q with (K) satisfy (Q), and we define $B : G \times G \to F$ by (4.2). Note that $q(x+y) = q(y+x)$, $B(0,y) = B(x,0) = 0 = q(0)$, and $q(2x) = 4q(x)$ since q is even; that is, $q(x) = q(x)$.

From (4.2), (K), and (Q), we conclude that

$$\begin{aligned} 4B(x+y,2z) &= q(x+y+2z) - q(x+y-2z) \\ &= q(x+z+(y+z)) + q(x+z-(y+z)) \\ &\quad - q(x-z-(y-z)) - q(x-z+(y-z)) \\ &= 2q(x+z) + 2q(y+z) - 2q(x-z) - 2q(y-z) \\ &= 8B(x,z) + 8B(y,z) \quad \text{for } x, y, z \in G. \end{aligned} \tag{4.3}$$

Now, $y = 0$ in (4.3) yields $B(x,2z) = 2B(x,z)$, so that (4.3) is

$$B(x+y,z) = B(x,z) + B(y,z) \quad \text{for } x, y, z \in G;$$

that is, B is additive in the first variable. The symmetry of B implies the additivity of B in the second variable.

The same conclusion can be obtained in relation to the Jensen equation (J) as follows (see Aczél, Chung, and Ng [40] and Aczél and Dhombres [44]). Use (K), (Q), and (4.2) to get

$$\begin{aligned} 2B(x+y,z) + 2B(x-y,z) &= q(x+y+z) + q(x-y+z) \\ &\quad - q(x+y) - q(x-y) - 2q(z) \\ &= 2q(x+z) + 2q(y) - 2q(x) - 2q(y) - 2q(z) \\ &= 4B(x,z) \quad \text{for } x, y, z \in G, \end{aligned}$$

which for fixed z is Jensen. Since $B(0,z) = 0$, we conclude from Theorem 1.50 that $B(x,z)$ is additive in the first variable. The symmetry of B gives the biadditivity of B.

Finally, from (4.2) and (Q), we have

$$2B(x,x) = q(2x) - 2q(x) = 4q(x) - 2q(x) = 2q(x),$$

which is (4.1).

To prove the uniqueness of B, suppose there exists another symmetric, biadditive $B_1 : G \times G \to F$ such that $q(x) = B_1(x,x)$ for $x \in G$. If we define $B_2(x,y) = B(x,y) - B_1(x,y)$ for $x, y \in G$, then B_2 is also a symmetric, biadditive function with $B_2(x,x) = 0$ for all $x \in G$. Hence $B_2(x+y, x+y) = 0$ for $x, y \in G$, implying that $B_2(x,y) = 0$; that is, $B_1 = B$. Thus B is unique.

This concludes the proof of the theorem.

Remark 4.2. Note that $F = \mathbb{R}$ or $\mathbb{C}$ may be replaced by any Abelian group divisible by 2 in Theorem 4.1.

Remark 4.3. The condition (K) is equivalent to

$$q(x+y+2z) = q(x+z+y+z) \quad \text{for } x, y, z \in G. \tag{4.4}$$

Condition (K) implying (4.4) is obvious. Suppose (4.4) holds. Replace y by $y - z$ in (4.4) to get (K) (also see Kurepa [588]).

Proposition 4.4. *Let $q : G \to F$ satisfy* (Q) *and* (K)*, where G is a group and $F = \mathbb{R}$ or $\mathbb{C}$. Then, B given by* (4.2) *satisfies*

(i) $B(x+y-x-y, z) = 0$ *for* $x, y, z \in G$,
(ii) $B(x+y, z) = B(y+x, z)$ *for* $x, y, z \in G$,
(iii) $B(x+y-x, z) = B(y, z)$ *for* $x, y, z \in G$,

and q satisfies

$$q(x+y+z) = q(x+y) - q(z) + q(y+z) - q(x) + q(z+x) - q(y) \tag{4.5}$$

for $x, y, z \in G$. As a matter of fact, (i), (ii), *and* (iii) *are equivalent to each other.*

Proof. By (4.2) and (K), and noting that $q(-x) = q(x)$ (q even),

$$\begin{aligned} B(x+y-x-y, z) &= \frac{1}{4}[q(x+y-x-y+z) - q(x+y-x-y-z)] \\ &= \frac{1}{4}[q(x+(-x-y+z)+y) - q(x+(-x-y-z)+y)] \\ &= \frac{1}{4}[q(-y+z+y) - q(-y-z+y)] \\ &= 0, \end{aligned}$$

which is (i) (which can also use the additivity of B in the first variable).

By using the additivity in the first variable, we get

$$0 = B(x+y-x-y, z) = B(x+y, z) + B(-x-y, z),$$

which is (ii). Finally, from (ii), we obtain

$$B(x+y,z) = B(y+x,z) = B(y,z) + B(x,z);$$

that is,

$$B(x+y,z) - B(x,z) = B(y,z),$$

which is (iii).

Since B is biadditive,

$$B(x+y,z) = B(x,z) + B(y,z);$$

that is, by using (4.2), we have

$$\frac{1}{2}[q(x+y+z) - q(x+y) - q(z)]$$
$$= \frac{1}{2}[q(x+z) - q(x) - q(z)] + \frac{1}{2}[q(y+z) - q(y) - q(z)],$$

which is (4.5) (see also Kannappan [464]).

Remark 4.5. Let G' be the commutator subgroup of G. Then $\overline{G} = G/G'$ is a commutative group. If $q : G \to F$ satisfies (Q) and (K), then $q = \overline{q} \circ j$, where j is the canonical homomorphism of G into $\overline{G}$ and $\overline{q} : \overline{G} \to \mathbb{C}$ satisfies (Q).

To see this, for $j : G \to \overline{G}$ defined by $j(x) = x + G'$ for $x \in G$, define

$$\overline{q} : \overline{G} \to \mathbb{C}$$

by

$$\overline{q}(x + G') = q(x) \quad \text{for } x \in G.$$

This $\overline{q}$ is well defined. Note that $q(x) = q(x+u)$ for any commutator u because of (K). If $x + G' = y + G'$, then $x - y \in G'$ or $x - y = u$, a commutator. Then $q(x) = q(y+u) = q(y)$. Now,

$$\begin{aligned}\overline{q}(x+G'+y+G') + \overline{q}(x+G'-(y+G')) &= \overline{q}(x+y+G') + \overline{q}(x-y+G') \\ &= q(x+y) + q(x-y) \\ &= 2q(x) + 2q(y) \\ &= 2\overline{q}(x+G') + 2\overline{q}(y+G').\end{aligned}$$

Result 4.6. *Let $f : G \to F$ satisfy* (K). *Then $H = \{x \in G : f(x) = f(0)\}$ contains G' and $-x + a + x \in H$ for $x \in G$ and $a \in H$.*

Now,

$$f(x+y-x-y) = f(x+(-x-y)+y) = f(0) \Rightarrow G' \subseteq H.$$

For $x \in G,\ a \in H$,

$$f(-x+a+x) - f(a) = 0 \Rightarrow -x+a+x \in H.$$

Proposition 4.7. *Let V be a vector space over a field F (which can be $\mathbb{R}$ or $\mathbb{C}$) of characteristic zero. Then $q : V \to F$ satisfying* (Q) *has the properties that*

$$q \text{ is even, } q(0) = 0,\ q(rx) = r^2 q(x), \quad \text{for } x \in V,\ r \in Q, \tag{4.6}$$

and

$$q(u) + q(v) = 2q\left(\frac{u+v}{2}\right) + 2q\left(\frac{u-v}{2}\right) \quad \text{for } u, v \in V. \tag{4.7}$$

Proof. Changing y to $-y$ in (Q), we get $q(y) = q(-y)$; that is, q is an even function. Set $y = 0$ in (Q) to get $q(0) = 0$. With $y = x$ in (Q), we have $q(2x) = 2^2 q(x)$ for $x \in V$. By induction and evenness of q, it follows that

$$q(nx) = n^2 q(x) \quad \text{for } x \in V,\ n \in Z.$$

Also, we have

$$q\left(\frac{x}{n}\right) = \frac{1}{n^2} q(x), \quad n \in Z^*,\ x \in V.$$

Hence, for any rational $\frac{m}{n}$ and $x \in V$, we obtain

$$q\left(\frac{m}{n}x\right) = m^2 q\left(\frac{x}{n}\right) = \left(\frac{m}{n}\right)^2 q(x).$$

Replacing x by $\frac{u+v}{2}$ and y by $\frac{u-v}{2}$ for $u, v \in V$, we indeed obtain (4.7).

Remark 4.8. Equation (4.7) holds when the domain of q is a 2-divisible group G and q satisfies (K).

4.2 General Solution on a Complex Linear Space—Sesquilinear Solution

Remark 4.9. By Theorem 4.1, for the quadratic function q in Proposition 4.7, there exists a unique, symmetric, biadditive function $B : V \times V \to F$ such that $q(x) = B(x, x)$. In general, this B is not bilinear or sesquilinear. The question arises as to what condition to impose on q in order for B to be bilinear or sesquilinear.

Definitions. Let V be a linear space over $\mathbb{C}$. A mapping $B : V \times V \to \mathbb{C}$ is said to be *bilinear* provided B is biadditive and linear in each component; that is,

$$B(\lambda x, y) = \lambda B(x, y),\ B(x, \lambda y) = \lambda B(x, y), \quad \text{for } x, y \in V,\ \lambda \in \mathbb{C}.$$

The mapping B is called *sesquilinear*, *antilinear*, or *conjugate linear* if B is biadditive and $B(\lambda x, y) = \lambda B(x, y)$ and $B(x, \mu y) = \overline{\mu} B(x, y)$ for $x, y \in V,\ \lambda, \mu \in \mathbb{C}$.

Evidently, if B in Remark 4.9 is sesquilinear, then

$$q(\lambda x) = |\lambda|^2 B(x, x) = |\lambda|^2 q(x) \quad \text{for } x \in V,\ \lambda \in \mathbb{C}. \tag{4.8}$$

This condition seems to be adequate, and the corresponding result for vector spaces over the reals is false.

Theorem 4.10. (Kurepa [584, 585], Vrbová [823]). *Suppose $q : V \to \mathbb{C}$ satisfies* (Q) *and* (4.8)*, where V is a complex linear space. Then there exists a unique sesquilinear form $B_1 : V \times V \to \mathbb{C}$ such that*

$$\begin{aligned} B_1(x,y) &= \frac{1}{4}[q(x+y)-q(x-y)]+\frac{i}{4}[q(x+iy)-q(x-iy)] \\ &= B(x,y)-iB(ix,y) \quad \textit{for } x,y\in V, \end{aligned} \tag{4.9}$$

where $B(x,y)=\frac{1}{4}[q(x+y)-q(x-y)]$ is given by (4.2)*.*

Proof. Kurepa in [584, 585] proved this result, and Vrbová in [823] gave a simple proof. Also see [44]. Here we present a different simple, direct proof of the same.

By Theorem 4.1, there is a symmetric, biadditive $B : V \times V \to \mathbb{C}$ given by (4.1) and (4.2).

Use of (4.2) and (4.8) yields

$$\begin{cases} B(\lambda x,\lambda y)=|\lambda|^2B(x,y),\ B(\lambda x,y)=|\lambda|^2B\left(x,\dfrac{y}{\lambda}\right), \\ B(x,\lambda y)=|\lambda|^2B\left(\dfrac{x}{\lambda},y\right), \quad \text{for } x,y\in V \text{ and for } \lambda\ (\neq 0)\in\mathbb{C}. \end{cases} \tag{4.10}$$

Further,

$$\begin{cases} B(ix,iy)=B(x,y), \\ B(ix,y)=B(x,-iy)=-B(x,iy), \quad \text{for } x,y\in V. \end{cases} \tag{4.11}$$

Now (4.9) and (4.11) give $B_1(ix,iy)=B_1(x,y)$, which in turn yields

$$B_1(x,iy)=-B_1(ix,y) \quad \text{for } x,y\in V. \tag{4.12}$$

Note that B_1 is biadditive. From (4.9), (4.10), and (4.11), we have

$$\begin{cases} B_1(\lambda x,\lambda y)=|\lambda|^2B_1(x,y), \\ B_1(\lambda x,y)=|\lambda|^2B_1\left(x,\dfrac{y}{\lambda}\right), \\ B_1(x,\lambda y)=|\lambda|^2B_1\left(\dfrac{x}{\lambda},y\right), \quad \text{for } x,y\in V,\ \lambda\ (\neq 0)\in\mathbb{C}, \\ B_1(x,iy)=B(x,iy)-iB(ix,iy)=-B(ix,y)-iB(x,y) \\ \qquad\quad =-i(B(x,y)-iB(ix,y))=-iB_1(x,y). \end{cases} \tag{4.13}$$

Thus combining (4.13) with (4.12), we obtain

$$B_1(ix,y)=iB_1(x,y)=-B_1(x,iy) \quad \text{for } x,y\in V. \tag{4.14}$$

Using (4.13) and (4.14) for $x, y \in V$ and $t \in \mathbb{R}$, we get

$$\begin{aligned}
B_1((t+i)x, y) &= B_1(tx, y) + i\,B_1(x, y) \quad \text{for } x, y \in V,\ t \in \mathbb{R} \\
&= (1+t^2)B_1\left(x, \frac{y}{t+i}\right) \\
&= (1+t^2)B_1\left(x, \frac{t-i}{1+t^2}y\right) \\
&= (1+t^2)\left[B_1\left(x, \frac{t}{1+t^2}y\right) + i\,B_1\left(x, \frac{1}{1+t^2}y\right)\right] \\
&= \frac{t^2}{1+t^2}B_1\left(\frac{1+t^2}{t}x, y\right) + \frac{i}{(1+t^2)}B_1((1+t^2)x, y) \\
&= \frac{t^2}{1+t^2}\left[B_1\left(\frac{x}{t}, y\right) + B_1(tx, y)\right] \\
&\quad + \frac{i}{1+t^2}[B_1(x, y) + B_1(t^2x, y)] \\
&= \frac{1}{1+t^2}B_1(x, ty) + \frac{t^2}{1+t^2}B_1(tx, y) \\
&\quad + \frac{i}{1+t^2}[B_1(x, y) + B_1(t^2x, y)];
\end{aligned}$$

that is,

$$\begin{aligned}
(1+t^2)B_1(tx, y) + (1+t^2)i\,B_1(x, y) &= B_1(x, ty) + t^2B_1(tx, y) \\
&\quad + i\,B_1(x, y) + i\,B_1(t^2x, y),
\end{aligned}$$

meaning

$$B_1(tx, y) + it^2B_1(x, y) = B_1(x, ty) + i\,B_1(t^2x, y) \quad \text{for } x, y \in V,\ t \in \mathbb{R}. \tag{4.15}$$

Changing t into $-t$ in (4.15) and adding or subtracting the resultant, we have

$$B_1(t^2x, y) = t^2B_1(x, y),\ B_1(tx, y) = B_1(x, ty), \quad \text{for } x, y \in V,\ t \in \mathbb{R},$$

from which we can conclude (since B_1 is biadditive) that

$$B_1(tx, y) = tB_1(x, y) = B_1(x, ty) \quad \text{for } x, y \in V,\ t \in \mathbb{R}. \tag{4.16}$$

Thus B_1 is sesquilinear. Indeed, for $\lambda = t + is,\ t, s \in \mathbb{R}$, from (4.14) and (4.16) it follows that

$$\begin{aligned}
B_1(\lambda x, y) = B_1((t+si)x, y) &= B_1(tx, y) + B_1(six, y) \\
&= tB_1(x, y) + si\,B_1(x, y) = \lambda B_1(x, y)
\end{aligned}$$

and

$$\begin{aligned} B_1(x, \lambda y) &= B_1(x, (t + si)y) \\ &= B_1(x, ty) + B_1(x, siy) \\ &= tB_1(x, y) - siB_1(x, y) \\ &= \overline{\lambda} B_1(x, y). \end{aligned}$$

This completes the proof. □

4.3 Regular Solutions

We will consider regularity solutions of (Q).

Result 4.11. (Refer to Kurepa [576].) *Suppose $q : \mathbb{R} \to \mathbb{R}$ satisfies* (Q)*. If q is continuous or continuous at a point, bounded on $[0, d[$ $(d > 0)$, or bounded on a set of positive measure or measurable, then q has the form*

$$q(x) = cx^2 \quad \text{for } x \in \mathbb{R}, \tag{4.17}$$

where c is a real constant.

Remark 4.12. For continuous q, (4.17) follows from Proposition 4.7 since $q(rx) = r^2 q(x)$ for $x \in \mathbb{R}$, $r \in Q$.

Also, from Theorem 4.1, $q(x) = B(x, x)$, where B is biadditive. Now, from (4.2), we see that B is continuous in each variable so that, for fixed y,

$$B(x, y) = c(y)x = cyx$$

(using additivity in the second variable), which leads to (4.17).

Remark 4.13. A mapping $q : \mathbb{R} \to \mathbb{C}$ satisfying (Q) and continuous is of the form (4.17), where c is a complex constant.

Result 4.14. [576]. *Let V be a Banach space over $\mathbb{R}$. A mapping $q : V \to \mathbb{R}$ satisfying* (Q) *and continuous at one point is continuous everywhere and is bounded on every bounded sphere.*

Theorem 4.15. (Baker [89]). *Suppose $q : \mathbb{C} \to \mathbb{C}$ is a solution of* (Q)*. Then q is continuous or continuous at a point if and only if there exist complex constants a, b, and d such that*

$$q(z) = az^2 + b|z|^2 + d\overline{z}^2 \quad \text{for } z \in \mathbb{C}. \tag{4.18}$$

Proof. If q is of the form (4.18), it is easy to check that q satisfies (Q). We now prove the converse.

By Theorem 4.1, there exists a biadditive $B : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ such that $q(z) = B(z, z)$, for $z \in \mathbb{C}$, and B is continuous in each variable. By Theorem 1.23, for fixed $w \in \mathbb{C}$,

$$B(z, w) = c_1(w)z + c_2(w)\overline{z} \quad \text{for } z \in \mathbb{C}.$$

Applying once again the additivity in the second variable, we have

$$c_1(w+v)z + c_2(w+v)\overline{z} = c_1(w)z + c_2(w)\overline{z} + c_1(v)z + c_2(v)\overline{z} \quad \text{for } z, w, v \in \mathbb{C}.$$

By choosing z real and purely imaginary, we see that $c_1 + c_2$ and $c_1 - c_2$ are additive on $\mathbb{C}$ and are continuous. Hence

$$(c_1 + c_2)(w) = a_1 w + a_2 \overline{w}, \ (c_1 - c_2)(w) = b_1 w + b_2 \overline{w},$$

where a_1, a_2, b_1, b_2 are complex constants. Thus

$$B(z, w) = d_1 z w + d_2 z\overline{w} + d_3 \overline{z} w + d_4 \overline{z}\,\overline{w} \quad \text{for } z, w \in \mathbb{C}, \tag{4.19}$$

where d_i $(i = 1 \text{ to } 4)$ are complex constants. This proves the theorem.

Remark 4.16. If q in Theorem 4.15 is differentiable, then $q(z) = cz^2$, where c is a complex constant.

Result 4.17. [576]. *Let H be a Hilbert space over $\mathbb{R}$. Then, for every continuous $q : H \to \mathbb{R}$ satisfying* (Q)*, there exists a unique, symmetric, bounded, linear operator $T : H \to H$ such that*

$$q(x) = \langle Tx, x \rangle \quad \textit{for } x \in H.$$

Further, if $q(Tx) = q(x)$ for every orthogonal operator $T : H \to H$ and $x \in H$, then

$$q(x) = c\|x\|^2 \quad \textit{for } x \in H,$$

where c is a real constant.

Result 4.18. [89]. *Let V be a complex vector space and $q : V \to \mathbb{C}$ be a quadratic functional satisfying* (Q) *such that, for $x \in V, \ \lambda \in \mathbb{C}$,*

$$|q(\lambda x)| = |\lambda^2 q(x)|.$$

Then, one of the following holds for all $x \in V, \ \lambda \in \mathbb{C}$:

$$q(\lambda x) = \lambda^2 q(x); \quad q(\lambda x) = |\lambda|^2 q(x); \quad q(\lambda x) = \overline{\lambda^2} q(x).$$

Result 4.19. [89]. (Decomposition of a quadratic functional.) *Let $q : V \to \mathbb{C}$ be a solution of* (Q)*, where V is a complex linear space. For each $x \in V$, the mapping $\lambda \to q(\lambda x)$ $(\lambda \in \mathbb{C})$ is continuous* (*that is, continuous along rays*) *if and only if there exist three unique quadratic functionals $q_1, q_2, q_3 : V \to \mathbb{C}$ such that $q = q_1 + q_2 + q_3$,*

$$q_1(\lambda x) = \lambda^2 q_1(x), \ q_2(\lambda x) = |\lambda|^2 q_2(x), \ q_3(\lambda x) = \overline{\lambda^2} q_3(x),$$

for $x \in V, \ \lambda \in \mathbb{C}$.

4.4 Generalizations and Equivalent Forms of (Q)

Now we consider some generalizations and equivalent forms of (Q).

Theorem 4.20. *Let $f : G \to \mathbb{C}$ be a nonconstant solution of*

$$f(x+y) + f(x-y) = af(x) + bf(y) \quad \text{for } x, y \in G, \tag{4.20}$$

where a and b are constants and G is an Abelian 2-divisible group. Then either f is Jensen (that is, it satisfies (J)*) or satisfies* (Q)*.*

Proof. Change y into $-y$ in (4.20) to get $bf(y) = bf(-y)$. If $b = 0$, then (4.20) gives $a = 2$ and is thus Jensen (put $y = 0$).

If $b \neq 0$, then f is even. Interchange x and y in (4.20) to obtain $(a-b)(f(x) - f(y)) = 0$.

Since f is nonconstant, $a = b$. Now $x = 0$ (or $y = 0$) in (4.20) gives $(a-2)f(y) = -af(0)$. If $a \neq 2$, then f is constant, which is not the case. Thus f satisfies (Q). □

Theorem 4.21. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a continuous solution of*

$$f(z+w) + f(z-\overline{w}) = 2f(z) + 2f(w) \quad \text{for } z, w \in \mathbb{C}. \tag{4.21}$$

Then f has the form

$$f(z) = a(z+\overline{z})^2 + c(z-\overline{z}) \quad \text{for } z \in \mathbb{C}, \tag{4.18}$$

where a, c are complex constants.

Proof. First restrict f to $\mathbb{R}$. Then f satisfies (Q) on $\mathbb{R}$ and is continuous. Then by Remark 4.12 we have $f(x) = c_1 x^2$ for $x \in \mathbb{R}$, where c_1 is a complex constant.

Set $w = it$, $t \in \mathbb{R}$ in (4.21) to get

$$f(z+it) = f(z) + f(it) \quad \text{for } z \in \mathbb{C},\ t \in \mathbb{R}.$$

First, taking $z = is$, $s \in \mathbb{R}$, we get "additivity" of f, and the continuity yields $f(it) = c_2 t$, where c_2 is complex. Thus, for $z = s + it$, we have

$$\begin{aligned} f(z) &= f(s+it) = f(s) + f(it) = c_1 s^2 + c_2 t \\ &= a(z+\overline{z})^2 + b(z-\overline{z}), \end{aligned}$$

which is (4.18).

Usually in statistics, in particular in estimation theory, a quadratic expression has to be maximized subject to some constraints. This can of course be done by calculus methods. However, if the arguments of the quadratic forms are themselves matrices, this method becomes very cumbersome and requires a lot of indices. Another method, known as the quasi inner product method is more often used. It is used in statistical estimation and mathematical programming. Drygas [230] proved

a Jordan and von Neumann type characterization theorem for quasi-inner products. In this characterization of a quasi inner product, the functional equation

$$f(x) + f(y) = f(x-y) + 2\left[f\left(\frac{x+y}{2}\right) - f\left(\frac{x-y}{2}\right)\right]$$

played an important role. By changing y to $-y$ in the equation above and adding the resultant to it, we obtain

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y), \tag{4.22}$$

an equation similar to (Q). □

Theorem 4.22. *Let G be a group and F be a field of characteristic different from* 2. *Let $f : G \to F$ satisfy* (4.22) *for $x, y \in G$ and the condition* (K). *Then there exist an additive A and a biadditive $B : G \times G \to F$ such that*

$$f(x) = B(x,x) + A(x) \quad \text{for } x \in G. \tag{4.23}$$

Proof. Either $x = 0$ or $y = 0$ in (4.22) gives $f(0) = 0$.

Interchange x and y in (4.22) and subtract the resultant equation from (4.22) to get

$$F(x-y) = F(x) + F(-y) \quad \text{for } x, y \in G,$$

where $F(x) = f(x) - f(-x)$ for $x \in G$. Thus F is additive (change $-y$ to y), say $2A$. Then $f(-x) = f(x) - 2A(x)$ and (4.22) becomes

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) - 2A(y);$$

that is, $q(x) = f(x) - A(x)$ satisfies (Q). By Theorem 4.1, (4.23) follows.

From Proposition 4.4, it is easy to see that if $q : G \to F$ satisfies (Q) and (K), then (4.5) holds. But the converse is not true. We have the following result. □

Theorem 4.23. [464]. *The general solution $q : G \to F$ (G a group and F a field of characteristic different from* 2) *of* (4.5) *with* (K) *is of the form*

$$q(x) = B(x,x) + A(x), \tag{4.23}$$

where $A : G \to F$ is additive and $B : G \times G \to F$ is biadditive.

Proof. Set $z = 0$ in (4.5) to get $q(0) = 0$. Putting $z = -x$ in (4.5), we have

$$q(y) + q(x) + q(y) + q(-x) = q(x+y) + q(y-x),$$

which is (4.22). The result now follows from Theorem 4.22.

Now consider the functional equation

$$\begin{aligned} f(x+y-z) + f(x) + f(-y) + f(z) = {} & f(-x+y) \\ & + f(-y+z) + f(-z+x), \end{aligned} \tag{4.24}$$

where $f : G \to F$. The only solution of (4.24) is $f = 0$.

To see this, let $z = x$ to get

$$f(y) + f(-y) + 2f(x) = f(-x+y) + f(-y+x) + f(0). \tag{4.25}$$

Now $y = 0$ in (4.25) gives $f(x) + f(0) = f(-x)$; that is, $f(0) = 0$ and f is even. Again (4.25) shows that f is additive. Since f is even, $f = 0$. □

But if we make a slight change in an argument in (4.24), the equation is equivalent to (Q). This leads to the following theorem.

4.5 Equivalence to (Q)

Theorem 4.24. *Let G be a 2-divisible group and F be either $\mathbb{R}$ or $\mathbb{C}$ or a field of characteristic different from 2. Suppose $f : G \to F$ satisfies* (K). *Then the following are equivalent:*

a) f *satisfies* (Q),
b) $f(x+y-z) + f(x) + f(-y) + f(z) = f(-x-y) + f(-y+z) + f(-z+x)$,
c) $f(x-y-z) + f(x) + f(-y) + f(-z) = f(x-y) + f(y+z) + f(-z+x)$,
d) $f(x+y+z) + f(x-y-z) + f(-x-y+z) + f(-x+y-z)$
$= 4f(x) + 4f(y) + 4f(z)$,

for all $x, y, z \in G$.

Proof. We prove the implications in the order stated.

To prove that (a) $\Rightarrow$ (b), since f satisfies (Q) and (K), from Proposition 4.7 and Remark 4.8 we have that f is even, $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$, and f satisfies (4.7).

$$\begin{aligned}\text{left side of (b)} &= 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x+y}{2} - z\right) + 2f\left(\frac{x-y}{2}\right) + 2f\left(\frac{x+y}{2}\right) \\ &= f(x+y) + 2\left[2f\left(\frac{x-z}{2}\right) + 2f\left(\frac{y-z}{2}\right)\right] \\ &= f(-x-y) + f(-z+x) + f(-y+z),\end{aligned}$$

which is the right side of (b).

To prove that (b) $\Rightarrow$ (c), set $z = 0$ in (b) to get $f(x+y) + f(0) = f(-x-y)$; that is, $f(0) = 0$ and f is even. Now change y to $-y$ in (b) to get (c).

To prove that (c) $\Rightarrow$ (d), put $x = 0$ in (c) to have $f(-y-z) + f(0) = f(y+z)$, which implies that $f(0) = 0$ and f is even.

Use that f is even and f satisfies (K) in the following calculations. First replace, z by $x + z$ in (c) to get

$$f(x+y+z) = -f(x-y) + f(y+z) + f(z+x) + f(x) + f(y) - f(z). \tag{4.26}$$

Second, change x and z to $-x$ and $-z$ in (c) to obtain

$$f(-x-y+z) = f(x+y) + f(y-z) + f(-z+x) - f(x) - f(y) - f(z) \tag{4.27}$$

for $x, y, z \in G$. Lastly, replace x and y by $-x$ and $-y$ in (c) to have

$$f(-x+y-z) = f(x-y) + f(-y+z) + f(z+x) - f(x) - f(y) - f(z). \tag{4.28}$$

Now adding (4.26), (c), (4.27), and (4.28), we have

$$\text{left side of (d)} = f(x+y) + f(x-y) + 2(f(y+z) + f(y-z)) + 2(f(z+x) + f(z-x)) - 2f(x) - 2f(y) - 4f(z). \tag{4.29}$$

Changing x to y, y to z, and z to x in (4.29), we obtain

$$\text{left side of (d)} = f(y+z) + f(y-z) + 2(f(z+x) + f(z-x)) + 2(f(x+y) + f(x-y)) - 2f(y) - 2f(z) - 4f(x). \tag{4.30}$$

Equate the right sides of (4.29) and (4.30) to have

$$f(y+z) + f(y-z) = f(x+y) + f(x-y) - 2f(x) + 2f(z), \tag{4.31}$$

which permuting x, y, z yields

$$\begin{aligned} f(z+x) + f(z-x) &= f(y+z) + f(y-z) - 2f(y) + 2f(x) \\ &= f(x+y) + f(x-y) - 2f(y) + 2f(z). \end{aligned} \tag{4.32}$$

Using (4.31) and (4.32) in (4.30), we obtain

$$\begin{aligned} \text{left side of (d)} &= 5(f(x+y) + f(x-y)) - 6f(x) - 6f(y) + 4f(z) \\ &= 5(f(y+z) + f(y-z)) - 6f(y) - 6f(z) + 4f(x). \end{aligned} \tag{4.33}$$

Fix y and define

$$h(x) = 5(f(x+y) + f(x-y)) - 6f(x) - 6f(y).$$

Then (4.33) yields $h(x) + 4f(z) = h(z) + 4f(x)$; that is, $h(x) - 4f(x) = \text{constant} = t(y)$, say. Finally, (4.33) becomes

$$\text{left side of (d)} = 4f(x) + t(y) + 4f(z).$$

The symmetry of the left side in x, y, z shows that $t(y)$ is $4f(y)$, which shows that (d) holds.

To complete the cycle, we will now prove that (d) $\Rightarrow$ (a). Let $y = 0 = z$ in (d) to have $f(-x) = f(x) + 4f(0)$. Thus $f(0) = 0$ and f is even. Now put $z = 0$ in (d) to have (Q). This completes the proof of this theorem. □

4.6 Generalizations

First we consider a generalization of (d) in Theorem 4.24 and prove the following theorem.

Theorem 4.25. *Let G be a 2-divisible group and F be either $\mathbb{R}$ or $\mathbb{C}$ or a field of characteristic different from 2. Suppose $f_i : G \to F$ ($i = 1$ to 7) satisfy* (K) *and the functional equation*

$$\begin{aligned} f_1(x+y+z) + f_2(x-y-z) + f_3(-x-y+z) \\ + f_4(-x+y-z) = 4f_5(x) + 4f_6(y) + 4f_7(z) \end{aligned} \tag{4.34}$$

for $x, y, z \in G$. Then they are given by

$$\begin{cases} f_1(x) = q(x) + (A_2 + A_3 + A_4)(x) + b_1 \\ \qquad = B(x,x) + (A_2 + A_3 + A_4)(x) + b_1, \\ f_2(x) = B(x,x) + (2A_1 - A_2 - A_3 + A_4)(x) + b_2, \\ f_3(x) = B(x,x) + (A_2 - A_3 - A_4)(x) + b_3, \\ f_4(x) = B(x,x) + (2A_1 - A_2 + A_3 - A_4)(x) + b_4, \\ f_5(x) = B(x,x) + A_4(x) + b_5, \\ f_6(x) = B(x,x) + A_3(x) + b_6, \\ f_7(x) = B(x,x) + (A_2 - A_1)(x) + b_7, \end{cases} \tag{4.35}$$

with $\sum_{i=1}^{4} b_i = 4\sum_{j=5}^{7} b_j$, where A_i is additive ($i = 1$ to 4), q satisfies (Q)*, and $B : G \times G \to F$ is biadditive.*

Proof. Replacing x, y, and z in (4.34) by $-x, -y$, and $-z$, respectively, adding and subtracting the resultant from (4.34), and using (K), we get

$$\begin{aligned} f_1^e(x+y+z) + f_2^e(x-y-z) + f_3^e(-x-y+z) \\ + f_4^e(-x+y-z) = 4f_5^e(x) + 4f_6^e(y) + 4f_7^e(z), \end{aligned} \tag{4.34e}$$

$$\begin{aligned} f_1^o(x+y+z) + f_2^o(x-y-z) + f_3^o(-x-y+z) \\ + f_4^o(-x+y-z) = 4f_5^o(x) + 4f_6^o(y) + 4f_7^o(z), \end{aligned} \tag{4.34o}$$

for $x, y, z \in G$, where f_i^e, f_i^o ($i = 1$ to 7) are the even and odd parts of f_i ($i = 1$ to 7). That is, f_i^e and f_i^o also satisfy (4.34). Note that f_i^e is even, f_i^o is odd, and $f_i^o(0) = 0$.

Finding f_i^e is relatively easy by reducing to the Pexiderized Jensen equation several times and reducing to (d) of Theorem 4.24. We do that first.

Setting $z = -y$ and $z = y$ separately in (4.34e), we get

$$f_1^e(x) + f_2^e(x) + f_3^e(-x-2y) + f_4^e(-x+2y) = 4f_5^e(x) + 4(f_6^e + f_7^e)(y) \tag{4.36}$$

and

$$f_1^e(x+2y) + f_2^e(x-2y) + f_3^e(x) + f_4^e(x) = 4f_5^e(x) + 4(f_6^e + f_7^e)(y). \tag{4.37}$$

Subtracting (4.36) from (4.37), we have

$$(f_1^e - f_3^e)(x+2y) + (f_2^e - f_4^e)(x-2y) = (f_1^e + f_2^e - f_3^e - f_4^e)(x)$$

for $x, y \in G$, which is a Pexiderized Jensen equation. So there exists an additive A_1 on G such that

$$f_1^e - f_3^e = A_1 + c_1, \quad f_2^e - f_4^e = A_1 + c_2, \quad c_1, c_2 \text{ are constants.}$$

Since f_i^e $(i = 1 \text{ to } 4)$ are even and A_1 is odd, we have

$$f_1^e - f_3^e = c_1, \quad f_2^e - f_4^e = c_2. \tag{4.38}$$

With $y = 0$, (4.36) gives

$$f_1^e + f_2^e + f_3^e + f_4^e = 4f_5^e + c_3, \quad c_3 \text{ is a constant.} \tag{4.39}$$

Similarly, letting $y = -x$ and $y = x$ separately in (4.34e), we obtain

$$f_1^e - f_4^e = c_4, \quad f_3^e - f_2^e = c_5, \quad f_1^e + f_2^e + f_3^e + f_4^e = 4f_7^e + c_6, \tag{4.40}$$

where c_4, c_5, and c_6 are constants.

In (4.34e), $x = 0 = z$ gives

$$f_1^e + f_2^e + f_3^e + f_4^e = 4f_6^e + c_7. \tag{4.41}$$

From (4.38) to (4.41), we see that f_2^e to f_7^e differ from f_1^e by a constant, and substituting these values of f_2^e to f_7^e in terms of f_1^e in (4.34e), we have

$$\begin{aligned} f_1^e(x+y+z) + f_1^e(x-y+z) + f_1^e(-x-y+z) \\ + f_1^e(-x+y-z) = 4f_1^e(x) + 4f_1^e(y) + 4f_1^e(z) + c, \end{aligned}$$

where c is a constant.

With $x = 0 = y = z$ in the above, we get $c = -8f_1^e(0)$, and then it is easy to see that $f_1^e - f_1^e(0)$ satisfies (d) of Theorem 4.24. Thus, we get from Theorem 4.24 and Theorem 4.1 that

$$f_i^e(x) = q(x) + b_i = B(x,x) + b_i \quad \text{for } x \in G \tag{4.42}$$

$(i = 1 \text{ to } 7)$, where q satisfies (Q), B is biadditive, and $\sum_{i=1}^{4} b_i = 4\sum_{j=5}^{7} b_j$.

Now, to determine f_i^o is a little bit tedious. Again we reduce (4.34o) to the Jensen equation several times.

Letting $z = -y$ and $z = y$ separately in (4.34o), we have

$$f_1^o(x) + f_2^o(x) - f_3^o(x+2y) - f_4^o(x-2y) = 4f_5^o(x) + 4(f_6^o - f_7^o)(y), \tag{4.43}$$

$$f_1^o(x+2y) + f_2^o(x-2y) - f_3^o(x) - f_4^o(x) = 4f_5^o(x) + 4(f_6^o + f_7^o)(y). \tag{4.44}$$

Subtracting (4.43) from (4.44), we have

$$(f_1^o + f_3^o)(x+2y) + (f_2^o + f_4^o)(x-2y) = 8f_7^o(y) + \left(\sum_{i=1}^{4} f_i^o\right)(x). \tag{4.45}$$

Changing y to $-y$ in (4.45) and adding the resultant equation to (4.45), we get

$$\left(\sum_{i=1}^{4} f_i^o\right)(x+2y) + \left(\sum_{i=1}^{4} f_i^o\right)(x-2y) = 2\left(\sum_{i=1}^{4} f_i^o\right)(x),$$

which is Jensen. Since $f_i^o(0) = 0$,

$$f_1^o + f_2^o + f_3^o + f_4^o = A_1, \tag{4.46}$$

where A_1 is additive on G.

First $x = 0$ in (4.45) gives

$$(f_1^o - f_2^o + f_3^o - f_4^o)(2y) = 8f_7^o(y),$$

and then $x = 0 = y$ in (4.34o) gives

$$(f_1^o - f_2^o + f_3^o - f_4^o)(z) = 4f_7^o(z),$$

so that $f_7^o(2y) = 2f_7^o(y)$, and with (4.46) we obtain

$$2(f_1^o + f_3^o) = 4f_7^o + A_1. \tag{4.47}$$

Now (4.46) and (4.47) in (4.45) yields

$$(f_1^o + f_3^o)(x+2y) - (f_1^o + f_3^o)(x-2y) = 2(f_1^o + f_3^o)(2y),$$

which is Jensen. Noting that $(f_1^o + f_3^o)(0) = 0$, we get

$$f_1^o + f_3^o = A_2, \quad f_2^o + f_4^o = A_1 - A_2, \quad 4f_7^o = 2A_2 - A_1, \tag{4.48}$$

where A_2 is additive on G.

Letting $y = x$ and $y = -x$ separately in (4.34o) and subtracting the latter from the former, we obtain

$$(f_1^o + f_4^o)(2x+z) - (f_3^o + f_4^o)(2x-z) = A_1(z) + 8f_6^o(x). \tag{4.49}$$

Put $z = 0$ in (4.49) to get

$$(f_1^o + f_4^o - f_2^o - f_3^o)(2x) = 8f_6^o(x) \quad \text{for } x \in G.$$

Letting z become $-z$ in (4.49) and adding the resultant to (4.49), we have

$$\begin{aligned}(f_1^o + f_4^o - f_2^o - f_3^o)(2x+z)&\\ +(f_1^o + f_4^o - f_3^o - f_2^o)(2x-z) &= 16\, f_6^o(x)\\ &= 2(f_1^o + f_4^o - f_2^o - f_3^o)(2x),\end{aligned}$$

which is Jensen, so that, as before, we get

$$f_1^o - f_2^o - f_3^o + f_4^o = A_3, \quad 4\, f_6^o = A_3, \tag{4.50}$$

where A_3 is additive on G.

From (4.46) and (4.50), we have

$$2(f_1^o + f_4^o) = A_1 + A_3, \quad 2(f_2^o + f_3^o) = A_1 - A_3. \tag{4.51}$$

Letting $y = 0 = z$, (4.34o) gives

$$f_1^o + f_2^o - f_3^o - f_4^o = 4\, f_5^o. \tag{4.52}$$

Finally, put $z = 0$ in (4.34o). Then replace y by $-y$ and add the resultant equation to obtain

$$\begin{aligned}(f_1^o + f_2^o - f_3^o - f_4^o)(x+y)&\\ +(f_1^o + f_2^o - f_3^o - f_4^o)(x-y) &= 8\, f_5^o(y)\\ &= 2(f_1^o + f_2^o - f_3^o - f_4^o)(y) \quad \text{(use (4.52))},\end{aligned}$$

which is Jensen, so that

$$f_1^o + f_2^o - f_3^o - f_4^o = A_4 = 4\, f_5^o, \tag{4.53}$$

where A_4 is additive.

Use (4.50) and (4.53) to have

$$\begin{aligned}2(f_1^o - f_3^o) &= A_4 + A_3,\\ 2(f_2^o - f_4^o) &= A_4 - A_3.\end{aligned}$$

These equations with (4.48) yield

$$\begin{cases} f_1^o = \dfrac{A_2}{2} + \dfrac{A_3 + A_4}{4}, \;\; f_3^o = \dfrac{A_2}{2} - \dfrac{A_3 + A_4}{4}, \\ f_2^o = \dfrac{A_1 - A_2}{2} + \dfrac{A_4 - A_3}{4}, \;\; f_4^o = \dfrac{A_1 - A_2}{2} - \dfrac{A_4 - A_3}{4}. \end{cases} \tag{4.54}$$

Now (4.48), (4.50), (4.53), and (4.54) give f_i^o ($i = 1$ to 7). These together with (4.42) give the result sought, (4.35). This completes the proof of this theorem. □

Remark 4.26. Note that (4.34) determines seven unknown functions without any regularity assumptions on f_i.

Result 4.27. [464]. *Let V be a linear space over the field F of char $= 0$ (or different from 2). Suppose $f_i : V \to F$ ($i = 1$ to 7) satisfy the functional equation*

$$f_1(x+y+z) + f_2(x) + f_3(y) + f_4(z) = f_5(x+y) + f_6(y+z) + f_7(z+x).$$

Then there exists biadditive $B : V \times V \to F$, additive $A, A_i : V \to F$ ($i = 1$ to 3) such that f_i's are given by

$$\begin{aligned}
f_1(x) &= B(x,x) + A(x) + C_1 + C_3 + C_5,\\
f_2(x) &= B(x,x) + A(x) - A_1(x) - A_3(x) + C_3 - C_6,\\
f_3(x) &= B(x,x) + A(x) - A_1(x) - A_2(x) - C_2 - C_5,\\
f_4(x) &= B(x,x) + A(x) - A_2(x) - A_3(x) + C_1 - C_4,\\
f_5(x) &= B(x,x) + A(x) - A_1(x) - C_2 + C_3 + C_5,\\
f_6(x) &= B(x,x) + A(x) - A_2(x) + C_1 - C_4 + C_5,\\
f_7(x) &= B(x,x) + A(x) - A_2(x) - A_3(x) + C_1 + C_3 - C_6,
\end{aligned}$$

where C_i's are arbitrary constants.

4.7 More Generalizations

We determine the general solutions of the functional equation

$$f_1(x+y) + f_2(x-y) = f_3(x) + f_4(y) \quad \text{for } x, y \in G, \tag{4.55}$$

where $f_i : G \to F$ ($i = 1$ to 4), G is a 2-divisible group, and F is a commutative field of characteristic different from 2. The motivation came from a result due to Drygas [230]. Also, this equation is a generalization of the quadratic functional equation (Q). Special cases of this equation include the additive equation (A), the Jensen equation (J), the Pexider equation, and many more. Here we determine the general solution of (4.55) without any regularity assumptions on f_i. Note that equation (4.55), like (4.34), determines several unknown functions, in this case four. We prove the following theorem. Even though it is solved in Ebanks, Kannappan, and Sahoo [241], here we present a different proof.

Theorem 4.28. *Let G be a 2-divisible group and F be a commutative field of characteristic different from 2. The general solution of* (4.55) *with f_1 and f_2 satisfying the condition* (K) *is given by*

$$\begin{cases}
f_1(x) = B(x,x) - (A_1 - A_2)(x) + b_1,\\
f_2(x) = B(x,x) - (A_1 + A_2)(x) + b_2,\\
f_3(x) = 2B(x,x) - 2A_1(x) + b_3,\\
f_4(x) = 2B(x,x) + 2A_2(x) + b_4,
\end{cases} \tag{4.56}$$

with $b_1 + b_2 = b_3 + b_4$, where $B : G \times G \to F$ is a symmetric biadditive function and $A_i : G \to F$ ($i = 1, 2$) are homomorphisms (that is, additive).

Proof. Interchange x and y in (4.55) to get

$$f_1(y+x)+f_2(y-x)=f_3(y)+f_4(x) \quad \text{for } x,y\in G. \tag{4.57}$$

First subtract (4.57) from (4.55) and use (K) to have

$$f_2^o(x-y)=\alpha(x)-\alpha(y) \quad \text{for } x,y\in G, \tag{4.58}$$

where $f_2^o(x)=f_2(x)-f_2(-x)$, $\alpha(x)=f_3(x)-f_4(x)$. f_2^o is odd.

Set $y=0$ in (4.58) to obtain $\alpha(x)=f_2^o(x)+c_1$, so that (4.58) becomes

$$f_2^o(x-y)=f_2^o(x)-f_2^o(y);$$

that is, f_2^o is additive, say

$$f_2^o=A_1. \tag{4.59}$$

Then $\alpha(x)=A_1(x)+c_1$.

Second, add (4.57) and (4.55) and use (K) to have

$$2f_1(x+y)+f_2^e(x-y)=\beta(x)+\beta(y) \quad \text{for } x,y\in G, \tag{4.60}$$

where $f_2^e(x)=f_2^2(x)+f_2(-x)$, $\beta(x)=f_3(x)+f_4(x_1)$.

Now $y=0$ in (4.60) gives

$$2f_1(x)+f_2^e(x)=\beta(x)+c_2, \quad c_2=\beta(0). \tag{4.61}$$

Eliminating f_1 from (4.61) and (4.60), we have

$$-f_2^e(x+y)+f_2^e(x-y)+\beta(x+y)+c_2=\beta(x)+\beta(y). \tag{4.62}$$

Changing y to $-y$ in (4.62), we obtain

$$-f_2^e(x-y)+f_2^e(x+y)+\beta(x-y)+c_2=\beta(x)+\beta(-y). \tag{4.63}$$

Adding (4.63) and (4.62), we get

$$\beta(x+y)+\beta(x-y)+2c_2=2\beta(x)+\beta(y)+\beta(-y) \quad \text{for } x,y\in G.$$

This equation is (4.22). Noting that β also satisfies the condition (K) from (4.60), from Theorem 4.22, β is given by

$$\beta(x)=B(x,x)+A(x)+c_2 \quad \text{for } x\in G. \tag{4.64}$$

From (4.59) and (4.64), we get

$$\begin{cases} 2f_3(x)=B(x,x)+(A+A_1)(x)+c_1+c_2, \\ 2f_4(x)=B(x,x)+(A-A_1)(x)-c_1+c_2. \end{cases} \tag{4.65}$$

Using $\beta(x)$ given by (4.64) in (4.63), we have

$$f_2^e(x+y) - f_2^e(x-y) = 2B(x,y),$$

which with $y = x$ gives

$$f_2^e(x) = \frac{1}{2}B(x,x) + c_3. \tag{4.66}$$

From (4.59) and (4.66), we get

$$2f_2(x) = \frac{1}{2}B(x,x) + A_1(x) + c_3. \tag{4.67}$$

Now (4.61), (4.64), and (4.66) yield

$$2f_1(x) = \frac{1}{2}B(x,x) + A(x) + 2c_2 + c_3. \tag{4.68}$$

Equations (4.65), (4.67), and (4.68) give the solution sought, (4.56). This proves the theorem. □

4.8 Another Form of Quadratic Function

Oresme in 1347 verbally described for the quadratic function a functional equation that can be written as

$$\frac{s((n+1)t) - s(nt)}{s(nt) - s((n-1)t)} = \frac{2n+1}{2n-1} \tag{4.69}$$

for all positive integers n and positive real t. The same equation with the same solution reappears almost three centuries later with Galileo (1638) in connection with falling bodies. In the following theorem, we obtain the continuous solution of (4.69).

Theorem 4.29. *Let $s : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy* (4.69) *for $t \in \mathbb{R}_+$ and $n \in Z_+$ be continuous. Then $s(t) = c_1 t^2 + c_2$ for $t \in \mathbb{R}_+$, where c_1, c_2 are constants.*

Proof. If s is a solution of (4.69), so is $s - s(0)$. Without loss of generality, take $s(0) = 0$. Set $n = 1$ in (4.69) to get

$$s(2t) - s(t) = 3s(t) \quad \text{or} \quad s(2t) = 2^2 s(t).$$

Similarly, $n = 2$ gives $s(3t) = 3^2 s(t)$. So we will prove by induction that $s(nt) = n^2 s(t)$ for positive n and $t \in \mathbb{R}_+$. Suppose $s(nt) = n^2 s(t)$. Then, by induction,

$$\begin{aligned}(2n-1)s((n+1)t) &= (2n+1)(s(nt) - s((n-1)t)) + (2n-1)s(nt)\\ &= (2n+1)(n^2 - (n-1)^2)s(t) + (2n-1)n^2 s(t)\\ &= (2n-1)(n+1)^2 s(t).\end{aligned}$$

That is,

$$s((n+1)t) = (n+1)^2 s(t) \quad \text{for } n \in Z_+,\ t \in \mathbb{R}_+.$$

Also

$$s\left(\frac{t}{n}\right) = \frac{1}{n^2} s(t) \quad \text{for } t \in \mathbb{R}_+,\ n \in R_+.$$

Thus

$$s\left(\frac{m}{n}t\right) = \frac{1}{n^2} s(mt) = \left(\frac{m}{n}\right)^2 s(t) \quad \text{for } m, n \in Z_+,\ t \in \mathbb{R}_+.$$

So

$$s(rt) = r^2 s(t) \quad \text{for } r \in Q,\ t \in \mathbb{R}_+.$$

Invoking continuity of s, we have that

$$s(xt) = x^2 s(t) \quad \text{for } x, t \in \mathbb{R}_+.$$

Setting $t = 1$, we have $s(t) = ct^2$. This proves the theorem. □

4.9 Entire Functions and Quadratic Equations

In this section, we will consider three functional equations studied by Haruki [348]:

$$|f(z+w)| + |f(z-w)| = 2|f(z)| + 2|f(w)|, \qquad z, w \in \mathbb{C}, \tag{4.70}$$

$$|f(z+w)|^2 + |f(z-w)|^2 = 2|f(z)|^2 + 2|f(w)|^2, \quad z, w \in \mathbb{C}, \tag{4.71}$$

where $f : \mathbb{C} \to \mathbb{C}$ is an entire function, and (4.76).

First we treat equation (4.70) and present two proofs.

Theorem 4.30. [348]. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function. Then f is a solution of equation* (4.70) *if and only if $f(z) = bz^2$, where b is a complex constant.*

Proof. For the first proof, we need the following two results about additive and biadditive functions.

Suppose $A : \mathbb{C} \to \mathbb{R}$ satisfies (A) and is continuous. Then

$$A(z) = A(x + iy) = ax + dy,$$

where a and d are reals, $z = x + iy,\ x, y \in \mathbb{R}$.

If $A : \mathbb{C} \to \mathbb{R}$, we can consider A also to be from $\mathbb{C}$ to $\mathbb{C}$. Since A is continuous, by Theorem 1.23,

$$A(z) = a_1 z + a_2 \overline{z} \quad \text{for } z \in \mathbb{C},$$

where a_1 and a_2 are complex constants. But A is real, so $a_1 z + a_2 \overline{z} = \overline{a}_1 \overline{z} + \overline{a}_2 z$; that is, $(a_1 - \overline{a}_2)z = (\overline{a}_1 - a_2)\overline{z}$. Setting $z = 1$ and i, we get that $a_2 = \overline{a}_1$. Thus

$$\begin{aligned} A(z) &= a_1 z + \overline{a}_1 \overline{z} = (a_1 + \overline{a}_1)x + i(a_1 - \overline{a}_1)y \\ &= ax + by, \end{aligned}$$

where a and b are reals.

Next we consider the biadditive function $B : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$. Let $B : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ be biadditive and continuous in each variable. Then

$$B(x + iy, u + iv) = a_1xu + a_2xv + a_3yu + a_4yv \quad \text{for } x, y, u, v \in \mathbb{R},$$

where a_i $(i = 1 \text{ to } 4)$ are real constants.

By the part above (for fixed w and for fixed z), we get

$$\begin{aligned} B(z, w) &= a_1(w)x + a_2(w)y \\ &= b_1(z)u + b_2(z)v \quad \text{for } z, w \in \mathbb{C}, \end{aligned} \tag{4.72}$$

$z = x + iy$, $w = u + iv$, $a_1(w)$, $a_2(w)$, $b_1(z)$, $b_2(z)$ are real functions, and $x, y, u, v \in \mathbb{R}$.

$y = 0$ in (4.72) gives $a_1(w)x = b_1(x)u + b_2(x)v$. Then

$$\begin{aligned} a_1(w + w')x &= b_1(x)(u + u') + b_2(x)(v + v'), \quad w' = u' + iv' \\ &= a_1(w)x + a_1(w')x \quad \text{for } w, w' \in \mathbb{C},\ x \in \mathbb{R}. \end{aligned}$$

Thus, $a_1(w)$ is real, continuous, and additive on $\mathbb{C}$. So,

$$a_1(w) = d_1u + d_2v,$$

where d_1, d_2 are real. Similarly,

$$a_2(w) = d_3u + d_4v,$$

where d_3, d_4 are real. Substitution of these values of $a_1(w)$ and $a_2(w)$ in (4.72) yields the required form of $B(z, w)$.

For the first proof of (4.70), let $q : \mathbb{C} \to \mathbb{R}$ be defined by

$$q(z) = |f(z)| \quad \text{for } z \in \mathbb{C}.$$

Since f satisfies (4.70), q satisfies (Q) on $\mathbb{C}$. Then, by Theorem 4.1, there is a biadditive $B : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ such that $q(z) = B(z, z)$ for $z \in \mathbb{C}$.

Use of the biadditive result above yields

$$q(z) = ax^2 + 2hxy + by^2,$$

where $z = x + iy$ and a, h, b are real constants.

This conclusion can also be obtained from Theorem 4.15, noting that q is real.

Now we have $|f(z)| = q(z) = ax^2 + 2hxy + by^2$ with $a = b = h = 0$ or $a > 0$, $ab - h^2 > 0$. If $a > 0$, $ab - b^2 > 0$, then we get $\Delta \log |f(z)| = 0$ for $0 < |z| < \infty$, where Δ stands for the Laplacian operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Thus we obtain $a = b$ and $h = 0$; that is, $|f(z)| = a(x^2 + y^2) = a|z|^2$. This proves the theorem. □

For a second, alternate proof from H. Haruki [348], let M be the absolute maximum of $|f(x)|$ in $|z| = 1$. Putting $q(z) = |f(z)| - M|z|^2$, where $q : \mathbb{C} \to \mathbb{R}_+$, by (4.70) we have

$$q(z + w) + q(z - w) = 2q(z) + 2q(w). \tag{Q}$$

Since $q(z) \leq 0$ in $|z| = 1$ and $q(0) = 0$, there exists a complex constant z_0 with $|z_0| < 1$ such that $q(z_0)$ is the absolute maximum of $q(z)$ in $|z| \leq 1$. (We use Weierstrass's theorem.)

Putting $z = z_0$ in (Q), we have

$$q(z_0 + w) + q(x_0 - w) = 2q(z_0) + 2q(w). \tag{4.73}$$

Putting $\delta = \text{dist}\,(z_0, |z| = 1)$ (> 0), by (4.73) we have, for $|w| \leq \delta$,

$$q(z_0) + q(z_0) \geq 2q(z_0) + 2q(w).$$

Hence we have, for $|w| \leq \delta$,

$$q(w) \leq 0. \tag{4.74}$$

Putting $w = z$ in (Q) and using $q(0) = 0$, we have, for $|z| < +\infty$,

$$q(2z) = 4q(z). \tag{4.75}$$

By (4.74) and (4.75), we have $q(z) \leq 0$ in $|z| < +\infty$. Hence we have $|f(z)| \leq M|z|^2$ in $|z| < +\infty$. Hence, we have $f(z) = Cz^2$, where C is a complex constant with $|C| \leq M$. Thus the theorem is proved. □

Now we consider the functional equation (4.71).

Theorem 4.31. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function that satisfies equation* (4.71) *for $z, w \in \mathbb{C}$. Then $f(z) = cz$, where c is a complex constant.*

Proof. Define $q : \mathbb{C} \to \mathbb{R}$ by

$$q(z) = |f(z)|^2.$$

Then (4.71) becomes (Q). By Theorem 4.1, $q(z) = B(z, z)$, where $B : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ is biadditive. By following the proof of Theorem 4.30, we see that

$$|f(z)|^2 = q(z) = a|z|^2, \quad a > 0.$$

This proves the theorem. □

Note that in both the theorems we are using the fact that if $f, g : \mathbb{C} \to \mathbb{C}$ are entire functions such that $|f(z)| \leq m|g(z)|$, where $m \geq 0$ is a real constant, then $f(z) = cg(z)$ for a complex constant c.

Result 4.31a. (H. Haruki [348]). *If $f : \mathbb{C} \to \mathbb{C}$ is an entire function and satisfies equation* (4.71)*, where $z \in \mathbb{C}$ and w is real* (*or purely imaginary*)*, then the solution of* (4.71) *is cz, where c is an arbitrary complex constant and only this.*

Result 4.32. [348]. *If the entire functions $f, g, \varphi, \psi : \mathbb{C} \to \mathbb{C}$ satisfy the functional equation*

$$|f(z + w)| + |g(z - w)| = 2|\varphi(z)| + 2|\psi(w)| \quad \textit{for } z, w \in \mathbb{C}, \tag{4.76}$$

then all solutions of (4.76) *are the following and only these:*

(i) $f(z) = a,\ g(z) = b,\ \varphi(z) = c,\ \psi(z) = d$ *with* $|a| + |b| = 2(|c| + |d|)$,

(ii) $f(z) = (az + b)^2,\ g(z) = e^{i\theta_1}(az + c),\ \varphi(z) = e^{i\theta_2}\left(az + \frac{b+c}{2}\right)^2$,
$\psi(z) = e^{i\theta_3}\left(az + \frac{b-c}{2}\right)^2$,

where $a \neq 0,\ b, c$ *are arbitrary complex constants, and* θ_i $(i = 1, 2, 3)$ *are real constants. Compare this with* Theorem 4.28.

4.10 Summary by Stetkaer [750]

4.10.1 The Quadratic Functional Equation

A norm $\|\cdot\|$ on a vector space V stems from an inner product $(\cdot,\cdot)$ on V (i.e., $\|x\|^2 = (x, x)$ for all $x \in V$) if and only if it satisfies the parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in V. \tag{JvN}$$

This is classical knowledge, going back to the theorem of Apollonius in Euclidean geometry: The sum of the squares of the lengths of the two diagonals of a parallelogram is equal to the sum of the squares of the lengths of the four sides. A special case of Apollonius's Theorem, is Pythagoras's Theorem, which asserts the same for rectangles. Jordan and von Neumann [398] proved the result for vector spaces.

Generalizing the parallelogram identity from a vector space V to a group G, we are led to the *quadratic functional equation*

$$f(xy) + f(xy^{-1}) = 2f(x) + 2f(y), \quad x, y \in G, \tag{Q}$$

where $f : G \to H$ is to be determined. We allow G to be any group and the range space H of f to be any Abelian group.

To describe the results about the quadratic functional equation on groups in a short way, we introduce the following (nonstandard) terminology.

Definition. Let G be a group and H be an Abelian group. We will say that a map $f : G \to H$ is a *quadratic function* if there exists a symmetric bimorphism $B : G \times G \to H$ such that $f(x) = B(x, x)$ for all $x \in G$.

We choose the word function to avoid confusion with earlier works in which a *quadratic form* or a *quadratic functional* by definition is any solution of the quadratic functional equation. Any quadratic function is a solution of the quadratic functional equation (Q), but the converse is not true in general even for $H = \mathbb{C}$.

The basic result for Abelian groups is the theorem below. It is due to Aczél [40, 44]. In Kannappan [471, Result 1], it is mentioned that its assumption of G being Abelian can be replaced by Kannappan's condition (K).

Theorem. [40]. *Let G be an Abelian group and let H be an Abelian group in which every equation of the form $2x = h \in H$ has one and only one solution $x \in H$. Then*

any solution $f : G \to H$ of the quadratic functional equation on G is a quadratic function.

Motivated by the original situation (JvN), Kannappan [471] studied the equation (Q) when G is the additive group of a linear space, to find out when solutions arise from bilinear functionals. So there is not only the group structure to take into account but also the multiplication by scalars. For literature about the vector space situation, we refer to [44, 471] and the recent monograph [187] and their references because here we shall discuss the pure group case, where the group may even be non-Abelian.

Kannappan [471] also finds equivalent forms of the quadratic functional equation when the solutions $f : G \to F$ are defined on a 2-divisible group G, take their values in a field F of characteristic different from 2, and satisfy Kannappan's condition (K). In this setting, he solves certain generalizations of the quadratic functional equation.

The key to the studies on a non-Abelian group G is the following observation.

Observation. If $f : G \to H$ is a solution to the quadratic equation, then its Cauchy difference Cf satisfies Jensen's functional equation (J) in each of its variables when the other variable is fixed.

In a recent preprint, Friis and Stetkaer [293] showed the following results:

1. Any solution of the quadratic functional equation on a group G is a function of $G/[G, G']$, as is the case for solutions of Jensen's functional equation.
2. If $[G, G'] = G'$ holds, which it does for certain non-Abelian groups such as $GL(n, \mathbb{R})$, $n \geq 2$, then all solutions of the quadratic equation on G are quadratic functions.
3. A solution of the quadratic equation on a product of groups satisfies Kannappan's condition if and only if its restriction to each of the subgroups satisfies Kannappan's condition.

5

Characterization of Inner Product Spaces

The inner product space, characterization of i.p.s., parallelepiped law and generation, different characterization of i.p.s. and its generalizations, and different orthogonality are considered in this chapter.

We repeat the customary definition of an inner product space. Recall that a linear (vector) space V is an inner product space (i.p.s.) provided there is a mapping (called the inner product (i.p.)) $\langle\ ,\ \rangle : V \times V \to F$, a field ($F = \mathbb{R}$ or $\mathbb{C}$ depending on whether V is a real or complex vector space) satisfying the following conditions:

(i) $\langle v, v\rangle \geq 0$ with equality if, and only if, $v = 0$ for $v \in V$;
(ii) *symmetry:*

$$\langle u, v\rangle = \langle v, u\rangle \quad \text{(real symmetry) in the real case;}$$
$$\langle v, u\rangle = \overline{\langle u, v\rangle} \quad \text{(conjugate symmetry) in the complex case for } u, v \in V;$$

(iii)*additivity* in each variable:

$$\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle;$$
$$\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle \quad \text{for } u, v, w \in V;$$

(iv)*homogeneity:*

$$\langle \lambda u, v\rangle = \lambda\langle u, v\rangle, \quad \lambda \in \mathbb{R} \text{ or } \mathbb{C},\ u, v \in V;$$
$$\langle u, \lambda v\rangle = \overline{\lambda}\langle u, v\rangle, \quad \lambda \in \mathbb{C},\ u, v \in V.$$

The inner product gives rise to a norm $\|\cdot\|$ (metric) on V given by $\|v\| = \sqrt{\langle v, v\rangle}$, leading to a normed linear space (n.l.s.) V. What is of interest is the converse question. In the class of n.l.s., the i.p.s. are particularly interesting. It is well known that most natural geometric properties may fail to hold in a general n.l.s. unless the space is an i.p.s. It is not without interest to determine when a linear metric space or n.l.s. V is isometric with a Hilbert space or an i.p.s. In other words, when can one define in it a bilinear symmetric i.p. from which its given metric can be derived in

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the customary way? One might then wish to investigate suitable conditions on the n.l.s. V that can be imposed on the norm under which the given norm on V could have arisen from an inner product. Numerous basic characterizations of i.p.s. under various conditions were given starting with Fréchet [291], (typical classical result of) Jordan and von Neumann [398], etc. Conditions have been found with motivations that are highly geometrical, and others are primarily algebraic. There is an interesting survey book by Amir [73] that lists quite a number of characterizations. Since then, many new characterizations have been found. Some of them have been given in terms of the functions (*normal derivatives*) (Amir [73], Alsina and Garcia-Roig [70]) $\rho'_{\pm} : V \times V \to \mathbb{R}$ defined by

$$\rho'_{\pm}(x, y) = \lim_{t \to 0^{\pm}} \frac{\|x + ty\|^2 - \|x\|^2}{t},$$

height functions [70] $h_1 : V \times V \to \mathbb{R}$ by

$$h_1(x, y) = y + \frac{\|y\|^2 - \rho'_+(x, y)}{\|x - y\|^2}(x - y),$$

etc., and also in terms of the classical orthogonal relations.

Numerous equivalent definitions of orthogonality of elements of abstract Euclidean spaces can be given (Amir [73], James [389]). The most natural is the *Pythagorean relation*: $x \perp^p y$ whenever $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

Other Orthogonal Relations

Blaschke-Birkhoff-James orthogonal relation: $x \perp^B y$ whenever $\|x\| \leq \|x + ty\|$ for all $t \in \mathbb{R},\ x, y \in V$.

Robert's relation: $x \perp^R y \Leftrightarrow \|x + ky\| = \|x - ky\|$ for all $k \in \mathbb{R},\ x, y \in V$.

James relation (isosceles orthogonality): $x \perp^J y$ whenever $\|x - y\| = \|x + y\|$.

The concept of orthogonality has certain desired properties, the most important being that every two-dimensional subspace contains nonzero orthogonal elements.

Broadly, characterizations fall into (i) equalities, (ii) normal derivatives, (iii) approximations, (iv) various orthogonality conditions, (v) inequalities, (vi) functional equations, etc. Item (v) inequalities will be treated later, in Chapter 16.

The usual arguments for the characterizations of i.p.s. deal with rather sophisticated geometric properties of a unit ball (smoothness, etc.). A functional approach favours direct arguments, and some of these may be valid in the case of vector spaces over fields other than the real field.

Functional equations are instrumental in many characterizations. It is fruitful to look at the matter from this point of view. One of the objectives of this chapter is to bring out the involvement of functional equations in various characterizations of i.p.s. (Aczél [44], Dhombres [218], Day [214], Rassias [684]).

We start off with a result of Fréchet [291]. In Theorems 5.1 and 5.2, Fréchet discovered a necessary and sufficient algebraic condition from which he derived this

more abstract criterion: An n.l.s. is an i.p.s. if, and only if, every three-dimensional subspace V' of V is isometric with a Euclidean (three-dimensional) space. The next interesting movement forward was from Jordan and von Neumann [398], who described another necessary and sufficient algebraic condition that implies that Fréchet's abstract condition can be weakened as follows: An n.l.s. is an i.p.s. if and only if every two-dimensional subspace V' of V is isometric with a Euclidean (two-dimensional) space. The basic algebraic (norm) condition that results from the above that makes the n.l.s. an i.p.s. (and the oldest prototype of all these results) is the *parallelogram identity*, also known as the *Jordan–von Neumann identity* (*or Appolonius Law or the norm equation*)

$$\text{(JvN)} \qquad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for } x, y \in V.$$

This translates into a well-known functional equation known as the *quadratic functional equation*,

$$\text{(Q)} \qquad q(x+y) + q(x-y) = 2q(x) + 2q(y) \quad \text{for } x, y \in V,$$

studied in Chapter 4. (Because of this fact, most of the computations centre around (JvN) and (Q).)

In a real two-dimensional space, it is well known that the space is an i.p.s. if and only if the set of points of norm one is an ellipse [212] (a geometrical characterization).

First we consider Fréchet's result and then follow it with the Jordan–von Neumann result.

5.1 Fréchet's Equation

The functional equation resulting from Fréchet's condition [588, 464, 463, 465, 580] is

$$\text{(F)} \qquad p(x+y+z) = p(x+y) - p(z) + p(y+z) - p(x) + p(z+x) - p(y) \quad \text{for } x, y, z \in G.$$

We obtain the solution of (F) on a general setting.

Theorem 5.1. (Kannappan [465, 471]). *Let G be a group* (*not necessarily Abelian*), *H an Abelian group, and $p : G \to H$ satisfy the functional equation* (F). *Then p satisfies*

$$8p(x) = 4A(x) + B(x,x) \quad \textit{for } x \in G, \tag{5.1}$$

where $A : G \to H$ is additive and $B : G \times G \to H$ is biadditive.

Proof. Set $z = 0$ in (F) to get $p(0) = 0$.

With $z = -(x+y)$ in (F), (F) gives

$$p(x+y) - p(-(x+y)) = p(x) - p(-x) + p(y) - p(-y);$$

that is,

$$A(x) = p(x) - p(-x) \quad \text{for } x \in G \tag{5.2}$$

(satisfies (A)). Set $z = -y$ in (F) to obtain

$$p(x+y) + p(-y+x) = 2p(x) + p(y) + p(-y) \quad \text{for } x, y \in G$$

(similar to (4.22)). Replacing x by $-x$ and adding the resultant to the equation above, we have

$$q(x+y) + q(-x+y) = 2q(x) + 2q(y) \quad \text{for } x, y \in G, \tag{5.3}$$

where

$$q(x) = p(x) + p(-x) \quad \text{for } x \in G. \tag{5.4}$$

Remark. If G is Abelian, (5.3) is (Q) and (5.4) and (5.2) give (5.1).

Note that $q(0) = 0$, q is even, $q(2x) = 4q(x)$, $q(x+y) = q(y+x)$, and

$$2p(x) = q(x) + A(x), \quad \text{for } x \in G. \tag{5.5}$$

Since p and A are solutions of (F), so is q; that is,

$$\begin{aligned} q(x+y+z) = {} & q(x+y) - q(z) + q(y+z) \\ & - q(x) + q(z+x) - q(y) \quad \text{for } x, y, z \in G \end{aligned} \tag{5.6}$$

(see (4.5)).

Define $B : G \times G \to H$ by

$$B(x, y) = q(x+y) - q(x-y) \quad \text{for } x, y \in G. \tag{5.7}$$

From (5.6), (5.7), $q(x+y) = q(y+x)$, and that q is even, we have

$$\begin{aligned} B(x+y, z) &= q(x+y+z) - q(x+y-z) \\ &= q(y+z) - q(y-z) + q(z+x) - q(-z+x) \\ &= B(x, z) + B(y, z); \end{aligned}$$

that is, B is additive in the first variable. By the symmetry of B, it follows that B is biadditive.

From (5.7) and (5.3), it is easy to see that $4q(x) = B(x, x)$, and then from (5.5) we obtain (5.1). This proves the theorem. □

Remark. Historically, (Q) is connected to i.p.s. Here again we obtain (Q) as follows.

Changing y to $-y$ in (5.4) and using that q is even and $q(x+y) = q(y+x)$, we get (Q) from (5.3).

Remark. If H is a 2-divisible Abelian group, then (5.1) becomes $p(x) = A(x) + B(x, x)$ (see also Theorem 4.29 and (4.27)).

Now we consider a generalization of (F):

$$\text{(F1)} \qquad \begin{aligned} p(x+y+z) &= r(x+y) - r(z) + s(y+z) \\ &\quad - s(x) + t(z+x) - t(y) \quad \text{for } x, y, z \in G. \end{aligned}$$

Theorem 5.1a. (Kannappan [470]). *Let G be a group* (*not necessarily Abelian*) *and H an Abelian group. Let $p, r, s, t : G \to H$ satisfy the functional equation* (F1). *The most general solution of* (F1) *is given by*

$$16p(x) = 16(r(x)-a) = 16(s(x)-b) = 16(t(x)-c) = 4A(x)+B(x,x) \quad \textit{for } x \in G,$$

where A is additive, B is biadditive, and a, b, c are constants.

Proof. Set $x = y = z = 0$ in (F1) to get $p(0) = 0$. Now put $y = z = 0$ in (F1) to obtain

$$p(x) = r(x) - a - s(x) + b + t(x) - c \quad \text{for } x \in G, \tag{5.8}$$

where $r(0) = a,\ s(0) = b,\ t(0) = c$.

From (5.8) and (F1) with $z = 0$, we get

$$\alpha(x+y) = \alpha(x) - \alpha(y) \quad \text{for } x, y \in G,$$

where $\alpha(x) = p(x) - r(x) + a$.

Obviously, $\alpha(0) = 0$ and $\alpha(y) = -\alpha(y)$ (put $x = 0$ above); that is, $2\alpha(y) = 0$. Hence

$$2p(x) = 2(r(x) - a).$$

Similarly, we get $2p(x) = 2(s(x) - b) = 2(t(x) - c)$. Thus $2p(x)$ is a solution of (F), and from Theorem 5.1, we have

$$16p(x) = 4A(x) + B(x,x),$$

where A is additive and B is biadditive. This proves the theorem. □

Remark. If H is a 2-divisible Abelian group in Theorem 5.1a, then

$$p(x) = r(x) - a = s(x) - b = t(x) - c = A(x) + B(x,x).$$

5.1.1 The Parallelepiped Law

Theorem 5.1b. *Any Banach space $\mathcal{B}$* (*over $\mathbb{R}$*) *that satisfies*

$$\|x+y+z\|^2 = \|x+y\|^2 + \|y+z\|^2 + \|z+x\|^2 - \|x\|^2 - \|y\|^2 - \|z\|^2 \tag{5.9}$$

for all $x, y, z \in \mathcal{B}$ is a Hilbert space (*i.p.s.*) *and conversely.*

Proof. The converse is easy to verify. Let us assume that (5.9) holds. Define $q : V \to \mathbb{R}$ by $q(x) = \|x\|^2$. Then $q(0) = 0,\ q$ is even, and q satisfies (5.6). Then, by Theorem 5.1, we have

$$q(x) = B(x, x) + A(x) \quad \text{for } x \in V,$$

where B is biadditive and A is additive. Since q is even, $A = 0$, so that $q(x) = B(x, x)$.

Note that

$$\begin{aligned} B(x, y) &= \frac{1}{2}[q(x + y) - q(x) - q(y)] \\ &= \frac{1}{4}[q(x + y) - q(x - y)] \\ &= \frac{1}{2}(\|x + y\|^2 - \|x\|^2 - \|y\|^2) \quad \text{for } x, y \in V. \end{aligned}$$

In general, B is biadditive only. Here B is bilinear. Indeed, for $t, s \in \mathbb{R},\ x, y \in V$,

$$\|tx + y\| = \|(t - s)x + sx + y\| \leq |t - s|\,\|x\| + \|sx + y\|$$

or

$$|\,\|tx + y\| - \|sx + y\|\,| \leq |t - s|\,\|x\|.$$

Let $r_n \in Q$ such that $r_n \to t$ as $n \to \infty$. Then

$$|\,\|r_n x + y\| - \|tx + y\|\,| \leq |r_n - t|\,\|x\| \to 0 \text{ as } n \to \infty;$$

that is, $\|r_n x + y\| \to \|tx + y\|$ as $r_n \to t$ for $x, y \in V$. Note that since B is biadditive, $B(r_n x, y) = r_n B(x, y)$, for $r_n \in Q,\ x, y \in V$. Now

$$\begin{aligned} r_n B(x, y) &= B(r_n x, y) \\ &= \frac{1}{2}\left[\|r_n x + y\|^2 - \|r_n x\|^2 - \|y\|^2\right] \\ &\to \frac{1}{2}\left[\|tx + y\|^2 - \|tx\|^2 - \|y\|^2\right] \quad \text{as } r_n \to t \\ &= B(tx, y). \end{aligned}$$

But then $r_n B(x, y) \to tB(x, y)$ as $r_n \to t$. Hence, $B(tx, y) = tB(x, y)$; that is, B is bilinear and so V is an i.p.s.

If we use $B(x, y) = \frac{1}{4}[\|x + y\|^2 - \|x - y\|^2]$, then, for $r_n \in Q,\ t \in \mathbb{R}$,

$$\begin{aligned} r_n B(x, y) &= B(r_n x, y) \\ &= \frac{1}{4}\left[\|r_n x + y\|^2 - \|r_n x - y\|^2\right] \\ &\to \frac{1}{4}\left[\|tx + y\|^2 - \|tx - y\|^2\right] \quad \text{as } r_n \to t \\ &= B(tx, y). \end{aligned}$$

This completes the proof. □

5.1.2 Parallelogram Identity

Next in line is the Jordan–von Neumann result [398]. By putting $z = -y$ in (5.9), they obtained the identity (JvN), showed that the new condition is as strong as equation (5.9), and proved Theorem 5.2.

Theorem 5.2. (Jordan and von Neumann [398]). *An n.l.s. V over $\mathbb{R}$ satisfying* (JvN) *must be a Hilbert space (that is, an i.p.s.).*

Proof. Suppose (JvN) holds. Define $q : V \to \mathbb{R}$ by $q(x) = \|x\|^2$. Then q satisfies (Q), and by Theorem 4.1, $q(x) = B(x, x)$, where B is a symmetric, biadditive function. Then use the argument in Theorem 5.1b to show that B is bilinear. Q.E.D.

Subsequent authors have found norm conditions weaker than (JvN) that require a Banach space to be an i.p.s. Hereafter the central equation will be (Q); that is, most of the characterizations will follow from the parallelogram identity (JvN) and (Q). In a way, the equation (Q) arises in many characterizations of i.p.s.

Consider the three-dimensional *parallelepiped identity* (Röhmel [708], Kannappan [470])

$$\begin{aligned}\|x+y+z\|^2 + \|x+y-z\|^2 + \|x-y+z\|^2 + \|x-y-z\|^2 \\ = 2(\|x+y\|^2 + \|x-y\|^2 + \|x+z\|^2 \\ + \|x-z\|^2 + \|y+z\|^2 + \|y-z\|^2) \\ - 4(\|x\|^2 + \|y\|^2 + \|z\|^2) \quad \text{for } x, y, z \in V \text{ an n.l.s.}\end{aligned} \tag{5.10}$$

Put $y + z = 0$ in (5.10) to get

$$\|2x+z\|^2 + \|2x-z\|^2 = 4\|x+z\|^2 + 4\|x-z\|^2 - 6\|z\|^2. \tag{5.11}$$

Replace z by $2z$ in (5.11) and use (5.10) to have

$$\begin{aligned}4\|x+z\|^2 + 4\|x-z\|^2 &= 4\|x+2z\|^2 + 4\|x-2z\|^2 - 24\|z\|^2 \\ &= 16\|x+z\|^2 + 16\|x-z\|^2 - 24\|z\|^2 - 24\|x\|^2,\end{aligned}$$

which is (JvN). Hence an n.l.s. satisfying (5.10) is an i.p.s. □

As a generalization (functional equation version) of (5.10), we treat the functional equation

$$\begin{aligned}\phi(x+y+z) + \phi(x+y-z) + \phi(x-y+z) + \phi(x-y-z) \\ = 2(\phi(x+y) + \phi(x-y) + \phi(x+z) + \phi(x-z) \\ + \phi(y+z) + \phi(y-z)) - 4(\phi(x) + \phi(y) + \phi(z)) \\ \text{for } x, y, z \in V,\end{aligned} \tag{5.10a}$$

where $\phi : V \to \mathbb{R}$ and V is an n.l.s. over $\mathbb{R}$ [470].

Theorem 5.3. [470]. *Suppose $\phi : V \to \mathbb{R}$ satisfies the functional equation* (5.10a). *Then $q(x) = \phi(2x) - 16\phi(x)$ satisfies* (Q).

Proof. First z to $-z$ in (5.10a) give $\phi(z) = \phi(-z)$; that is, ϕ is even, and then $z = 0$ in (5.10a) gives $\phi(0) = 0$.

Let $z = y$ and $z = x + y$ separately in (5.10a) to obtain

$$\phi(x+2y)+\phi(x-2y) = 4[\phi(x+y)+\phi(x-y)]-6\phi(x)-8f(y)+2\phi(2y) \tag{5.12}$$

and

$$\begin{aligned}\phi(2x+2y)+\phi(2x)+\phi(2y)+2[\phi(x+y)-\phi(x-y)]\\ +2[\phi(x)+\phi(y)]-2[\phi(x+2y)+\phi(2x+y)] = 0 \quad \text{for } x, y \in V.\end{aligned} \tag{5.13}$$

Replace y by $-y$ in (5.13), add the equation thus obtained to (5.13), and use (5.12) twice (once with x and y interchanged) to get

$$\begin{aligned}\phi(2x+2y)+\phi(2x-2y)-2\phi(2x)-2\phi(2y)-16\phi(x+y)\\ -16\phi(x-y)+32\phi(x)+32\phi(y) = 0 \quad \text{for } x, y \in V,\end{aligned}$$

which transforms to (Q) by defining $q(x) = \phi(2x) - 16\phi(x)$. This proves the theorem. □

Remark. The solution of (5.10) can be obtained from (5.10a).

Suppose (5.10) holds in V. Define $\phi(x) = \|x\|^2$ for $x \in V$. Then ϕ satisfies (5.10a) and we get $q(x) = -12\|x\|^2$; that is, the norm satisfies (JvN) and hence V is an i.p.s.

5.1.3 More on Fréchet's Result

Putting successively $x = u + 2v$, $y = v$ and $x = u + v$, $y = v$ in (JvN) and subtracting the latter from the former, we obtain

$$\|u + 3v\|^2 - 3\|u + 2v\|^2 + 3\|u + v\|^2 - \|u\|^2 = 0 \quad \text{for } u, v \in V. \tag{5.14}$$

This equation is a special case ($k = 3$) of

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \|u + jv\|^2 = 0 \quad \text{for } u, v \in V,$$

treated by Johnson [396] (see also [708]). The known proof that (5.14) implies V is an i.p.s. is based on Fréchet's result [284] that if a continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfies

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} g(r + js) = 0 \quad \text{for } r, s \in \mathbb{R} \tag{5.15}$$

(that is, the $(n + 1)$th-order difference of g is zero), then g is a polynomial of degree at most n. Here, we give a simple and direct proof in Theorem 5.4 that (5.14) implies that V is an i.p.s. and obtain the general solution of (5.15) for $n = 3$ without assuming any regularity condition on g in Theorem 5.4a.

Theorem 5.4. *An n.l.s. V satisfying* (5.14) *is an i.p.s.*

Proof. Replacing u by $u - v$ and u by $u - 2v$ separately in (5.14), we get

$$\|u + 2v\|^2 - 3\|u + v\|^2 + 3\|u\|^2 - \|u - v\|^2 = 0,$$
$$\|u - 2v\|^2 - 3\|u - v\|^2 + 3\|u\|^2 - \|u + v\|^2 = 0.$$

Adding the two equations above, we obtain

$$\|u + 2v\|^2 + \|u - 2v\|^2 = 4\|u + v\|^2 + 4\|u - v\|^2 - 6\|u\|^2. \tag{5.16}$$

We have after replacing $2v$ by v in (5.16)

$$\|u + v\|^2 + \|u - v\|^2 = \|2u + v\|^2 + \|2u - v\|^2 - 6\|u\|^2,$$

which, by using (5.16) again (this time interchanging u and v), gives

$$\|u + v\|^2 + \|u - v\|^2 = 4\|u + v\|^2 + 4\|u - v\|^2 - 6\|v\|^2 - 6\|u\|^2;$$

that is, (JvN). This proves the theorem. □

Now we turn to finding the general solution of (5.15) for $n = 3$. First note that the case $n = 2$ reduces to Jensen's equation and that $g(x) = A(x) + d$, where A is additive; that is, g is a "polynomial" of degree at most 1.

Equation (5.15) with $n = 3$ yields

$$g(x + 3y) - 3g(x + 2y) + 3g(x + y) - g(x) = 0 \quad \text{for } x, y \in \mathbb{R}.$$

Theorem 5.4a. *Suppose $f : V \to \mathbb{R}$ has all its third-order differences zero; that is, f satisfies*

$$f(u + 3y) - 3f(u + 2v) + 3f(u + v) - f(u) = 0 \quad \textit{for } u, v \in V. \tag{5.14a}$$

Then f has the form

$$f(x) = d + A(x) + q(x) = d + A(x) + B(x, x) \quad \textit{for } x \in V, \tag{5.17}$$

where d is a constant, A is additive, q satisfies (Q)*, and B is symmetric and biadditive; that is, f is a "polynomial" of degree at most* 2.

Proof. Replacing u by x and v by $x - y$ in (5.14a), we get

$$f(x + 2y) = 3f(x + y) + f(x - y) - 3f(x), \quad x, y \in V. \tag{5.18}$$

Interchange x and y in (5.18) to get

$$f(2x + y) = 3f(x + y) + f(y - x) - 3f(y) \quad \text{for } x, y \in V. \tag{5.19}$$

With $y = 0$, (5.19) yields

$$f(2x) = 3f(x) + f(-x) - 3f(0) \quad \text{for } x \in V. \tag{5.20}$$

Replacing x by $2x$ in (5.19) and using (5.19) and (5.20), we obtain

$$2f(x+y)+f(x-y)+f(y-x) = 3f(x)+f(-x)+3f(y)+f(-y)-4f(0). \tag{5.21}$$

Changing y into $-y$ in (5.21) and subtracting the resultant equation from (5.21), we have

$$J(y+x) + J(y-x) = 2J(y),$$

which is Jensen, where

$$J(x) = f(x) - f(-x).$$

Since J is odd and $J(0) = 0$, we get

$$f(x) - f(-x) = A(x), \quad \text{where } A \text{ is additive.} \tag{5.22}$$

Substituting (5.22) into (5.21), we have

$$2f(x+y) + 2f(x-y) - A(x-y) = 4f(x) - A(x) + 4f(y) - A(y) - 4f(0);$$

that is,

$$q(x+y) + q(x-y) = 2q(x) + 2q(y),$$

where

$$q(x) = f(x) - \frac{1}{2}A(x) - f(0)$$

(q satisfies (Q)). Now (5.17) follows from Theorem 4.1. This completes the proof of this theorem. □

Remark. The general solution of (5.21) can be obtained from [241] by identifying $2f = f_1$, $f(x) + f(-x) = f_2(x)$, and $3f(x) + f(-x) - 2f(0) = f_3(x) = f_4(x)$.

Remark. If in Theorem 5.14a $V = \mathbb{R}$ and f is continuous and satisfies (5.14a), then f is a polynomial of degree at most 2 (see Fréchet [288, 290]).

5.1.4 More Characterizations

Result 5.5. (Amir [73]). *Let $\phi : V^3 \to \mathbb{R}$ and V an n.l.s. over $\mathbb{R}$ be such that*

$$\phi(x,y,z) = \|x+y+z\|^2 + \|x+y-z\|^2 - \|x-y+z\|^2 - \|x-y-z\|^2, \tag{5.23}$$

for $x, y, z \in V$, is independent of z. Then V is an i.p.s.

Proof. Set $z = x + y$ and $z = 0$ to have

$$4\|x+y\|^2 - 4\|x\|^2 - 4\|y\|^2 = 2\|x+y\|^2 - 2\|x-y\|^2,$$

which is (JvN). So, V is an i.p.s. □

Theorem 5.5a. *Suppose $\phi : V^3 \to \mathbb{R}$, $q : V \to \mathbb{R}$, and V an n.l.s. over $\mathbb{R}$ are such that*

$$\phi(x,y,z) = q(x+y+z) + q(x+y-z) - q(x-y+z) - q(x-y-z), \quad \text{for } x,y,z \in V, \tag{5.23a}$$

is independent of z. Then there is an additive $A : V \to \mathbb{R}$ and biadditive $B : V^2 \to \mathbb{R}$ such that

$$q(x) = B(x,x) + A(x) + b \quad \text{for } x \in V.$$

This is a functional equation version of (5.23).

Proof. $z = x + y$ and $z = x - y$ in (5.23a) yield

$$q(2(x+y)) + q(0) - q(2x) - q(-2y) = q(2x) + q(2y) - q(2(x-y)) - q(0) \quad \text{for } x,y \in V;$$

that is,

$$q(x+y) + q(x-y) = 2q(x) + q(y) + q(-y) - 2q(0),$$

which is (4.22). Thus there exist a biadditive B and an additive A (Theorem 4.22) such that

$$q(x) - q(0) = B(x,x) + A(x) \quad \text{for } x \in V.$$

□

Remark. If $q(x) = \|x\|^2$ in Result 5.5a, then $q(0) = 0$, $A = 0$ (since q is even), q satisfies (5.23), and $q(x) = B(x,x)$. So, q satisfies (JvN) and V is an i.p.s.

Theorem 5.6. [73]. *Let V be an n.l.s. and $\phi : \mathbb{R} \to \mathbb{R}$ be continuous with $\phi(0) = 0$, $\phi(1) = 1$ and satisfy*

$$\phi(\|x+y\|) + \phi(\|x-y\|) = 2\phi(\|x\|) + 2\phi(\|y\|) \quad \text{for } x,y \in V. \tag{5.24}$$

Then V is an i.p.s.

Proof. From (5.24), it is easy to check that $\phi(n\|x\|) = n^2\phi(\|x\|)$ for $n \in Z_+$, $x \in V$. Replacing x by $\frac{x}{n}$, we see that $\phi\left(\frac{1}{n}\|x\|\right) = \frac{1}{n^2}\phi(\|x\|)$ or $\phi(r\|x\|) = r^2\phi(\|x\|)$, r positive rational, and $x \in V$. Since ϕ is continuous, $\phi(t\|x\|) = t^2\phi(\|x\|)$, $t > 0$, $t \in \mathbb{R}$, $x \in V$. Choosing x with $\|x\| = 1$, we get $\phi(t) = t^2\phi(1) = t^2$ since $\phi(1) = 1$. So, (5.24) becomes (JvN) and V is an i.p.s.

This could also be achieved as follows. Define $q : V \to \mathbb{R}$ by

$$q(x) = \phi(\|x\|) \quad \text{for } x \in \mathbb{R}.$$

Then q satisfies (Q), q is continuous, and there exists a symmetric, biadditive $B : V \times V \to \mathbb{R}$ such that

$$q(x) = B(x,x),$$
$$B(x,y) = \frac{1}{2}(q(x+y) - q(x-y)), \quad \text{for } x,y \in V.$$

To prove the bilinearity of B, notice that q is continuous and $B(rx, y) = rB(x, y)$, $r \in Q$ (see Theorem 5.1b).

Thus V is an i.p.s. □

Result 5.7. (Sundaresan [780], Rassias [679], Amir [73]). *Let V be an n.l.s. Suppose for $n \geq 2$ that*

$$\sum_{i=1}^{n} \|x_i - \overline{x}\|^2 = \sum_{i=1}^{n} \|x_i\|^2 - n\|\overline{x}\|^2 \tag{5.25}$$

holds for $x_1, x_2, \ldots, x_n \in V$ with $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. Then V is an i.p.s.

Here (JvN) *is used.*

Next we will prove a functional equation analogue of (5.25) that will cover Result 5.7.

Theorem 5.7a. (Kannappan [466]). *Let V be a linear space over $\mathbb{R}$ and $\phi, \psi : V \to \mathbb{R}$ satisfy the functional equation*

$$\sum_{i=1}^{n} \phi(x_i - \overline{x}) = \sum_{i=1}^{n} \phi(x_i) - n\psi(\overline{x}) \tag{5.25a}$$

for $x_1, x_2, \ldots, x_n \in V$ and $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ for a fixed $n \geq 3$. Then

$$\begin{aligned} \phi(x) &= q(x) + A(x) + c = B(x, x) + A(x) + c, \\ \psi(x) &= B(x, x) + A(x), \end{aligned} \tag{5.26}$$

for $x \in V$, where q satisfies (Q), *A satisfies* (A), *B is biadditive, and c is a constant.*

Proof. Set $x_1 = x_2 = \cdots = x_n = x$ in (5.25a) to get

$$\psi(x) = \phi(x) - \phi(0) \quad \text{for } x \in V. \tag{5.27}$$

Now (5.25a) can be written as

$$\sum_{i=1}^{n} \phi(x_i - \overline{x}) = \sum_{i=1}^{n} \phi(x_i) - n\phi(\overline{x}) + n\phi(0). \tag{5.28}$$

First let n be even, say $2k$. Put $x_1 = \cdots = x_k = x$, $x_{k+1} = \cdots = x_{2k} = y$ in (5.28) to have

$$\phi\left(\frac{x-y}{2}\right) + \phi\left(\frac{y-x}{2}\right) = \phi(x) + \phi(y) - 2\phi\left(\frac{x+y}{2}\right) + 2\phi(0). \tag{5.29}$$

Next, let n be odd, say $2k+1$. Set $x_1 = \cdots = x_k = x$, $x_{k+1} = \cdots = x_{2k} = y$, $x_{2k+1} = \frac{x+y}{2}$ in (5.28) to obtain

$$k\phi\left(\frac{x-y}{2}\right)+k\phi\left(\frac{y-x}{2}\right)+\phi(0)$$
$$=k\phi(x)+k\phi(y)+\phi\left(\frac{x+y}{2}\right)-(2k+1)\phi\left(\frac{x+y}{2}\right)+(2k+1)\phi(0),$$

which is (5.29). Now we solve (5.29).

Replacing x by $2x$ and y by $2y$ in (5.29), we have

$$2\phi(x+y)+\phi(x-y)+\phi(y-x)=\phi(2x)+\phi(2y)+2\phi(0). \tag{5.30}$$

Define

$$2q(x)=\phi(x)+\phi(-x)-2\phi(0) \quad \text{for } x\in V, \tag{5.31}$$

so that (5.30) can be rewritten as

$$\phi(2x)+\phi(2y)=2q(x-y)+2\phi(x+y);$$

that is,

$$\phi(u+v)+\phi(u-v)=2\phi(u)+2q(v) \quad \text{for } u,v\in V. \tag{5.32}$$

Note that q is even and $q(0)=0$. Change u to $-u$ in (5.32) to yield

$$\phi(-u+v)+\phi(-u-v)=2\phi(-u)+2q(v). \tag{5.33}$$

Adding (5.33) and (5.32) and utilizing (5.31), we obtain

$$q(u+v)+q(u-v)=2q(u)+2q(v); \tag{Q}$$

that is, q is quadratic so that $q(x)=B(x,x)$, for a biadditive B, by Theorem 4.1. Subtracting (5.33) from (5.32), we get

$$A(u+v)+A(u-v)=2A(u),$$

which is Jensen, where

$$A(u)=\phi(u)-\phi(-u) \quad \text{for } u\in V. \tag{5.34}$$

Note that $A(0)=0$ and A is odd. Thus A is additive. Adding (5.31) and (5.34) and from (5.27), we obtain the result sought, (5.26). This proves the theorem. □

Remark. In Result 5.7a, ϕ and ψ are obtained without assuming any regularity condition. It is easy to see that Result 5.7 is an easy consequence of Result 5.7a.

Theorem 5.7b. (Kannappan [466]). *Let V be an n.l.s. Suppose $\phi,\psi:\mathbb{R}_+\to\mathbb{R}$ satisfy the functional equation*

$$\sum_{i=1}^{n}\phi(\|x_i-\overline{x}\|)=\sum_{i=1}^{n}\phi(\|x_i\|)-n\psi(\|\overline{x}\|) \tag{5.25b}$$

for $x_1, x_2, \ldots, x_n \in V$ *and* $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ *for a fixed* $n \geq 2$. *Then*

$$\phi(\|x\|) = B(x,x) + c, \quad \psi(\|x\|) = B(x,x), \quad \text{for } x \in V, \tag{5.26a}$$

where B *is biadditive and* c *is a constant.*

Proof. As in Result 5.7a, we consider two cases, where n is even and n is odd.

Case 1. Let $n = 2k$. Set $x_1 = x_2 = \cdots = x_k = x$, $x_{k+1} = x_{k+2} = \cdots = x_{2k} = y$ in (5.25b) to get

$$2\phi\left(\frac{\|x-y\|}{2}\right) = \phi(\|x\|) + \phi(\|y\|) - 2\psi\left(\frac{\|x+y\|}{2}\right);$$

that is,

$$2\phi(\|x-y\|) = \phi(2\|x\|) + \phi(\|2y\|) - 2\psi(\|x+y\|) \quad \text{for } x, y \in V, \tag{5.35}$$

and

$$\phi(\|x\|) - \psi(\|x\|) = \phi(0) \quad (\text{put } y = x \text{ in the above}), \tag{5.26}$$

and so

$$2\phi(\|x+y\|) + 2\phi(\|x-y\|) = \phi(\|2x\|) + \phi(\|2y\|) + 2\phi(0) \quad \text{for } x, y \in V.$$

Case 2. Let $n = 2k+1$ be odd.

Let $x_1 = x_2 = \cdots = x_k = x$, $x_{k+1} = \cdots = x_{2k} = y$, and $x_{2k+1} = \frac{x+y}{2}$ in (5.25b) to obtain

$$2k\phi\left(\frac{\|x-y\|}{2}\right) + \phi(0) = k\phi(\|x\|) + k\phi(\|y\|) + \phi\left(\frac{\|x+y\|}{2}\right) - (2k+1)\psi\left(\frac{x+y}{2}\right)$$

and (5.27), and thus (5.35).

Now we solve (5.35). Set $x + y = u$, $x - y = v$, and $q(x) = \phi(\|x\|) - \phi(0)$ in (5.35) to have (Q) satisfied by q. Then, from Theorem 4.1, it follows that $q(x) = B(x,x)$ for $x \in V$, where B is biadditive, and then (5.26a) holds. This proves the result. □

Theorem 5.7c. [466]. *Let* V *be an n.l.s. over* $\mathbb{R}$, $g : \mathbb{R}_+ \to \mathbb{R}$ *be continuous, and* $h : \mathbb{R}_+ \to \mathbb{R}$ *satisfy the functional equation*

$$\sum_{i=1}^{n} g(\|x_i - \overline{x}\|^2) = \sum_{i=1}^{n} g(\|x_i\|^2) - nh(\|\overline{x}\|^2) \tag{5.25c}$$

for $n \geq 2$, $x_1, x_2, \ldots, x_n \in V$, $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. *Then* V *is an i.p.s.*

Proof. If g is a solution of (5.25c), so is $g - c$. So, without loss of generality, assume $g(0) = 0$. We can use either Result 5.7b or Result 5.7a (see [466]) to prove this result. We use Result 5.7b. Define $\phi, \psi : V \to \mathbb{R}$ by

$$\phi(\|x\|) = g(\|x\|^2), \ \psi(\|x\|) = h(\|x\|^2), \quad x \in V.$$

Then (5.25c) becomes (5.25b), and by Result 5.7b we get

$$g(\|x\|^2) = \phi(\|x\|) = B(x, x) = \psi(\|x\|) = h(\|x\|^2) \quad \text{for } x \in V$$

since $c = 0$ and $A = 0$ (since ϕ is even), where B is a symmetric, biadditive function. Now g satisfies

$$g(\|x + y\|^2) + g(\|x - y\|^2) = 2g(\|x\|^2) + 2g(\|y\|^2)$$

for $x, y \in V$.

By induction, it follows that $g(\|nx\|^2) = n^2 g(\|x\|^2)$, $g(r^2t^2) = r^2 g(t^2)$, for $r \in Q$, $t \in \mathbb{R}_+$, and by continuity $g(p^2t^2) = p^2 g(t^2)$ for any $p, t \in \mathbb{R}_+$; that is, $g(p^2) = cp^2$ for some constant c, $p \in \mathbb{R}_+$. Then the norm $\|\cdot\|$ satisfies (JvN) so that V is an i.p.s. Hence we get the result. □

Next we consider the functional equation

$$\phi(t) = f(x + ty) \quad \text{for } x, y \in V, \ t \in \mathbb{R}. \tag{5.36}$$

In Amir [73] and Röhmel [708], it is proven that if $\phi(t) = \|x + ty\|^2$ for $x, y \in V$ an n.l.s., and $t \in \mathbb{R}$ is a polynomial in t of degree 2, then V is an i.p.s. Now we prove the following result.

Theorem 5.8. (Kannappan [470]). *Suppose that the function $f : V \to \mathbb{R}$ is such that, for each x, y in V, the function $\phi : \mathbb{R} \to \mathbb{R}$ given by* (5.36) *is a polynomial function of degree* 2 *in t; that is, there exist $a(x, y)$, $b(x, y)$, and $c(x, y)$ such that*

(5.37)
$$\phi(t) = f(x + ty) = a(x, y)t^2 + b(x, y)t + c(x, y), \quad \text{for } t \in \mathbb{R} \text{ and } x, y \in V,$$

holds. Then f and ϕ are given by

$$f(x) = q(x) + A(x) + d = B(x, x) + A(x) + d \quad \text{for } x \in V, \tag{5.36a}$$

$$\phi(t) = q(y)t^2 + [2B(x, y) + A(y)]t + q(x) + A(x) + d \tag{5.36b}$$

for $t \in \mathbb{R}$, $x, y \in V$, where A is additive, B is biadditive, q satisfies (Q)*, and d is a constant.*

Proof. Since $\phi(t)$ is a quadratic function of t, we can write (5.37) as

$$\phi(t) = f(x + ty) = at^2 + bt + c, \tag{5.38}$$

where a, b, c are functions of x and y. (Since we will not substitute anything into a, b, c as such, we don't write their arguments x, y explicitly.)

Setting $t = 0$ in (5.38), we get

$$f(x) = c.$$

Replacing t by $-t$ in (5.38), we have

$$f(x - ty) = at^2 - bt + c. \tag{5.38a}$$

Subtracting and adding (5.38a) and (5.38), we have

$$f(x + ty) - f(x - ty) = 2bt \tag{5.39}$$

and

$$f(x + ty) + f(x - ty) = 2at^2 + 2c = 2at^2 + 2f(x), \tag{5.40}$$

respectively.

Letting $t = 1$ and 2 separately in (5.39) and (5.40) and eliminating b and a, we obtain

$$f(x + 2y) - f(x - 2y) = 2(f(x + y) - f(x - y))$$

and

$$f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] - 6f(x),$$

from which follows

$$f(x + 2y) = 3f(x + y) + f(x - y) - 3f(x) \quad \text{for } x, y \in V, \tag{5.14a}$$

which is (5.18).

Now, from the proof of Theorem 5.4a, it follows that

$$f(x) = q(x) + \frac{1}{2}A(x) + f(0),$$

which is (5.36a). Note that $t = 1$ in (5.39) and (5.38a) give b and a by

$$\begin{aligned} f(x + y) - f(x - y) &= 2b, \\ f(x + y) + f(x - y) - 2f(x) &= 2a, \quad \text{for } x, y \in V. \end{aligned} \tag{5.41}$$

Now the form of $\phi(t)$, (5.36b), follows from (5.37), (5.36a), (5.41), and Theorem 4.1. This completes the proof of this theorem. □

Corollary. *Suppose V is an n.l.s. and $x, y \in V$. Let $\phi, F : \mathbb{R} \to \mathbb{R}$ be such that $\phi(t) = F(\|x + ty\|^2)$ for $t \in \mathbb{R}$ is a polynomial in t of degree 2. Then V is an i.p.s.*

Corollary. [470]. *Suppose $\phi, g : \mathbb{R} \to \mathbb{R}$ and g is continuous such that $\phi(t) = g(\|x + ty\|)$ is a polynomial of degree 2 in t for $x, y \in V$. Then V is an i.p.s.*

5.1.5 Some Generalizations

5.1.5.1 Solution of a Generalization of the Fréchet Equation

We now devote our attention to the study of the following functional equation that is a generalization of the Fréchet condition (F):

$$\begin{aligned} p(x+y+z) &= p_1(x+y) - p_2(z) + p_3(y+z) \\ &\quad - p_4(x) + p_5(z+x) - p_6(y) \quad \text{for } x, y, z \in G. \end{aligned} \tag{5.42}$$

We prove the following theorem regarding (5.42).

Theorem 5.9. (Kannappan [471]). *Suppose that* $p, p_i : G \to H$ $(i = 1$ *to* $6)$ *satisfy the functional equation* (5.42)*. Let* G *be a group* (*which need not be Abelian*) *and* H *an Abelian group. Then there exist additive functions* $A, A_i : G \to H$ $(i = 1, 2, 3)$ *and a biadditive map* $B : G \times G \to H$ *such that* p, p_i *satisfy*

$$\left.\begin{aligned} 8p(x) &= 4A(x) + B(x,x) + 8c, \\ 8p_1(x) &= 4A(x) + B(x,x) - 8A_1(x) - 8(c_1 + c_2 - c - b_2), \\ 8p_2(x) &= 4A(x) + B(x,x) - 8A_2(x) - 8A_3(x) + 8b_2, \\ 8p_3(x) &= 4A(x) + B(x,x) - 8A_3(x) - 8(c_5 + c_6 - c - b_4), \\ 8p_4(x) &= 4A(x) + B(x,x) - 8A_1(x) - 8A_2(x) + 8b_4, \\ 8p_5(x) &= 4A(x) + B(x,x) - 8A_2(x) - 8(c_3 + c_4 - c - b_6), \\ 8p_6(x) &= 4A(x) + B(x,x) - 8A_1(x) - 8A_3(x) + 8b_6, \end{aligned}\right\} \tag{5.42a}$$

where $c, b_2, b_4, b_6, c_1, c_2, c_3, c_4, c_5, c_6$ *are constants.*

Proof. Letting $z = 0$ in (5.42), we obtain

$$p(x+y) - p_1(x+y) + b_2 = p_3(y) - p_6(y) + p_5(x) - p_4(x),$$

where $p_2(0) = b_2$, which is a Pexider equation. So, there is an additive A_1 such that

$$\begin{aligned} (p - p_1)(x) &= A_1(x) + c_1 + c_2 - b_2, \\ (p_3 - p_6)(x) &= A_1(x) + c_1, \\ (p_5 - p_4)(x) &= A_1(x) + c_2, \end{aligned} \tag{5.43}$$

where c_1, c_2 are arbitrary constants.

Similarly, setting $y = 0$ and $z = 0$ separately in (5.42) results in the Pexider equations

$$p(x+z) - p_5(x+z) + b_6 = (p_1 - p_4)(x) + (p_3 - p_2)(z)$$

and

$$p(y+z) - p_3(y+z) + b_4 = (p_1 - p_6)(y) + (p_5 - p_2)(z),$$

respectively, with $p_4(0) = b_4$, $p_6(0) = b_6$, and the solutions

$$\text{(5.43a)} \qquad \begin{cases} (p - p_5)(y) = A_2(y) + c_3 + c_4 - b_6, \\ (p_1 - p_4)(y) = A_2(y) + c_4, \\ (p_3 - p_2)(y) = A_2(y) + c_3, \end{cases}$$

$$\text{(5.43b)} \qquad \begin{cases} (p - p_3)(z) = A_3(z) + c_5 + c_6 - b_4, \\ (p_1 - p_6)(z) = A_3(z) + c_5, \\ (p_5 - p_2)(z) = A_3(z) + c_6, \end{cases}$$

where A_2, A_3 are additive and c_3, c_4, c_5, c_6 are constants.

Now use (5.43), (5.43a), and (5.43b) to express p_1 to p_6 in terms of p as

$$\text{(5.42b)} \qquad \begin{cases} p_1(x) = p(x) - A_1(x) - c_1 - c_2 + b_2, \\ p_5(y) = p(y) - A_2(y) - c_3 - c_4 + b_6, \\ p_3(z) = p(z) - A_3(z) - c_5 - c_6 + b_4, \\ p_2(y) = p_3(y) - A_2(y) - c_3 \\ \qquad\quad = p(y) - A_3(y) - A_2(y) - c_3 - c_5 - c_6 + b_4, \\ p_4(y) = p_1(y) - A_2(y) - c_4 \\ \qquad\quad = p(y) - A_1(y) - A_2(y) - c_4 - c_1 - c_2 + b_2, \\ p_6(x) = p_3(x) - A_1(x) - c_1 \\ \qquad\quad = p(x) - A_3(x) - A_1(x) - c_1 - c_5 - c_6 + b_4. \end{cases}$$

Substitution of these p_1 to p_6 given by (5.42b) into (5.42) using that A_1, A_2, and A_3 are additive results in

$$\text{(F)} \quad p(x + y + z) = p(x + y) - p(z) + p(y + z) - p(x) + p(z + x) - p(y) + c$$

with $p(0) = c$, $b_2 = c - c_3 - c_5 - c_6 + b_4$, $b_6 = c - c_1 - c_5 - c_6 + b_4$, and $c_1 + c_2 + c_4 = c_3 + c_5 + c_6$. Thus, by Theorem 5.1 there exists an additive A and a biadditive B such that

$$8p(x) = 4A(x) + B(x, x) + 8c.$$

Now the equation (5.42a) we seek is obtained from (5.42b) and the equation above. This completes the proof of this theorem. □

The aim now is to solve the equation (a generalization of (5.23) and (5.23a))

$$\text{(5.23b)} \quad \phi(x, y, z) = f(x + y + z) + g(x + y - z) + h(x - y + z) + k(x - y - z),$$

with ϕ independent of z, for $x, y, z \in V$.

Theorem 5.10. (Kannappan [467]). *Suppose* $\phi : V^3 \to \mathbb{R}$, *and* $f, g, h, k : V \to \mathbb{R}$ *satisfying* (5.23b) *does not depend on* z *for* $x, y, z \in V$. *Then* f, g, h, k *are given by*

(5.23c)
$$\begin{cases} f(x) = -q(x) + A_3(x) + b_1, \\ g(x) = -q(x) + A_3(x) - A_1(x) + b_2, \\ h(x) = q(x) + A_2(x) + b_3, \\ k(x) = q(x) + A_2(x) + A_1(x) + b_4, \end{cases}$$

for $x \in V$, where q satisfies (Q)*, A_1, A_2, and A_3 are additive, and b_1, b_2, b_3, and b_4 are constants.*

Proof. Since ϕ is independent of z, set $z = x + y$, 0, and $-x - y$ separately in (5.23b) to obtain

$$\begin{aligned} & f(2x+2y) + g(0) + h(2x) + k(-2y) \\ &= f(x+y) + g(x+y) + h(x-y) + k(x-y) \\ &= f(0) + g(2x+2y) + h(-2y) + k(2x) \quad \text{for } x, y \in V. \end{aligned} \tag{5.44}$$

Replace $2x, 2y$ by x, y in (5.44) to get

$$(f-g)(x+y) = (k-h)(x) + (h-k)(-y) + c_1, \tag{5.45}$$

where $c_1 = f(0) - g(0)$, which is Pexider. Hence there is an additive A_1 such that

$$f(x) - g(x) = A_1(x) + c_1, \tag{5.46}$$

$$k(x) - h(x) = A_1(x) + c_2, \quad \text{for } x \in V,\ c_2 = k(0) - h(0). \tag{5.47}$$

From (5.46), (5.47), and (5.44), we have

$$2f(x+y) + 2h(x-y) = f(0) + f(2x+2y) + h(-2y) + h(2x). \tag{5.48}$$

Let $y = x$ in (5.48) and then replace $2x$ by x to yield

$$2f(x) + 2h(0) = f(0) + f(2x) + h(-x) + h(x).$$

Now use the equation above in (5.48) to get

$$2h(x-y) + h(x+y) + h(-x-y) = h(2x) + h(-2y) + 2h(0). \tag{5.49}$$

Interchange x and y in (5.49) and subtract the resultant from (5.49) to have

$$2A_2(x-y) = A_2(2x) - A_2(2y),$$

which is Jensen, where

$$A_2(x) = h(x) - h(-x) \quad \text{for } x \in V. \tag{5.50}$$

Note that A_2 is odd, $A_2(0) = 0$, and thus that A_2 is additive. From (5.49) and (5.50), we obtain

$$2h(x-y) + 2h(x+y) - A_2(x+y) = h(2x) + h(2y) - A_2(2y) + 2h(0),$$

which can be rewritten as

$$\begin{aligned} 2h(x+y) - A_2(x+y) - 2h(0) + 2h(x-y) - A_2(x-y) - 2h(0) \\ = h(2x) - \frac{1}{2}A_2(2x) - h(0) + h(2y) - \frac{1}{2}A_2(2y) - h(0); \end{aligned}$$

that is, q satisfies the quadratic equation (Q), where

$$q(x) = h(x) - \frac{1}{2}A_2(x) - h(0) \quad \text{for } x \in V.$$

Thus we obtain h, and (5.47) gives k. It follows from (5.41) that $q(0) = 0$ and q is even.

Now we determine f and g. Now, letting $z = x - y$ and $-x - y$ separately in (5.23b), we get

$$f(2x) + g(2y) + h(2(x-y)) + k(0) = f(0) + g(2x+2y) + h(-2y) + k(2x),$$

which by replacing $2x$ by x and $2y$ by y and using h and (5.47) becomes

$$f(x) + f(y) + q(x-y) = f(0) + f(x+y) + q(y) + q(x);$$

that is,

$$f(x) + f(y) - q(x+y) + 2q(x) + 2q(y) = f(0) + f(x+y) + q(y) + q(x)$$

or

$$(f(x) + q(x) - f(0)) + (f(y) + q(y) - f(0)) = f(x+y) + q(x+y) - f(0),$$

which is additive. Thus

$$A_3(x) = f(x) + q(x) - f(0), \quad \text{for } x \in V,$$

is additive. Now we obtain f, g from (5.46). This completes the proof of the theorem. □

Remark. Note that the solution of (5.23b) is obtained without assuming any regularity condition on the functions.

5.1.6 Some More Characterizations

The functional equation

$$g(g(x)y + g(y)x) = g(x)g(y)g(x+y) \tag{5.51}$$

related to (Q) has been studied by many authors in connection with a characterization of i.p.s. under some additional conditions satisfied by g (see [44, 218, 822, 471]). The following two results are proved in [822].

Result 5.11. (Volkmann [822]). *Suppose $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous solution of the functional equation* (5.51) *for $x, y \in \mathbb{R}_+$ with $g(0) = 0$ and $g(x) > 0$ for $x \neq 0$. Then $g(x) = cx^2$.*

Result 5.11a. [822]. *The continuous solutions $g : \mathbb{R} \to \mathbb{R}$ of equation* (5.51) *for $x, y \in \mathbb{R}$ are given by $g(x) = 0,\ g(x) = \pm 1,\ g(x) = cx^2$, and*

$$g(x) = \begin{cases} cx^2, & x > 0 \\ 0, & x \leq 0 \end{cases}, \quad g(x) = \begin{cases} 0, & x > 0 \\ cx^2, & x \leq 0 \end{cases}, \quad \text{where } c > 0.$$

Result 5.11b. (Aczél and Dhombres [44], Dhombres [218]). *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous solution of* (5.51) *for $x, y \in \mathbb{R}$ with $g(0) = 0,\ g(x) > 0$ for $x \neq 0$, and $\lim_{x\to\infty} g(x) = \infty$. Then $g(x) = cx^2$.*

Here we obtain the continuous solution of (5.51) under different conditions first by a simple and direct method and then with the general solution by reducing it to the quadratic equation (Q). As a matter of fact, we prove the following theorem.

Theorem 5.12. (Kannappan [471]). *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that g is* 1–1 *both on the nonnegative reals and the nonpositive reals; that is, $\{g/\mathbb{R}_+ : \mathbb{R}_+ \to \mathbb{R}$ and $g/\mathbb{R}_- : \mathbb{R}_- \to \mathbb{R}$ are injective$\}$. Then g is a solution of the functional equation* (5.51) *holding for all $x, y \in \mathbb{R}$ if, and only if,*

$$g(x) = cx^2 \quad \text{for } x \in \mathbb{R}, \tag{5.51a}$$

where c is a constant.

Proof. Setting $x = 0 = y$ in (5.51) gives $g(0) = g(0)^3$. So, either $g(0) = 1$ or $g(0) = -1$ or $g(0) = 0$.

Let $g(0) = 1$. Then $x = 0$ in (5.51) yields $g(y) = g(y)^2$ for all $y \in \mathbb{R}$. Hence $g(y) = 0$ or 1, contradicting that g is 1–1 on $\mathbb{R}_+$.

Now consider the case $g(0) = -1$. With $x = 0$ in (5.51), we get $g(-y) = -g(y)^2$ for all $y \in \mathbb{R}$; that is, $g(y) = -g(-y)^2 = -g(y)^4$, again contradicting the one-to-oneness of g on $\mathbb{R}_+$.

Thus $g(0) = 0$. Taking $y = -x$ in (5.51), we have $g(-g(x)x + g(-x)x) = 0$; that is, $-xg(x) + xg(-x) = 0$ for all x since $g(0) = 0$ and g is 1–1 on $\mathbb{R}^+$ and $\mathbb{R}^-$. Therefore, g is even; that is, $g(-x) = g(x)$ for $x \in \mathbb{R}$.

First set $y = x$ and then separately $y = -2x$ in (5.51), and then use that g is even to obtain

$$g(2xg(x)) = g(x)^2 g(2x) = g(-2xg(x) + g(2x)x).$$

Since g is even and 1–1 on nonnegative reals, we can conclude either $-2xg(x) = -2xg(x) + g(2x)x$ (that is, $g(2x) = 0$, for $x \neq 0$, which is not possible) or $2xg(x) = -2xg(x) + g(2x)x$; that is,

$$g(2x) = 2^2 g(x) \quad \text{for all } x \in \mathbb{R}. \tag{5.52}$$

Now put $y = 2x$ and $y = -3x$ separately in (5.51) and use that g is even and (5.52) to have

$$\begin{aligned} g(2g(x)x + 4g(x)x) &= 4g(x)^2 g(3x) \\ &= g(-3xg(x) + xg(3x)) \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

As before, $6xg(x) = 3xg(x) - xg(3x)$, in which case $g(3x) = -3g(x)$ for all $x \in \mathbb{R}$. With $y = 3x$ in (5.51), we get $g(3xg(x) + g(3x)x) = 16g(x)^2 g(3x)$ (that is, $-48g(x)^3 = 0$ for all x), which cannot be. So, $6xg(x) = -3xg(x) + xg(3x)$, or $g(3x) = 3^2 g(x)$, for all $x \in \mathbb{R}$.

By induction, it follows that

$$g(nx) = n^2 g(x) \tag{5.52a}$$

for all integers n and $x \in \mathbb{R}$. Now (5.52a) yields $g(\frac{x}{n}) = \frac{1}{n^2} g(x)$ for $n \neq 0$, and

$$g\left(\frac{m}{n}x\right) = \left(\frac{m}{n}\right)^2 g(x) \quad \text{for rational } \frac{m}{n} \text{ and } x \in \mathbb{R}.$$

We invoke the continuity of g to obtain (5.51a), where $c = g(1)$ is a constant. This completes the proof of this theorem. □

Remark. [471]. Traditionally the solution (5.51a) has been associated with the quadratic equation (Q), so we will derive (Q) from (5.51) and the hypothesis of Theorem 5.12. Indeed, we prove Theorem 5.12 by linking (5.51) to the quadratic equation (Q).

Note that g is even and that $g(2x) = 4g(x)$. Changing y into $y - x$ in (5.51), we have

$$\begin{aligned} g(g(x)(y-x) + g(y-x)x) &= g(x)g(y-x)g(y) \\ &= g(-g(x)y + g(y)x) \quad (y \text{ to } -y \text{ in (5.51)}). \end{aligned}$$

Now, since g is even and 1–1 on $\mathbb{R}_+$, we can conclude that $g(x)(y - x) + xg(y - x) = g(x)y - xg(y)$ (that is, $xg(y - x) = x(g(x) - g(y))$ for all $x, y \in \mathbb{R}$), which for $y = 2x$ gives $xg(x) = x(g(x) - 4g(x))$ (that is, $g(x) = 0$ for $x \neq 0$), which cannot be. So

$$xg(x - y) = x(g(x) + g(y)) - 2yg(x) \quad \text{for } x, y \in \mathbb{R}. \tag{5.53}$$

Replacing y by $-y$ in (5.53) and adding the resultant equation to (5.53) and using g even, we get

$$x(g(x + y) + g(x - y)) = 2x(g(x) + g(y)) \quad \text{for all } x, y \in \mathbb{R}.$$

Hence g satisfies the equation (Q), the continuous solution of which is given by (5.51a) (see [463], Result 4.11).

Remark. [471]. The general solution $g : \mathbb{R} \to \mathbb{R}$ satisfying (5.51) with g one-to-one on $\mathbb{R}_+$ and $\mathbb{R}_-$ is given by

$$g(x) = B(x, x),$$

where $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a biadditive map.

Proof. From the proof in the remark above, we can conclude that g satisfies the quadratic equation (Q). Then, from Theorem 4.1, we get that $g(x) = B(x, x)$, where B is biadditive. □

Note that here we have obtained the solution of (5.51) without assuming any regularity condition on g.

The following remark will be of use in what follows.

Remark. Let V be a linear space over $\mathbb{R}$ and $f : V \to \mathbb{R}$ satisfy $f(tx) = |t| f(x)$ for $x \in V,\ t \in \mathbb{R}$. Then the following statements are equivalent:

(s1) $f(\pm x) = f(\pm y)$ implies $f(ax + by) = f(bx + ay)$,
(s2) $f(x) = f(y)$ implies $f(x + ty) = f(tx + y),\ t \neq 0, 1$,
(s3) $f(x + y) = f(x - y)$ implies $f(x + ty) = f(x - ty),\ t \neq \pm 1, 0$,

for $x, y \in V,\ a, b, t \in \mathbb{R}$.

Proof. (s1) $\Rightarrow$ (s2) is obvious.

Suppose (s2) holds and $f(x + y) = f(x - y)$. Then, by (s2),

$$f\left(x + y + \frac{1-t}{1+t}(x - y)\right) = f\left(\frac{1-t}{1+t}(x + y) + (x - y)\right)$$

or

$$f\left(\frac{2}{1+t}(x + ty)\right) = f\left(\frac{2}{1+t}(x - ty)\right),$$

which is (s3).

Suppose (s3) holds and $f(x) = f(y)$. Then

$$f\left(\frac{1}{2}((x + y) + (x - y))\right) = f\left(\frac{1}{2}((x + y) - (x - y)) - (x - y)\right).$$

Using (s3) we have

$$f\left(\frac{1}{2}\left(x + y + \frac{a-b}{a+b}(x - y)\right)\right) = f\left(\frac{1}{2}\left(x + y - \frac{a-b}{a+b}(x - y)\right)\right);$$

that is,

$$f\left(\frac{(ax + by)}{a + b}\right) = f\left(\frac{(bx + ay)}{a + b}\right),$$

which is (s1). □

If, instead of f, a norm $\|\ \|$ and a real n.l.s. V are used, the following are equivalent:

(r1) $\|x\| = \|y\|$ implies $\|ax + by\| = \|bx + ay\|$,

(r2) $\|x\| = \|y\|$ implies $\|x + ty\| = \|tx + y\|,\ t \neq 0, 1,$
(r3) $\|x + y\| = \|x - y\|$ implies $\|x + ty\| = \|x - ty\|,\ t \neq 0, \pm 1,$ for $x, y \in V,\ t, a, b \in \mathbb{R}.$

Now (r2) $\Rightarrow$ (r4), where

(r4) $x, y, z \in V,\ x + y + z = 0,\ \|x\| = \|y\| \Rightarrow \|x - z\| = \|y - z\|.$

Suppose (r2) holds. Let $x+y+z = 0$ with $\|x\| = \|y\|$. Then $\|x+2y\| = \|2x+y\|$ (by r2); that is, $\| - z + y\| = \|x - z\|$, which is (r4). (r4) says a triangle is isosceles if, and only if, two medians are equal. Note that the implication is symmetric. That is, $x - y + y - z + z - x = 0$ and $\|y - z\| = \|z - x\|$ imply $\|x\| = \|y\|$.

Ficken [271] proved the following classical theorem.

Result 5.13. [271]. *In order for an n.l.s. V over $\mathbb{R}$ to permit the definition of an i.p., it is necessary and sufficient that, whenever $\|x\| = \|y\|$ for $x, y \in V$ and $t \in \mathbb{R}$, we have*

$$\|x + ty\| = \|tx + y\|. \tag{5.54}$$

Now we will prove the functional equation version of Result 5.13. As usual, we will relate it to (Q).

Theorem 5.13a. (Aczél and Dhombres [44], Dhombres [218], Day [214]). *Suppose V is a real topological space and $f : V \to \mathbb{R}$ is a continuous function satisfying*

(i) $f(x) = 0$ *if, and only if,* $x = 0$ *and* $f(x) > 0$ *for all* $x \neq 0$,
(ii) $f(tx) = |t| f(x)$ *for* $x \in V,\ t \in \mathbb{R}$, *and*
(iii)*whenever* $f(x) = f(y)$ *for* $x, y \in V$, *we have*

$$f(x + ty) = f(tx + y) \quad \text{for all } t \in \mathbb{R}. \tag{5.54a}$$

Then f^2 satisfies (5.51), *and $q(x) = f(x)^2$ for $x \in V$ is a quadratic functional; that is, q satisfies* (Q).

Proof. The proof is achieved by two auxiliary results.

Proposition (i). *If $f(x) = f(y) = f(\alpha x + (1 - \alpha)y)$ for some $x, y \in V$ and some $\alpha \in]0, 1[$ and f is continuous and satisfies* (i), (ii), *and* (iii), *then $x = y$.*

Proof. The proof is based on induction and the liberal use of (5.54a).

Now $f(y) = f(\alpha x + (1 - \alpha)y) = f(\alpha(x - y) + y)$ with the use of (5.54a) implies

$$f(\{\alpha(x - y) + y\} + (\alpha - 1)y) = f((\alpha - 1)\{\alpha(x - y) + y\} + y);$$

that is,

$$f(\alpha x) = f(\alpha[(\alpha - 1)(x - y) + y])$$

or $f(x) = f((1 - \alpha)(x - y) - y)$, which is also $= f(y)$ (use (ii) noting that f is even and take $t = -1$).

Repeating the argument above, we get (using (5.54a))

$$f(\{(1-\alpha)(x-y)-y\}+(2-\alpha)y) = f((2-\alpha)\{(1-\alpha)(x-y)-y\}+y)$$

or

$$f((1-\alpha)x) = f((1-\alpha)[(2-\alpha)(x-y)-y]);$$

that is,

$$f(x) = f((2-\alpha)(x-y)-y) = f(y) \quad \text{for } x, y \in V.$$

Now we invoke induction. Suppose that

$$f(x) = f((n-\alpha)(x-y)-y) = f(y) \quad \text{for } x, y \in V,\ n \in Z_+.$$

Then, by (5.54a),

$$f([(n-\alpha)(x-y)-y]+(n+1-\alpha)y) = f((n+1-\alpha)[(n-\alpha)(x-y)-y]+y);$$

that is,

$$f((n-\alpha)x) = f((n-\alpha)[(n+1-\alpha)(x-y)-y]).$$

Thus, by (ii), $f(x) = f((n+1-\alpha)(x-y)-y)$ and induction holds. Hence

$$\frac{f(x)}{(n+1)-\alpha} = f\left(x-y-\frac{y}{(n+1)-\alpha}\right).$$

Allowing n tending to infinity and using the fact that f is continuous, it is easy to see that $f(x-y)=0$ or $x=y$ (use (i)). This proves Proposition (i). □

Remark. Note that the proof works as long as α is not a positive integer.

Proposition (ii). *For any $x, y \in V,\ x, y \neq 0$, and positive real r, there are no more than two distinct real values of t such that*

$$f(x+ty) = r,$$

where f satisfies (i), (ii), *and* (iii).

Proof. Suppose there exist three distinct t_i $(i = 1, 2, 3)$ such that

$$f(x+t_1y) = f(x+t_2y) = f(x+t_3y) \quad \text{for } x, y \in V. \tag{5.55}$$

Case (i). Suppose $\{x, y\}$ is linearly dependent, say $x = \lambda y$. Then (5.55) yields (use (ii))

$$|\lambda + t_1| = |\lambda + t_2| = |\lambda + t_3| \quad \text{since } f(y) \neq 0.$$

A possible case is $\lambda + t_1 = \lambda + t_2$ implies $t_1 = t_2$ and there is nothing to prove. Similarly, $\lambda + t_1 = \lambda + t_3$, and $\lambda + t_3 = \lambda + t_2$ imply $t_1 = t_3$ or $t_2 = t_3$ and there is nothing to prove.

It could be that $\lambda + t_1 = -(\lambda + t_2)$, implying $t_1 + t_2 = -2\lambda$ and $\lambda + t_1 = -(\lambda + t_3)$, giving $t_1 + t_3 = -2\lambda$ or $t_2 = t_3$. There is nothing to prove. The remaining cases can be argued similarly. In this case,

$$\begin{aligned} q(x+y)+q(x-y) &= f(x+y)^2 + f(x-y)^2 \\ &= [(1+\lambda)^2 + (1-\lambda)^2] f(y)^2 \\ &= (2+2\lambda^2) f(y)^2 \\ &= 2f(y)^2 + 2f(\lambda y)^2 \\ &= 2q(x) + 2q(y), \end{aligned}$$

which is (Q).

Case (ii). Suppose $\{x, y\}$ is linearly independent. Then $\{x + t_1 y, x + t_2 y\}$ is linearly independent. Since linear span $\{x, y\}$ = linear span $\{x + t_1 y, x + t_2 y\}$, $x + t_3 y = \alpha(x + t_1 y) + \beta(x + t_2 y)$ for some α, β. Without loss of generality, we may suppose that $x + t_3 y$ lies in the segment $]x + t_1 y, x + t_2 y[$. Then $\alpha + \beta = 1$ with $\alpha \in]0, 1[$, or $x + t_3 y = \alpha(x + t_1 y) + (1 - \alpha)(x + t_2 y)$, for $\alpha \in]0, 1[$.

Now (5.55) becomes

$$f(x + t_1 y) = f(x + t_2 y) = f(\alpha(x + t_1 y) + (1 - \alpha)(x + t_2 y)),$$

and, by Proposition (i), $x + t_1 y = x + t_2 y$, which is not the case. This proves Proposition (ii). □

Proof of Theorem 5.13a. Using (5.54a), for $x, y \neq 0,\ x \neq y$,

$$f\left(\frac{x}{f(x)}\right) = f\left(\frac{y-x}{f(x-y)}\right)$$

implies

$$f\left(f(x-y)\frac{x}{f(x)} + 2f(x)\frac{y-x}{f(x-y)}\right) = f(2x + y - x);$$

that is,

$$f(x+y) f(x) f(x-y) = f[2f(x)^2 y + (f(x-y)^2 - 2f(x)^2)x] \tag{5.56}$$

for $x, y \in V$ (including $x = y,\ x, y = 0$).

Note that (5.56) is "close" to (5.51). In order to get (5.51) precisely, start with

$$f\left(\frac{x}{f(x)}\right) = f\left(\frac{y}{f(y)}\right) \quad (x, y \neq 0)$$

and obtain

$$f\left(f(x)\frac{x}{f(x)} + f(y)\frac{y}{f(y)}\right) = f\left(f(y)\frac{x}{f(x)} + f(x)\frac{y}{f(y)}\right)$$

or

$$f(x+y) f(x) f(y) = f(f(y)^2 x + f(x)^2 y), \tag{5.56a}$$

which by squaring both sides yields

$$q(x)q(y)q(x+y) = q(q(y)x + q(x)y),$$

where $q(x) = f(x)^2$, which is (5.51).

Changing y to $-y$ in (5.56), we have

(5.56b) $$f(x+y)f(x)f(x-y) = f[2f(x)^2y + (2f(x)^2 - f(x+y)^2)x]$$

or use (5.56) by replacing x by $\frac{x}{2}$ and y by $y + \frac{x}{2}$ to have

$$f(x+y)f\left(\frac{x}{2}\right)f(-y) = f\left(2f\left(\frac{x}{2}\right)^2\left(y+\frac{x}{2}\right) + \left[f(-y)^2 - 2f\left(\frac{x}{2}\right)^2\right]\frac{x}{2}\right),$$

which is (5.56a).

Replacing y by $-y$ or x by $-x$ in (5.56a), we have

$$f(x-y)f(x)f(y) = f(f(x)^2y - f(y)^2x),$$

from which we obtain

(5.57)
$$f\left(\frac{f(x)^2y - f(y)^2x}{f(x)f(x-y)}\right) = f(y) = f\left(f(y)\frac{x}{f(x)}\right).$$

Now (5.54a) can be rewritten as (take $x, y \neq 0,\ x \neq y$)

$$\begin{aligned} f(x+y) &= f\left(\frac{f(x)^2y - f(y)^2x}{f(x)f(y)} + 2fy\frac{x}{f(x)}\right) \\ &= f\left(\frac{2(f(x)^2y - f(y)^2x)}{f(x)f(x-y)} \cdot \frac{f(x-y)}{2f(y)} + 2f(y)\cdot\frac{x}{f(x)}\right) \\ \text{and by (5.57) and (5.54a)} &= f\left(\frac{2(f(x)^2y - f(y)^2x)}{f(x)f(x-y)} + f(x-y)\frac{x}{f(x)}\right); \end{aligned}$$

that is,

(5.56c) $$\begin{aligned} &f(x+y)f(x-y)f(x) \\ &\quad = f(2f(x)^2y + (f(x-y)^2 - 2f(y)^2)x) \quad \text{for all } x, y \in V. \end{aligned}$$

Changing x to $-x$ or y to $-y$ in (5.56c) yields

(5.56d) $$\begin{aligned} &f(x+y)f(x-y)f(x) \\ &\quad = f(2f(x)^2y + (2f(y)^2 - f(x+y)^2)x) \quad \text{for } x, y \in V. \end{aligned}$$

Let us denote

$$\begin{aligned}\alpha &= f(x-y)^2 - 2f(y)^2,\\ \beta &= 2f(y)^2 - f(x+y)^2,\\ \gamma &= f(x-y)^2 - 2f(x)^2,\\ \delta &= 2f(x)^2 - f(x+y)^2,\end{aligned}$$

and invoke Proposition (ii).

Comparing (5.56), (5.56b), (5.56c), and (5.56d) and using Proposition (ii), we find that among the four numbers $\alpha, \beta, \gamma, \delta$ there are at most two distinct numbers. This leads to the following four cases:

Case 1. $\alpha = \delta$ implies that q satisfies (Q).

Case 2. $\beta = \gamma$ implies that q satisfies (Q).

Case 3. $\alpha = \beta$ and $\gamma = \delta$ yields $f(x+y)^2 + f(x-y)^2 = 4f(y)^2 = 4f(x)^2$; that is, (Q).

Case 4. $\alpha = \gamma$ and $\beta = \delta$. Then $f(x) = f(y)$. We have two subcases to deal with.

Subcase (i). Suppose $f(x+y) = f(x-y)$. Note that $\{x, y\}$ is linearly independent. We deduce from (5.56c) that

$$f(x) = f(y) = f\left(\frac{2f(y)^2}{f(x-y)^2}x + \left(1 - \frac{2f(y)^2}{f(x-y)^2}\right)y\right).$$

If $2f(y)^2 < f(x-y)^2$, then, by Proposition (i), $x = y$, which is a contradiction.

Suppose $f(x-y)^2 < 2f(y)^2$. We will show that this is also impossible.

Set $x + y = u$ and $x - y = v$. Since $\{x, y\}$ is linearly independent, so is $\{x+y, x-y\} = \{u, v\}$. Starting with u, v and following the argument above corresponding to (5.56c), we get

$$f(u+v)f(u-v)f(u) = f(2f(u)^2 v + (f(u-v)^2 - 2f(v)^2)u);$$

that is,

$$f(2x)f(2y)f(x+y) = f[2f(x+y)^2(x-y) + (f(2y)^2 - 2f(x-y)^2)(x-y)]$$

or, since $f(x+y) = f(x-y)$ and $f(x) = f(y)$, we get

$$f(x-y) = f(x+y) = f\left[\frac{f(x-y)^2}{2f(y)^2}(x-y) + \left(1 - \frac{f(x-y)^2}{2f(y)^2}\right)(x+y)\right].$$

Since $f(x-y)^2 < 2f(y)^2$, by Proposition (i), we have $x + y = x - y$, which is a contradiction. Hence $f(x-y)^2 = 2f(x)^2$ and (Q) holds.

Remark. If V is a real n.l.s. and f is replaced by the norm $\|\cdot\|$, the conclusion above can be obtained by using the triangle inequality in (5.56c).

Finally, we consider the following subcase.

Subcase (ii). $f(x+y) \neq f(x-y)$. Then start with $u = x+y,\ v = x-y$ and end up with cases 1, 2, 3, 4. Now case 4 is impossible since $f(u) \neq f(v)$. In all other cases, we get

$$q(u+v) + q(u-v) = 2q(u) + 2q(v);$$

that is,

$$q(2x) + q(2y) = 2q(x+y) + 2q(x-y),$$

which is (Q) since $q(tx) = f(tx)^2 = t^2 q(x)$.

Thus $q(x) = f(x)^2$ satisfies (Q) in all cases. □

Corollary. Proof of Result 5.13 [271]. *By* Theorem 5.13a, $q(x) = \|x\|^2$ *satisfies* (Q)*; that is,* (JvN). *Thus V is an i.p.s.*

A considerable weakening of (5.54) is [606]

(5.54b) $$\exists\, c > 1 \text{ such that } \|x + cy\| = \|y + cx\|,$$

for $x, y \in S_V$, the unit sphere of V.

5.2 Geometric Characterization

Now we present an application of Result 5.13 or 5.13a. It is well known that *the length of the median* of a triangle is determined by the lengths of the sides of the triangle. This is the geometrical interpretation of (5.58). The triangle with sides $x, y, x-y$ has $\frac{1}{2}(x+y)$ as its median through the origin; this means there exists a function $\ell : \mathbb{R}_+^3 \to \mathbb{R}_+$ for which

(5.58) $$\|x+y\| = \ell(\|x\|, \|y\|, \|x-y\|) \quad \text{for } x, y \in V,$$

where V is a real n.l.s. (Aczél and Dhombres [44], Amir [73], Aronszajn [78], Day [214], Dhombres [218]).

Theorem 5.14. [44, 218, 214, 78]. *Suppose V is a real n.l.s. such that the lengths of the sides of any triangle in V determine the lengths of the medians. Then the norm of V comes from an i.p.*

Proof. The geometrical property leads to the functional equation (5.58). We will now derive (5.54). The proof is by induction and is different from that in [44, p. 138]. First we will show that, for $x, y \in V$ with $\|x\| = \|y\|$,

(5.59) $$\|nx + y\| = \|x + ny\| \quad \text{for } n \in Z.$$

For $n = 1$, it is trivially true. Using (5.58), we obtain

$$\begin{aligned}\|2x + y\| &= \|(x + y) + x\| \\ &= \ell(\|x + y\|, \|x\|, \|y\|) \\ &= \ell(\|x + y\|, \|y\|, \|x\|) \quad (\text{since } \|x\| = \|y\|) \\ &= \|x + 2y\|.\end{aligned}$$

Suppose (5.59) holds for all $k \leq n$. Then

$$\begin{aligned}\|(n + 1)x + y\| &= \|(nx + y) + x\| \\ &= \ell(\|nx + y\|, \|x\|, \|(n - 1)x + y\|) \\ &= \ell(\|x + ny\|, \|y\|, \|(n - 1)y + x\|) \\ &= \|x + (n + 1)y\|,\end{aligned}$$

and so (5.59) holds for all positive n. Obviously (5.59) holds for all $n \in Z$.

For $\|x\| = \|y\|$, $\left\|\frac{x}{n}\right\| = \left\|\frac{y}{n}\right\|$, so that by (5.59) we get

$$\left\|x + \frac{y}{n}\right\| = \left\|\frac{x}{n} + y\right\| \quad \text{for all } n \in Z^*.$$

Now we will prove by induction on m, for fixed n and $\|x\| = \|y\|$,

(5.59a)
$$\begin{aligned}\|nx + my\| &= \|mx + ny\|, \\ \|nx + 2y\| &= \|(nx + y) + y\| \\ &= \ell(\|nx + y\|, \|y\|, \|nx\|) \\ &= \ell(\|x + ny\|, \|x\|, \|ny\|) \\ &= \|2x + ny\|.\end{aligned}$$

Suppose (5.59a) holds for all $k \leq m$.

$$\begin{aligned}\|nx + (m + 1)y\| &= \|(nx + my) + y\| \\ &= \ell(\|nx + my\|, \|y\|, \|nx + (m - 1)y\|) \\ &= \ell(\|mx + ny\|, \|x\|, \|(m - 1)x + ny\|) \\ &= \|(m + 1)x + ny\|.\end{aligned}$$

Thus (5.59a) is true. Also, it is easy to see that (5.59a) holds for all $m, n \in Z$. It follows that $\left\|\frac{n}{m}x + y\right\| = \left\|x + \frac{n}{m}y\right\|$ for $n, m\ (\neq 0) \in Z,\ x, y \in V$ with $\|x\| = \|y\|$.

Since $\|rx + y\| = \|x + ry\|$ for all $r \in Q$, and $\|\ \|$ is a continuous function, (5.54) holds for $x, y \in V,\ t \in \mathbb{R}$. Thus V is an i.p.s. □

Theorem 5.15. [44, 218]. *Let V be a linear space over reals and $q : V \to \mathbb{R}_+$ such that $q(0) = 0,\ q(x) \neq 0$ for $x \neq 0,\ x \in V$. Suppose that, for all $x \neq 0$ in V, there exists a subset I_x of $\mathbb{R}$ of positive Lebesgue measure M_x such that $q(\lambda x) \leq M_x$ for all $\lambda \in I_x$ (ray bounded). Further, if q is a solution of* (Q)*, then V is an i.p.s.*

Proof. Since q satisfies (Q) (by Theorem 4.1), there exists a symmetric, biadditive $B : V \times V \to \mathbb{R}$ such that

$$B(x, y) = \frac{1}{2}[q(x + y) - q(x) - q(y)] \quad \text{for } x, y \in V.$$

To show that B is bilinear, define $A : \mathbb{R} \to \mathbb{R}$ by

$$A(t) = B(tx, y) \quad \text{for } t \in \mathbb{R},\ x, y \in V.$$

Obviously, A is additive and for $t \in I_x$ we have

$$2A(t) = q(tx + y) - q(tx) - q(y) \geq -q(tx) - q(y) \geq -M_x - q(y);$$

that is, A is bounded below on a set of positive measure. Hence $A(t) = A(1)t$ for $t \in \mathbb{R}$; that is, $B(tx, y) = tB(x, y)$. So, B defines an i.p. in V. This proves the result. □

Now we prove the following generalizations.

5.2.1 Some Generalizations

First we treat the following two functional equations connected with (Q):

$$g(\lambda y + x) - g(\lambda y - x) = \lambda(g(y + x) - g(y - x)), \tag{5.60}$$

$$g(\lambda y + x) + h(\lambda y - x) = \lambda(k(y + x) + \ell(y - x)), \tag{5.60a}$$

for $x, y \in V$, where V is a linear space over a field F, with characteristic zero, $\lambda \in F$, and $g, h, k, \ell : V \to F$.

First we study equation (5.60). Let $q : \mathbb{R} \to \mathbb{R}$ be a quadratic functional (satisfying (Q)) and B be the associated biadditive B given by (4.1). The property $B(\lambda x, y) = \lambda B(x, y)$ for $x, y \in \mathbb{R},\ \lambda \in \mathbb{R}$ gives

$$q(\lambda x + y) - q(\lambda x - y) = \lambda(q(x + y)) - q(x - y),$$

which is (5.60). □

We now determine the solution of the functional equation (5.60) without assuming any regularity condition in Theorem 5.16. Even though its solution is obtained in [44, 320], here we present in Theorem 5.16 a simpler and direct proof.

Theorem 5.16. (Aczél and Dhombres [44], Gleason [320], Kannappan [465]). *Let V be a linear space over a field F of characteristic zero. Suppose $g : V \to F$ satisfies the functional equation*

$$g(\lambda y + x) - g(\lambda y - x) = \lambda(g(y + x) - g(y - x)) \quad \textit{for } x, y \in V,\ \lambda \in F. \tag{5.60}$$

Then $g - g(0)$ is quadratic (that is, it satisfies (Q)*), and there exists a unique symmetric, bilinear $B : V \times V \to F$ such that $g(x) = B(x, x) + c$ for $x \in V$, where c is a constant.*

Proof. If g is a solution of (5.60), then $g+$ is constant. So, without loss of generality, let us assume $g(0) = 0$.

Putting $\lambda = 0$ (or $y = 0$) in (5.60) gives that g is even; that is, $g(x) = g(-x)$ for $x \in V$.

Remark. If $V = \mathbb{R} = F$ in the theorem, then (5.60) with $y = 1$ yields $g(\lambda + x) - g(\lambda - x) = \lambda(g(1 + x) - g(1 - x)) = \lambda\alpha(x)$. The fact that g is even and the interchange of λ and x yield $\lambda\alpha(x) = x\alpha(\lambda)$; that is, $\alpha(x) = cx$ and $g(\lambda + x) - g(\lambda - x) = c\lambda x$, which with $x = \lambda$ gives $g(\lambda) = c\lambda^2$ (using $g(0) = 0$), where c is a constant, or we have

$$g(\lambda + x) - g(\lambda - x) = c\lambda x = \frac{c}{4}((\lambda + x)^2 - (\lambda - x)^2),$$

from which $g(0) = 0$ follows $g(\lambda) = c\lambda^2$ and we are done.

To continue the proof, now replace x by μx in (5.60) and use (5.60) and the fact that g is even to get

(5.60a)

$$g(\lambda y + \mu x) - g(\lambda y - \mu x) = \lambda\mu(g(y + x) - g(y - x)), \quad \text{for } x, y \in V,\ \lambda, \mu \in F.$$

Let $\lambda = \mu$ and $y = x$ in (5.60), and use $g(0) = 0$. We obtain (with $u = 2x$)

$$g(\lambda u) = \lambda^2 g(u) \quad \text{for } u \in V,\ \lambda \in F. \tag{5.61}$$

Remark. When $V = \mathbb{R} = F$ in the theorem, (5.61) yields $g(\lambda) = k\lambda^2$ and we are done.

To continue the proof, replacing x by $x + y$, putting $\lambda = 2$ into (5.60a), and using g even, we get

$$g(3y + x) = 2(g(2y + x) - g(x)) + g(y - x) \quad \text{for } x, y \in V. \tag{5.62}$$

Replace x by $2x$ and y by $y - x$ in (5.62) separately and use (5.61) to have

$$g(3y + 2x) = 8g(y + x) - 8g(x) + g(y - 2x) \tag{5.63}$$

and

$$g(3y - 2x) = 2g(2y - x) - 2g(x) + g(y - 2x), \tag{5.64}$$

respectively. First subtract (5.64) from (5.63) and utilize g even and (5.60a) to get

$$g(2y - x) = g(y + x) + 3g(y - x) - 3g(x) \quad \text{for } x, y \in V. \tag{5.65}$$

Second, add (5.63) and (5.64) and use (5.65) twice to obtain

$$\begin{aligned} &g(3y + 2x) + g(3y - 2x) \\ &\quad = 8g(y + x) - 10g(x) + 2g(2y - x) + 2g(y - 2x) \\ &\quad = 12(g(y + x) + g(y - x)) - 16g(x) - 6g(y). \end{aligned} \tag{5.66}$$

Finally, from (5.66), replacing x by $3x$ and y by $2y$ and using (5.61), (5.66), and g even, again we have

$$\begin{aligned}36(g(y+x)+g(y-x)) &= 12(g(2y+3x)+g(2y-3x)) - 144g(x) - 24g(y)\\ &= 12[12(g(y+x)+g(y-x)) - 16g(y) - 6g(x)]\\ &\quad - 144g(x) - 24g(y);\end{aligned}$$

that is, g satisfies (Q).

Then, by Theorem 4.1, there is a unique symmetric, biadditive $B : V \times V \to F$ such that $g(x) = B(x, x)$, where

$$4B(x, y) = g(x + y) - g(x - y) \quad \text{for } x, y \in V.$$

The use of (5.60) shows that $B(\lambda x, y) = \lambda B(x, y)$ for $x, y \in V,\ \lambda \in F$. So B is bilinear. This completes the proof of the theorem. □

Remark. Note that the general solution of (5.60) has been obtained without assuming any regularity conditions on g.

Finally, we determine the general solution of the most general equation (5.60a) connected with the Jordan–von Neumann quadratic functional equation in Theorem 5.16a.

Result 5.16a. (Kannappan [465]). *The functions $g, h, k, \ell : V \to F$ satisfy the functional equation*

(5.60a) $g(\lambda y + x) + h(\lambda y - x) = \lambda(k(y+x) + \ell(y-x)), \quad$ *for $x, y \in V,\ \lambda \in F$,*

if, and only if, there exists an additive (linear) A and a symmetric, bilinear B such that

$$\begin{aligned}g(x) &= B(x, x) + A(x) + b,\\ k(x) &= B(x, x) + A(x) + c,\\ h(x) &= -B(x, x) + A(x) - b,\\ \ell(x) &= -B(x, x) + A(x) - c,\end{aligned}$$

where b and c are constants.

Now we proceed to consider three related functional equations connected to i.p.s. and obtain their general solutions,

(5.67) $$g(\lambda x + y) + g(x - \lambda y) = (1 + \lambda^2)(g(x) + g(y)),$$

(5.68a) $$g(\lambda x + y) + h(x - \lambda y) = (1 + \lambda^2)(g(x) + h(y)),$$

and

(5.68b) $$g(\lambda x + y) + h(x - \lambda y) = (1 + \lambda^2)(k(x) + \ell(y)),$$

where $g, h, k, \ell : V \to F$, V is a linear space over a field F and $\lambda \in F$.

First we treat the functional equation (5.67). Even though it was solved in Rassias [683], here we present very simple and direct proofs.

Theorem 5.17. (Kannappan [464]). *Let V be a linear space over a field F of characteristic zero* (*or characteristic different from* 2). *If $g : V \to F$ satisfies the functional equation* (5.67) *for all $x, y \in V$ and $\lambda \in F$, then g satisfies* (Q) *and there exists a symmetric, bilinear $B : V \times V \to F$ such that $g(x) = B(x, x)$ for all $x \in V$.*

Proof. If $\lambda = 0$ in (5.67), there is nothing to prove. Let $\lambda \neq 0$.

Set $x = 0$ and $y = 0$ separately in (5.67) to have

$$g(-\lambda y) = (1 + \lambda^2)g(0) + \lambda^2 g(y) = g(\lambda y)$$

(change x to y in the second substitution); that is, g is even. Letting $y = 0$ in the above, we get $g(0) = 0$ and

$$g(\lambda x) = \lambda^2 g(x) \quad \text{for } x \in V,\ \lambda \in F. \tag{5.68}$$

Remark. If $V = \mathbb{R} = F$, the proof is finished.

Proof (i). With $\lambda = 1$, (5.67) becomes (Q). By Theorem 4.1, there exists a symmetric, biadditive $B : V \times V \to F$ such that

$$\begin{aligned} g(x) &= B(x, x) \quad \text{for } x \in V, \\ B(x, y) &= \frac{1}{2}[g(x + y) - g(x) - g(y)] \\ &= \frac{1}{4}[g(x + y) - g(x - y)] \quad \text{for } x, y \in V. \end{aligned} \tag{5.69}$$

In general, B is biadditive only. Here we show that B is bilinear.

Interchange x and y in (5.67) and use g even to have

$$g(\lambda x + y) + g(x - \lambda y) = g(\lambda y + x) + g(y - \lambda x)$$

or

$$\begin{aligned} B(\lambda x, y) &= g(\lambda x + y) - g(\lambda x - y) \\ &= g(\lambda y + x) - g(\lambda y - x) \\ &= B(x, \lambda y) \quad \text{for } x, y \in V,\ \lambda \in F. \end{aligned} \tag{5.70}$$

Further, by (5.69) and (5.70),

$$\begin{aligned} B(\lambda x, \lambda y) &= \lambda^2 B(x, y) \\ &= B(\lambda^2 x, y) \\ &= B(x, \lambda^2 y) \quad \text{for } x, y \in V,\ \lambda \in F. \end{aligned} \tag{5.71a}$$

Using (5.68), (5.70), (Q), and (5.71a), we show that B is homogeneous in the first variable. For $x, y \in V,\ \lambda \in F$, we have

$$
\begin{aligned}
&(\lambda+1)^2 g(x+y)+(\lambda-1)^2 g(x-y)\\
&\qquad = 2(1+\lambda^2)(g(x)+g(y))+2\lambda(g(x+y)-g(x-y))\\
&\qquad = 2(1+\lambda^2)(B(x,x)+B(y,y))+8\lambda B(x,y)
\end{aligned}
$$

and

$$
\begin{aligned}
&g((\lambda+1)(x+y))+g((\lambda-1)(x-y))\\
&\qquad = \frac{1}{2}[g(2x+2\lambda y)+g(2\lambda x+2y)]\\
&\qquad = 2[g(x+\lambda y)+g(\lambda x+y)]\\
&\qquad = 2[B(x+\lambda y,x+\lambda y)+B(\lambda x+y,\lambda x+y)]\\
&\qquad = 2[(1+\lambda^2)\{B(x,x)+B(y,y)\}+2B(x,\lambda y)+2B(\lambda x,y)]\\
&\qquad = 2(1+\lambda^2)[B(x,x)+B(y,y)]+8B(\lambda x,y).
\end{aligned}
$$

Hence, $B(\lambda x,y)=\lambda B(x,y)$ for $x,y\in V,\ \lambda\in F$. That is, B is bilinear.

Proof (ii). From (5.69) and (5.68), we have

$$B(\lambda x,x)=\frac{1}{4}[(\lambda+1)^2-(\lambda-1)^2]g(x)=\lambda B(x,x)\quad \text{for } x\in V,\ \lambda\in F.$$

Replacing x by $x+y$ in the above and using (5.70) and the symmetry of biadditive B, we get

$$B(\lambda x,y)=\lambda B(x,y);$$

that is, B is linear in the first component. Hence B is bilinear, and we are done.

Proof (iii). Replacing x by λx and y by $(\lambda+1)y+x$ in (Q) and using (5.68), we get

$$(\lambda+1)^2 g(x+y)+g((\lambda-1)x-(\lambda+1)y)=2\lambda^2 g(x)+2g((\lambda+1)y+x).$$

Changing x to $-x$ in the above and subtracting the resultant from the above and using the evenness of g and (5.69), we have

$$(\lambda+1)^2 B(x,y)-B((\lambda-1)x,(\lambda+1)y)=2B(x,(\lambda+1)y),$$

which by the use of (5.70) and the biadditivity of B yield $B(\lambda x,y)=\lambda B(x,y)$. This completes the proof of this theorem. □

Before considering (5.67) on a field F of char $F=2$, we study the remaining two equations.

Now we consider equation (5.68a) and prove the following theorem.

Theorem 5.17a. (Kannappan [464]). *Suppose $g,h:V\to F$, where F is a field of characteristic* 0 (*or different from* 2), *satisfy equation* (5.68a). *Then there is a unique symmetric bilinear $B:V\times V\to F$ such that*

$$g(x)=B(x,x)+b,\quad h(x)=B(x,x)-b,$$

where b is a constant.

Proof. Putting $\lambda = 0$ in (5.68a), we get $g(x) - h(x) = c$. This $g(x)$ in (5.68a) yields (5.67). Hence we get

$$h(x) + \frac{c}{2} = B(x, x) = g(x) - \frac{c}{2}.$$

This completes the proof of this theorem. □

Finally, we prove the following theorem regarding (5.68b).

Theorem 5.17b. [464]. *The general solutions $g, h, k, \ell : V \to F$, where F is a field of characteristic zero* (*or different from* 2), *of* (5.68b) *are given by*

$$g(x) = B(x, x) + b, \quad h(x) = B(x, x) - b,$$
$$k(x) = B(x, x) + c, \quad \ell(x) = B(x, x) - c,$$

where $B : V \times V \to F$ is a symmetric bilinear form and b and c are constants.

Proof. Put $\lambda = 0$ in (5.68b) to get $\ell(y) - g(y) = c = h(x) - k(x)$. Substituting these values of ℓ in terms of g and k in terms of h in (5.68b), we obtain (5.68a). The result sought follows from Theorem 5.17a. □

Remark. Note that no regularity condition is used in all these theorems.

We now consider the case where char $F = 2$.

Theorem 5.17c. (Rassias [683], Dhombres [220]). *Let V be a vector space over a perfect field F of characteristic* 2 (*other than F_2*) *and $g : V \to F$ satisfy* (5.67). *Then g is a solution of* (Q) *and $B(x, y) = g(x + y) - g(x) - g(y)$ is bilinear.*

Proof. Now (5.67) becomes

$$(5.68_1) \qquad g(\lambda x + y) + g(x + \lambda y) = (1 + \lambda^2)(g(x) + g(y)).$$

Now $g(x + y) + g(x - y) = g(x + y) + g(x + y) = 0 = 2(g(x) + g(y))$, and so g satisfies (Q).

When g is a solution of (5.67), so is $g - g(0)$. So, without loss of generality, assume $g(0) = 0$.

Now $x = 0$ or $y = 0$ in (5.68_1) yields (5.68), $g(\lambda x) = \lambda^2 g(x)$, for $\lambda \in F$, $x \in V$.

Define $B : V \times V \to F$ by

$$B(x, y) = g(x + y) - g(x) - g(y) \quad \text{for } x, y \in V.$$

Claim B is bilinear. First we will show that B is homogeneous in each variable. Note that B is symmetric, and since char $F = 2$, $-g(x) = g(x)$.

It is easy to check that $B(\lambda x, y) = B(x, \lambda y)$ and $\lambda^2 B(x, y) = B(\lambda x, \lambda y) = B(\lambda^2 x, y)$ for $\lambda \in F$, $x, y \in V$. Since F is a perfect field of char $F = 2$, for every $\mu \in F$ there is a $\lambda \in V$ such that $\mu = \lambda^2$. Hence

$$B(\mu x, y) = \mu B(x, y) \quad \text{for } \mu \in F,\ x, y \in V.$$

So, B is homogeneous in the first (and thus in the second) variable. Now we prove the biadditivity of B.

We observe that B from its definition as a Cauchy difference satisfies the cocycle equation

$$B(x, y) + B(x + y, z) = B(y, z) + B(y + z, x) \quad \text{for } x, y, z \in V.$$

Choose $\lambda \neq 0, 1$. Replace x, y by $\lambda x, \lambda y$ and y by λy separately to get

$$\lambda^2 B(x, y) + \lambda B(x + y, z) = B(\lambda y, z) + \lambda B(\lambda y + z, x),$$
$$\lambda B(x, y) + B(x + \lambda y, z) = \lambda B(y, z) + B(\lambda y + z, x),$$

which by subtraction or addition of λ times the second equation results in

$$B(x + \lambda y, z) = B(x + y, z) + (1 - \lambda)B(y, z) \quad \text{for } x, y, z \in V.$$

Replacing x by λx in the equation above and using it gives

$$\begin{aligned}\lambda B(x + y, z) &= B(\lambda x + y, z) + (1 - \lambda)B(y, z) \\ &= [B(x + y, z) + (1 - \lambda)B(x, z)] + (1 - \lambda)B(y, z)\end{aligned}$$

or

$$(1 - \lambda)[B(x + y, z) - B(x, z) - B(y, z)] = 0;$$

that is, since $\lambda \neq 1$, B is additive in the first variable. Hence B is biadditive, and this proves the result. □

5.2.2 Solution of an Equation Related to Ptolemaic Inequality

In Alsina and Garcia-Roig [70], the authors, motivated by the Ptolemaic inequality

$$\|x - y\| \, \|z\| \leq \|x - z\| \, \|y\| + \|z - y\| \, \|x\|,$$

for $x, y, z \in V$, an n.l.s., studied the equation

$$f(x - y) = f(x) f(y) f\left(\frac{x}{f(x)^2} - \frac{y}{f(y)^2}\right), \quad \text{for } x, y\ (\neq 0) \in \mathbb{R}, \tag{5.71}$$

in connection with i.p.s.

Here we obtain the solution of (5.71) in the following result under slightly different conditions and then under the same conditions as in [70] by presenting a simple and direct proof.

Theorem 5.18. (Alsina and Garcia-Roig [70], Kannappan [471]). *Let $f : \mathbb{R} \to \mathbb{R}_+$ be strictly increasing on $\mathbb{R}_+$ even and satisfy*

$$f(x) f\left(\frac{2x}{f(x)^2}\right) = 2, \quad \textit{for } x \neq 0, \tag{5.72}$$

and the functional equation (5.71)*. Then $f(x) = c|x|$ for $x \in \mathbb{R}$ and c a positive constant.*

Proof. First $y = x$ in (5.71) gives $f(0) = f(0)f(x)^2$ and $f(0) = f(0)^3$. Since f is strictly increasing, f is 1–1 on $\mathbb{R}_+$, $f(0) = 0$, and $f(x) \neq 0$ for $x \neq 0$. With $y = -x$, (5.71) using (5.72) and f even gives

$$f(2x) = f(x)^2 f\left(\frac{2x}{f(x)^2}\right) = 2f(x) \quad \text{for } x \in \mathbb{R}. \tag{5.73}$$

Set $y = -3x$ in (5.71) and utilize (5.72), (5.73), and f even to get

$$f(x)f(3x)f\left(\frac{x}{f(x)^2} + \frac{3x}{f(3x)^2}\right) = f(4x) = 4f(x)$$
$$= 2f(x)f(3x)f\left(\frac{6x}{f(3x)^2}\right)$$

$\left(2 \text{ by } f(3x)f\left(\frac{6x}{f(3x)^2}\right)\right)$; that is,

$$f\left(\frac{x}{f(x)^2} + \frac{3x}{f(3x)^2}\right) = f\left(\frac{12x}{f(3x)^2}\right)$$

since $f(x) \neq 0$ for $x \neq 0$. Then, since f is 1–1 on $\mathbb{R}^+$, either

$$\frac{x}{f(x)^2} + \frac{3x}{f(3x)^2} = -\frac{12x}{f(3x)^2},$$

which cannot be since $f(x) > 0$ for $x \neq 0$, or

$$\frac{x}{f(x)^2} + \frac{3x}{f(3x)^2} = \frac{12x}{f(3x)^2};$$

that is, since $f(x) > 0$ for $x \neq 0$,

$$f(3x) = 3f(x) \quad \text{for } x \in \mathbb{R}. \tag{5.74}$$

From (5.73) and (5.74), we obtain for all integers m, n

$$f(2^n 3^m) = 2^n 3^m f(1).$$

The set $\{2^n 3^m : n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}^+$, and since f is strictly increasing (this is the only place we need this), $f(x) = cx$ for $x \in \mathbb{R}_+$ and positive c. From f even, we obtain the solution sought. This completes the proof of this theorem. □

Remark. Suppose $f(x)f\left(\frac{x}{f(x)^2}\right) = 1$ for $x \neq 0$ holds instead of (5.73) in Theorem 5.18. Then

$$f(x) = f(2x - x) = f(2x)f(x)f\left(\frac{2x}{f(2x)^2} - \frac{x}{f(x)^2}\right);$$

that is, since $f(x) \neq 0$ for $x \neq 0$,

$$f(2x)f\left(\frac{2x}{f(2x)^2} - \frac{x}{f(x)^2}\right) = 1 = f(2x)f\left(\frac{2x}{f(2x)^2}\right),$$

from which it follows that $f(2x) = 2f(x)$. Now (5.72) holds and we obtain the solution $f(x) = c|x|$ as in [70].

5.2.3 Orthogonal Additivity and I.P.S.

There are several lines of approach for the characterization of i.p.s. among n.l.s. We have seen quite a few so far. We now turn our attention to geometry—in particular, to dimension and orthogonality. A number of definitions of orthogonality in vector spaces, in addition to the usual one for i.p.s., are known in the literature. Four were mentioned earlier, and we now introduce two more.

Definition. [693, 118, 692]. *Orthogonality space*. Let X be a real vector space of dimension ≥ 2 and $\perp$ be a binary relation on X. Then $X(\perp)$ is called an *orthogonality space* provided $\perp$ possesses the following properties:

(p1). $x \perp 0,\ 0 \perp x$ for all $x \in X$.
(p2). If $x, y \in X,\ x, y \neq 0$, and $x \perp y$, then x and y are linearly independent.
(p3). If $x, y \in X$ and $x \perp y$, then $sx \perp ty$ for all $s, t \in \mathbb{R}$.
(p4). If Y is a two-dimensional subspace of X and $x \in Y,\ \lambda \geq 0$, then there is a $y \in Y$ such that $x \perp y$ and $(x + y) \perp (\lambda x - y)$.

5.2.4 Diminnie Orthogonality

Let X be a real vector space with Hamel dimension $\dim X \geq 2$, $X^{(*)}$ stand for the topological dual space of X, and $F = \{\phi \in X^{(*)} : \|\phi\| \leq 1\}$. For fixed $x, y \in X$, define

$$D(\phi, \psi; x, y) := \phi(x)\psi(y) - \phi(y)\psi(x) \quad \text{for } \phi, \psi \in F$$

and

$$\|x, y\| = \sup\{D(\phi, \psi; x, y) : \phi, \psi \in F\} \quad \text{for all } x, y \in X.$$

Definition. For $x, y \in X$, x is said to be *Diminnie orthogonal* to y, in symbols $x \perp_{\mathrm{D}} y$, if

$$\|x, y\| = \|x\|\,\|y\|.$$

Several authors studied functional congruences applied to orthogonal pairs of independent variables, with orthogonality defined in some sense. We will see a couple of results in this direction. The congruences fall under *conditional equations.*

Definition. Let $X(\perp)$ be an orthogonality space and G be an Abelian group. Then a mapping $f : X \to G$ is called *orthogonally additive* with respect to $\perp$ if

(A′) $\qquad f(x + y) = f(x) + f(y) \quad$ for all $x, y \in X$ with $x \perp y$.

Result 5.19. (Rätz [693]). *Let $X(\perp)$ be an orthogonality space and G be an Abelian group. Then $f : X \to G$ is an odd solution of* (A′) *if, and only if, f is a solution of* (A).

Theorem 5.19a. (Rätz [694, 692]). *Suppose $X(\perp)$ is an orthogonality space, G is an Abelian group, and $q : X \to G$ is an even solution of* (A′)*. Then q is quadratic; that is, q satisfies* (Q)*.*

Proof. We will accomplish the proof in four stages by liberally using (p1) to (p4) and, in particular, (p3).

Step 1. For $x, y \in X,\ (x+y) \perp (x-y)$ implies that $q(x) = q(y)$.

$$\begin{aligned} q(x) &= q\left(\frac{x+y}{2} + \frac{x-y}{2}\right) \\ &= q\left(\frac{x+y}{2}\right) + q\left(\frac{x-y}{2}\right) && \text{by (p3) and (A}'\text{)} \\ &= q\left(\frac{x+y}{2}\right) + q\left(\frac{y-x}{2}\right) && (q \text{ even}) \\ &= q\left(\frac{x+y}{2} + \frac{y-x}{2}\right) && \text{(by (p3))} \\ &= q(y). \end{aligned}$$

Step 2. $q(2x) = 4q(x)$ for $x \in X$. Now use (p4) and (p3). For $x \in X$, there is a $y \in Y$ (a two-dimensional subspace of X containing x) with $x \perp y,\ (x+y) \perp (x-y)$.

$$\begin{aligned} q(2x) &= q(x+y+x-y) \\ &= q(x+y) + q(x-y) && \text{by (A}'\text{)} \\ &= q(x) + q(y) + q(x) + q(-y) && \text{by (A}'\text{) and (p3)} \\ &= 2q(x) + 2q(y) \\ &= 4q(x) && \text{by Step 1.} \end{aligned}$$

Step 3. For $x \in X,\ \lambda \in \mathbb{R},\ q(x+\lambda x) + q(x-\lambda x) = 2q(x) + 2q(\lambda x)$.

If $\lambda = 0$, there is nothing to prove. Let $\lambda \neq 0$. By (p4), there exists a $y \in X$ with $x \perp y$ and $(x+y) \perp (\lambda x - y)$, where $\lambda > 0$.

$$\begin{aligned} &q(x+\lambda x) + q(x-\lambda x) \\ &\quad = q(x+y+\lambda x-y) + q(x-\lambda x) \\ &\quad = q(x+y) + q(\lambda x - y) + q(x-\lambda x) && \text{by (A}'\text{)} \\ &\quad = q(x) + q(y) + q(\lambda x) + q(-y) + q(x-\lambda x) && \text{by (A}'\text{)} \\ &\quad = q(x) + q(\lambda x) + q(x-\lambda x) + 4q(y) - 2q(y) \\ &\quad = q(x) + q(\lambda x) + q(x-\lambda x+2y) - 2q(y) && \text{by Step 2 and (A}'\text{)} \\ &\quad = q(x) + q(\lambda x) + q(x+y) + q(-\lambda x+y) - 2q(y) \\ &\quad = 2q(x) + 2q(\lambda x). \end{aligned}$$

If $\lambda < 0$, consider $-\lambda$, and then the Step 3 relation holds.

Step 4. For $x \in X,\ r, s \in \mathbb{R},\ q(rx+sx) + q(rx-sx) = 2q(rx) + 2q(sx)$.

If either $r = 0$ or $s = 0$, it is trivially true. Let $r, s \neq 0$.

$$\begin{aligned} q(rx + sx) + q(rx - sx) &= q\left(rx + \frac{s}{r}(rx)\right) + q\left((rx) - \frac{s}{r}(rx)\right) \\ &= 2q(rx) + 2q\left(\frac{s}{r} \cdot rx\right) \\ &= 2q(rx) + 2q(sx). \end{aligned}$$

To prove that q satisfies (Q), again we are going to use (p3) and (p4).

Let $x, y \in X$. If either $x = 0$ or $y = 0$, (Q) holds since $q(0) = 0$ and q is even. Let $x, y \neq 0$.

Let Y be a two-dimensional subspace of X containing x and y. From (p4), we see that there is a $z \in Y$ such that $x \perp z$, $(x + z) \perp (x - z)$. Since $\{x, z\}$ is linearly independent, span $\{x, z\} = Y$. So, $y = rx + sz$ for some $r, s \in \mathbb{R}$. Now

$$\begin{aligned} q(x + y) + q(x - y) &= q(x + rx + sz) + q(x - rx - sy) \\ &= q((1 + r)x) + q(sz) \\ &\quad + q((1 - r)x) + q(-sz) && \text{by (A}'\text{)} \\ &= 2q(x) + 2q(rx) + 2q(sz) && \text{by Step 3} \\ &= 2q(x) + 2q(rx + sz) && \text{by (A}'\text{) and (p3)} \\ &= 2q(x) + 2q(y); \end{aligned}$$

that is, q is a solution of (Q). This concludes the proof. □

Corollary 5.19b. [693]. *Let $X(\perp)$ be an i.p.s. and G be an Abelian group. Then $f : X \to G$ is an even solution of* (A′) *if and only if there exists an additive mapping $A : \mathbb{R} \to G$ such that $f(x) = A(\|x\|^2)$ for $x \in X$.*

Corollary 5.19c. *Let $X(\perp)$ be an i.p.s. and G be a uniquely 2-divisible Abelian group. Then $f : X \to G$ is a solution of* (A′) *if and only if there is an additive map $A : \mathbb{R} \to G$ and $h : X \to G$ such that*

$$f(x) = A(\|x\|^2) + h(x) \quad \text{for } x \in X.$$

Corollary 5.19d. *Let $X(\perp)$ be an i.p.s. and G a real Hausdorff topological vector space, and $f : X \to G$ is a continuous solution of* (A′)*. Then there is a continuous function $h : X \to G$ and a constant $c \in G$ such that $f(x) = c\|x\|^2 + h(x)$ for $x \in X$.*

Now we determine all orthogonally additive functions with respect to a general orthogonality. Let V be a real i.p.s. with i.p. $\langle\ ,\ \rangle$ and orthogonality $\perp$ ($x \perp y$ if $\langle x, y\rangle = 0$). A mapping $f : V \to \mathbb{R}$ satisfying

$$\text{(A}''\text{)} \qquad f(x + y) = f(x) + f(y), \quad \text{for } x, y \in V, \text{ with } x \perp y,$$

is called an *orthogonally additive* function on V (or f satisfies a *conditional Cauchy equation* with condition $x \perp y$). We determine all continuous solutions of (A″) in the following theorem.

Theorem 5.20. (Aczél and Dhombres [44]). *Let V be a real i.p.s. of dimension ≥ 2. A continuous function $f : V \to \mathbb{R}$ is orthogonally additive (that is, it satisfies* (A″)*) if, and only if, there exists a continuous linear additive homogeneous $A : V \to \mathbb{R}$, a continuous bilinear $B : V \times V \to \mathbb{R}$, and a constant $c \in \mathbb{R}$ such that*

$$f(x) = B(x, x) + A(x) = c\|x\|^2 + A(x), \quad \text{for } x \in V. \tag{5.75}$$

Proof. It is easy to verify that f given by (5.75) is a solution of (A″).

Conversely, let f be a continuous solution of (A″). Every function $f : V \to \mathbb{R}$ can be written as

$$f(x) = f_e(x) + f_o(x) \quad \text{for } x \in V,$$

where $f_e(x) = \frac{f(x)+f(-x)}{2}$ is called the even part of f and $f_o(x) = \frac{f(x)-f(-x)}{2}$ is called the odd part of f. Note that f_e and f_o are continuous and satisfy (A″). We will show that f_o is linear and f_e satisfies (Q). We make use of the fact that in each two-dimensional subspace H there exist nonzero x, y in H such that $x \perp y$ and $(x + y) \perp (x - y)$. Note that $x \perp y$ implies $\alpha x \perp \beta y$ for $\alpha, \beta \in \mathbb{R}$.

(i) To prove that f_o is linear (that is, homogeneous and additive on V), it is enough to show that f_o is linear on any two-dimensional subspace H of V. Now there exist $x, y\ (\neq 0) \in H$ such that $x \perp y$ and $(x + y) \perp (x - y)$. First, we show that $f_o(rx) = r f(x)$ and $f_o(ry) = r f_o(y)$ for $r \in \mathbb{R}$.

$$\begin{aligned} f_o(2x) &= f_o(x + y + x - y) \\ &= f_o(x + y) + f_o(x - y) && \text{by (A″)} \\ &= f_o(x) + f_o(y) + f_o(x) + f_o(-y) && \text{again by (A″)} \\ &= 2 f_o(x) && \text{since } f_o \text{ is odd.} \end{aligned}$$

Similarly, $f_o(2y) = 2 f_o(y)$.

By induction, we show that $f_o(nx) = n f_o(x)$ and $f_o(ny) = n f_o(y)$, $n \in Z_+$. Suppose this result holds for all k, $0 \leq k < n$. Then

$$\begin{aligned} f_o(nx+(n-2)y) & \\ &= f_o((n-1)(x+y) + (x-y)) \\ &= f_o((n-1)(x+y)) + f_o(x-y) && \text{by (A″)} \\ &= f_o((n-1)x) + f_o((n-1)y) \\ &\quad + f_o(x) + f_o(-y) && \text{by (A″)} \\ &= (n-1)[f_o(x) + f_o(y)] + f_o(x) - f_o(y) && \text{by induction} \\ &= n f_o(x) + (n-2) f_o(y) \\ &= f_o(nx) + f_o((n-2)y) && \text{since } nx \perp (n-2)y. \end{aligned}$$

Hence $f_o(nx) = n f_o(x)$, and similarly $f_o(ny) = n f_o(y)$. Since f_o is odd, this holds for all $n \in Z$.

When $x \perp y$, so is $\frac{x}{n} \perp \frac{y}{n}$ for all n ($\neq 0$). Thus

$$f_o\left(n \cdot \frac{x}{n}\right) = n f_o\left(\frac{x}{n}\right) \quad \text{or} \quad f_o\left(\frac{x}{n}\right) = \frac{1}{n} f_o(x).$$

Hence, for any rational $\frac{m}{n}$,

$$f_o\left(\frac{m}{n}x\right) = \frac{m}{n} f_o(x) \quad \text{and} \quad f_o\left(\frac{m}{n}y\right) = \frac{m}{n} f_o(y).$$

Now invoke continuity of f_o to obtain

$$f_o(rx) = r f_o(x) \quad \text{and} \quad f_o(ry) = r f_o(y) \quad \text{for } r \in \mathbb{R},$$

from which follows $f_o(rx + sy) = r f_o(x) + s f_o(y)$ for $r, s \in \mathbb{R}$ since $rx \perp sy$.

Let $u, v \in H$. Then there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $u = \alpha x + \beta y$, $v = \gamma x + \delta y$. Let $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} f_o(\lambda u + v) &= f_o((\lambda\alpha + \gamma)x + (\lambda\beta + \delta)y) \\ &= (\lambda\alpha + \gamma) f_o(x) + (\lambda\beta + \delta) f_o(y) \quad \text{since } (\lambda\alpha + \gamma)x \perp (\lambda\beta + \delta)y \\ &= \lambda(\alpha f_o(x) + \beta f_o(y) + \gamma f_o(x) + \delta f_o(y) \\ &= \lambda f_o(\alpha x + \beta y) + f_o(\gamma x + \delta y) \\ &= \lambda f_o(u) + f_o(v). \end{aligned}$$

Thus, f_o is linear on H and therefore on V. Take $f_o = A$.

(ii) To prove that f_e satisfies (Q) on H, use the same technique as in (i). As before,

$$\begin{aligned} f_e(2x) &= f_e(x + y + x - y) \\ &= f_e(x + y) + f_e(x - y) && \text{use (A}''\text{)} \\ &= f_e(x) + f_e(y) + f_e(x) + f_e(-y) && \text{again use (A}''\text{)} \\ &= 2 f_e(x) + 2 f_e(y) && \text{since } f_e \text{ is even.} \end{aligned}$$

Similarly, $f_e(2y) = 2 f_e(x) + 2 f_e(y)$. Hence $f_e(2x) = f_e(2y)$ and then $f_e(x) = f_e(y)$ by applying them to $\frac{x}{2} \perp \frac{y}{2}$. So, $f_e(2x) = 4 f_e(x) = 2^2 f_e(x)$. Use induction to show that $f_e(nx) = f_e(ny) = n^2 f(x)$ for $n \in Z_+$.

Assume that this relation holds for all k, $0 \leq k < n$. Then, as before,

$$\begin{aligned} & f_e(nx+(n-2)y) \\ &= f_e((n-1)(x+y) + (x-y)) \\ &= f_e((n-1)(x+y)) + f_e(x-y) && \text{apply (A}''\text{)} \\ &= (n-1)^2[f_e(x) + f_e(y)] \\ &\quad + f_e(x) + f_e(-y) && \text{apply induction and (A}''\text{)} \\ &= [2(n-1)^2 + 2] f_e(x) && \text{since } f_e \text{ is even} \\ &= 2(n^2 - 2n + 2) f_e(x) \\ &= f_e(nx) + f_e((n-2)y) && \text{apply (A}''\text{)} \\ &= f_e(nx) + (n-2)^2 f_e(x). \end{aligned}$$

Hence $f_e(nx) = [2(n^2 - 2n + 2) - (n-2)^2] f_e(x) = n^2 f_e(x)$. Similarly, $f_e(ny) = n^2 f_e(x) = f_e(nx)$. Since f_e is even, $f_e(nx) = f_e(ny) = n^2 f(x)$ for all $n \in Z$. Applying this to $\frac{x}{n} \perp \frac{y}{n}$, we get

$$f_e\left(\frac{x}{n}\right) = f_e\left(\frac{y}{n}\right) = \frac{1}{n^2} f_e(x)$$

and then

$$f_e\left(\frac{m}{n}x\right) = \left(\frac{m}{n}\right)^2 f(x) \quad \text{for all } \frac{m}{n} \in Q.$$

Invoking the continuity of f_e, we have

$$f_e(rx) = f_e(ry) = r^2 f_e(x) \quad \text{for all } r \in \mathbb{R},$$

from which follows

$$f_e(rx + sy) = f_e(rx) + f_e(sy) = r^2 f_e(x) + s^2 f_e(y) \quad \text{for } r, s \in \mathbb{R}.$$

Let $u, v \in H$. Then there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $u = \alpha x + \beta y$, $v = \gamma x + \delta y$, and

$$\begin{aligned}
&f_e(u+v) + f_e(u-v) \\
&\quad = f_e((\alpha+\gamma)x + (\beta+\delta)y) + f_e((\alpha-\gamma)x + (\beta-\delta)y) \\
&\quad = (\alpha+\gamma)^2 f_e(x) + (\beta+\delta)^2 f_e(y) + (\alpha-\gamma)^2 f_e(x) \\
&\qquad + (\beta-\delta)^2 f_e(y) \quad \text{since } (\alpha \pm \gamma)x \perp (\beta \pm \delta)y \\
&\quad = [(\alpha+\gamma)^2 + (\alpha-\gamma)^2] f_e(x) + [(\beta+\delta)^2 + (\beta-\delta)^2] f_e(y) \\
&\quad = 2(\alpha^2 + \gamma^2) f_e(x) + 2(\beta^2 + \delta^2) f_e(y) \\
&\quad = 2[f_e(\alpha x) + f_e(\beta y) + f_e(\gamma x) + f_e(\delta y)] \\
&\quad = 2 f_e(u) + 2 f_e(v);
\end{aligned}$$

that is, f_e is a solution of (Q) on H and therefore on V. Then, by Theorem 4.1, there is a biadditive $B : V \times V \to \mathbb{R}$ such that $f_e(u) = B(u,u)$ for $u \in V$. By the continuity of f_e, B is bilinear (that is, an i.p. on V).

Indeed, for $r \in Q$, $rB(u,v) = B(ru,v) = \frac{1}{2}[f_e(ru+v) - f_e(ru) - f_e(v)]$. So, if $\lambda \in \mathbb{R}$ and $r \to \lambda$, then $\lambda B(u,v) = B(\lambda u, v)$.

Since V is an i.p.s., if $x \perp y$, since f_e is orthogonally additive, $B(x,y) = \frac{1}{2}(f_e(x+y) - f_e(x) - f_e(y)) = 0$. Let $u \in V$ and S be a two-dimensional subspace containing u with $\{e_1, e_2\}$ as its orthonormal basis. Then $(e_1 + e_2) \perp (e_1 - e_2)$.

$$\begin{aligned} f_e(e_1) &= f_e\left(\frac{e_1+e_2}{2}+\frac{e_1-e_2}{2}\right) \\ &= f_e\left(\frac{e_1+e_2}{2}\right)+f_e\left(\frac{e_1-e_2}{2}\right) && \text{by (A}''\text{)} \\ &= f_e\left(\frac{e_1}{2}\right)+f_e\left(\frac{e_2}{2}\right)+f_e\left(\frac{e_1}{2}\right)+f_e\left(\frac{-e_2}{2}\right) && \text{by (A}''\text{)} \\ &= \frac{1}{2}[\,f_e(e_1)+f_e(e_2)] \end{aligned}$$

since f_e is even. Similarly,

$$f_e(e_2)=\frac{1}{2}[\,f_e(e_1)+f_e(e_2)]=f_e(e_1)=c \quad \text{(say)}.$$

Let $u=\alpha e_1+\beta e_2$ for $\alpha,\beta\in\mathbb{R}$.

$$\begin{aligned} f_e(u) &= B(u,u) \\ &= B(\alpha e_1+\beta e_2,\alpha e_1+\beta e_2) \\ &= \alpha^2 B(e_1,e_1)+\beta^2 B(e_2,e_2) \quad \text{since } B(e_1,e_2)=0 \\ &= \alpha^2 f_e(e_1)+\beta^2 f_e(e_2) \\ &= c(\alpha^2+\beta^2) \\ &= c\|u\|^2. \end{aligned}$$

This proves (5.75) and concludes the proof of this theorem. □

Corollary. *If f in* Theorem 5.20 *is an odd solution of* (A″), *then f is additive. If f is an even solution of* (A″), *then f is quadratic; that is, a solution of* (Q).

Now we consider $\perp_B$, Birkhoff-James orthogonality.

Result 5.21. [218]. *Let V be a real n.l.s. of dimension at least* 2. *Let $f: V\to\mathbb{R}$ satisfy*

$$(\text{A}''') \qquad f(x+y)=f(x)+f(y) \quad \textit{for } x,y\in V \textit{ with } x\perp_B y$$

(*that is,* orthogonally additive *with respect to* $\perp_B$). *Then the odd part f_o is additive and the even part f_e is zero. The norm of V comes from an i.p. if, and only if,* (A‴) *has a nonadditive solution.*

Diminnie Orthogonality

There are cases in which characterizations of i.p.s. among n.l.s. underlie a dimension condition. A certain uniqueness property of the Diminnie orthogonality works only for dimension at least three but does collapse for dimension two. The following result in [702] shows that if orthogonal additive maps instead of the orthogonality relation itself are used, the two-dimensional case is no longer exceptional.

Result 5.22. *A real n.l.s. X with* $\dim X \geq 3$ *is an i.p.s.* $\Leftrightarrow \perp_B$ *is symmetric* $\Leftrightarrow \perp_D$ *is right unique.*

Let V be a real n.l.s. with $\dim V \geq 2$, $\perp_D$ *the Diminnie orthogonality in V, and G an Abelian group. A mapping $f : V \to G$ is said to be* orthogonally additive *with respect to* $\perp_D$ (*or satisfy a conditional Cauchy functional equation*) *provided*

$$(\mathrm{A}'''') \qquad f(x+y) = f(x) + f(y) \quad \text{for } x, y \in V \text{ with } x \perp_D y.$$

Result 5.23. (de Rham [702]). *Suppose $f : V \to G$ satisfies* (A''''). *Then the odd solutions of* (A'''') *are precisely the additive mappings from $V \to G$, every even solution is a solution of* (Q), *every solution of* (A'''') *is additive* $\Leftrightarrow 0$ *is the only even solution of* (A''''), *and V is an i.p.s.* $\Leftrightarrow$ *not every solution of the equation* (A'''') *with $G = \mathbb{R}$ is additive.*

Remark. While the relations $\perp_B$ and $\perp_D$ are not sufficient for a two-dimensional real n.l.s. to be an i.p.s., the condition concerning (A'''') is.

Some More Generalizations

We consider two generalizations of (5.25) and (5.25a).

Theorem 5.7d. *Let V be a real linear space and $\phi, \alpha, \psi : V \to \mathbb{R}$. Then these functions satisfy the functional equation*

$$\sum_{i=1}^{n} \phi(x_i - \overline{x}) = \sum_{i=1}^{n} \alpha(x_i) + \psi(\overline{x}) \tag{5.25d}$$

for $x_1, x_2, \ldots, x_n \in V$, $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, and fixed $n \geq 3$ if, and only if,

$$\begin{aligned} \phi(x) &= B(x,x) + A_1(x) + c, \\ \alpha(x) &= B(x,x) + A(x) + c, \\ \psi(x) &= -n(B(x,x) + A(x) + c_1 - c), \end{aligned} \tag{5.26a}$$

for $x \in V$, where A, A_1 satisfy (A), *B is a symmetric, biadditive function, and c, c_1 are constants.*

Proof. Setting $x_1 = x_2 = \cdots x_n = x$ in (5.25d) yields

$$\psi(x) = -n\alpha(x) + nc, \quad \text{where } \phi(0) = c. \tag{5.27a}$$

As in Result 5.7a, we consider two cases according to whether n is even or odd.

First let n be even, say $2k$. Let $x_1 = \cdots = x_k = x$, $x_{k+1} = \cdots = x_{2k} = y$ in (5.25d) to have

$$k\phi\left(\frac{x-y}{2}\right) + k\phi\left(\frac{y-x}{2}\right) = k\alpha(x) + k\alpha(y) + \psi\left(\frac{x+y}{2}\right)$$

or

(5.29d) $\phi(x-y)+\phi(y-x)=\alpha(2x)+\alpha(2y)-2\alpha(x+y)+2c \quad \text{for } x, y \in V.$

When n is odd, say $2k+1$, put $x_1 = \cdots = x_k = x$, $x_{k+1} = \cdots = x_{2k} = y$, $x_{2k+1} = \frac{x+y}{2}$, $\overline{x} = \frac{x+y}{2}$ in (5.25d) to get

$$\begin{aligned} k\phi\left(\frac{x-y}{2}\right) + k\phi\left(\frac{y-x}{2}\right) + \phi(0) & \\ &= k\alpha(x) + k\alpha(y) + \alpha\left(\frac{x+y}{2}\right) + \psi\left(\frac{x+y}{2}\right) \\ &= k\alpha(x) + k\alpha(y) + \alpha\left(\frac{x+y}{2}\right) + (2k+1)\phi(0) \\ &\quad - (2k+1)\alpha\left(\frac{x+y}{2}\right), \end{aligned}$$

which is the same as (5.29d).

So, we solve (5.29d), which can be rewritten as

(5.76) $$\alpha(u+v)+\alpha(u-v)=2\alpha(u)+2\beta(v)$$

with $x+y=u$, $x-y=v$, and

(5.77) $$2\beta(x)=\phi(x)+\phi(-x)-2c, \quad \text{for } x \in V.$$

Note that (5.76) can be written as (putting $u=0$ in (5.76))

$$\alpha(u+v)+\alpha(u-v)=2\alpha(u)+\alpha(v)+\alpha(-v)-2c,$$

which is (4.22), and, by Theorem 4.22, there is an additive A and a symmetric, biadditive B such that

$$\begin{gathered} \alpha(u) = B(u,u) + A(u) + c_1 \quad \text{for } u \in V, \\ 2\beta(x) = \alpha(x) + \alpha(-x) - 2c = \phi(x) + \phi(-x) - 2c = 2B(x,x), \end{gathered}$$

and thus

$$\psi(x) = -n[B(x,x) + A(x) + c_1 - c].$$

It remains to determine ϕ. Putting these values of α and ψ in (5.25d), we obtain

$$\begin{aligned} \sum_{i=1}^{n} \phi(x_i - \overline{x}) &= \sum_{i=1}^{n} [B(x_i, x_i) + A(x_i) + c_1] + nc - n[B(\overline{x}, \overline{x}) + A(\overline{x}) + c_1] \\ &= \sum_{i=1}^{n} B(x_i - \overline{x}, x_i - \overline{x}) + nc; \end{aligned}$$

that is,

$$\sum_{i=1}^{n}[\phi(x_i - \overline{x}) - B(x_i - \overline{x}, x_i - \overline{x}] = nc$$

or

$$\sum_{i=1}^{n} t(x_i - \overline{x}) = nc,$$

where $t(x) = \phi(x) - B(x, x)$ for $x \in V$ and $t(0) = c$; that is,

$$\sum_{i=1}^{n} t(y_i) = nc,$$

where $y_i = x_i - \overline{x},\ i = 1$ to n with $\sum_{i=1}^{n} y_i = 0$ (see Theorem 1.76).

Put $y_1 = x,\ y_2 = y,\ y_3 = -(x + y),\ y_4 = 0 = \cdots = y_n$ to have

$$t(x) + t(y) + t(-x - y) = 3c \quad \text{(which is (A))}.$$

Set $y = 0$ to get $t(x) + t(-x) = 2c$ or $t(-x) = -t(x) + 2c$, and then

$$t(x) + t(y) = t(x + y) + c \quad \text{(which is (A))}.$$

So, there is an additive A_1 such that

$$t(x) = A_1(x) + c = \phi(x) - B(x, x).$$

Hence

$$\phi(x) = B(x, x) + A_1(x) + c.$$

This proves (5.26a) and the result. □

6

Stability

In this chapter, the stability problem is treated in general. In particular, stability of the multiplicative function, logarithmic function, trigonometric functions, vector-valued functions, alternative Cauchy equation, polynomial equation, and quadratic equation is treated.

S.M. Ulam, in his famous lecture in 1940 to the Mathematics Club of the University of Wisconsin, presented a number of unsolved problems. This is the starting point of the theory of the stability of functional equations. One of the questions led to a new line of investigation, nowadays known as the stability problems. Ulam [807] discusses:

> ... the notion of stability of mathematical theorems considered from a rather general point of view: When is it true that by changing "a little" the hypothesis of a theorem one can still assert that the thesis of the theorem remains true or "approximately" true? ...
> For very general functional equations one can ask the following question. When is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation? Similarly, if we replace a given functional equation by a functional inequality, when can one assert that the solutions of the inequality lie near to the solutions of the strict equation?

Suppose G is a group, $H(d)$ is a metric group, and $f : G \to H$. For any $\varepsilon > 0$, does there exist a $\delta > 0$ such that

$$d(f(xy),\ f(x)f(y)) < \delta$$

holds for all $x, y \in G$ and implies there is a homomorphism $M : G \to H$ such that

$$d(f(x), M(x)) < \varepsilon \quad \text{for all } x \in G?$$

If the answer is affirmative, then we say that the Cauchy equation (M) is stable. These kinds of questions form the basics of stability theory, and D.H. Hyers [382] obtained

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the first important result in this field. Many examples of this have been solved and many variations have been studied since. Several investigations followed, and almost all functional equations are "stabilized". It is impossible to give a summary of all the results known in this growing area. This chapter is designed to provide an introduction to the formulation and solution of such problems. At the same time, we present different methods of solving problems of this type. We present as many interesting results as possible. Good sources are the book by Hyers, Isac, and Rassias [385] and the survey articles by Székelyhidi [798] and Ger [310].

6.1 Stability of the Additive Equation

We now present Hyers, Isac, and Rassias's [385] valuable contribution to Ulam's problem in the following theorem.

Theorem 6.1. [385]. *Let X and Y be Banach spaces (X could be n.l.s.) and $f : X \to Y$ be such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \tag{6.1}$$

for some $\delta > 0$ and for all $x, y \in X$. Then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{6.2}$$

exists for each x in X, and $A : X \to Y$ is the unique additive function such that

$$\|f(x) - A(x)\| \leq \delta \tag{6.3}$$

for any $x \in X$. Moreover, if $f(tx)$ is continuous in t ($\in \mathbb{R}$) for each fixed $x \in X$ (that is, if for each $x \in X$ the function $t \to f(tx)$ from $\mathbb{R}$ to Y is continuous for each fixed x), then A is linear. Moreover, if f is continuous at a point in X, then A is continuous everywhere in X.

Proof. For any $x \in X$, the inequality

$$\left\| \left(\frac{1}{2}\right) f(2x) - f(x) \right\| \leq \left(\frac{1}{2}\right) \delta \tag{6.4}$$

holds. Now, by induction,

$$\|2^{-n} f(2^n x) - f(x)\| \leq (1 - 2^{-n})\delta. \tag{6.5}$$

Indeed, replace x by $2x$ in (6.4) to have

$$\left\| \frac{1}{2} f(2^2 x) - f(2x) \right\| \leq \frac{1}{2}\delta;$$

that is,

$$\left\| \frac{1}{2} f(2^2 x) - 2 f(x) - f(2x) + 2 f(x) \right\| \leq \frac{1}{2}\delta$$

or

$$\left\| \frac{1}{2} f(2^2 x) - 2 f(x) \right\| - \| f(2x) - 2 f(x) \| \leq \frac{1}{2}\delta,$$

meaning

$$\left\| \frac{1}{2^2} f(2^2 x) - f(x) \right\| < \delta \left(\frac{1}{2} + \frac{1}{2^2} \right),$$

from which follows

$$\left\| \frac{1}{2^n} f(2^n x) - f(x) \right\| \leq \delta \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \right) = \delta \left(1 - \frac{1}{2^n} \right),$$

which is (6.5).

Next we show that the sequence $\left\{ \frac{1}{2^n} f(2^n x) \right\}$ is Cauchy for each $x \in X$. Choose $m > n$. Then

$$\begin{aligned} \left\| \frac{1}{2^n} f(2^n x) - \frac{1}{2^m} f(2^m x) \right\| &= \frac{1}{2^n} \left\| \frac{1}{2^{m-n}} f(2^{m-n} \cdot 2^n x) - f(2^n x) \right\| \\ \text{by (6.5)} \quad &\leq \frac{1}{2^n} \cdot \delta \left(1 - \frac{1}{2^{m-n}} \right) \\ &= \delta \left(\frac{1}{2^n} - \frac{1}{2^m} \right). \end{aligned}$$

Thus the sequence is Cauchy and, since Y is a Banach space (complete), there exists a limit function $A : X \to Y$ such that (6.2) holds.

The next step is to show that A is additive. Replace x and y in (6.1) by $2^n x$ and $2^n y$ to get

$$\left\| \frac{1}{2^n} f(2^n (x + y)) - \frac{1}{2^n} f(2^n x) - \frac{1}{2^n} f(2^n y) \right\| \leq \frac{1}{2^n}\delta$$

for $n \in Z_+^*$, $x, y \in X$. Letting $n \to \infty$, we observe that A is additive from the inequality above, and from (6.2) we obtain (6.3).

Our next goal is to show that A is unique. Suppose $A_1 : X \to Y$ is another additive function satisfying the inequality (6.3). Then, for $x \in X$,

$$\begin{aligned} \|A(x) - A_1(x)\| &= \frac{1}{n} \|A(nx) - f(nx) - A_1(nx) + f(nx)\| \\ &\leq \frac{2\delta}{n} \quad \text{(using (6.3))}, \end{aligned}$$

showing thereby that $A_1 = A$.

Finally, suppose f is continuous at $y \in X$. Let $\{x_n\}$ in X be such that $x_n \to 0$ as $n \to \infty$. Then, for any integer $m \in Z_+^*$,

$$\begin{aligned}\|A(x_n+y)-A(y)\| &= \|A(x_n)\| \\ &= \frac{1}{m}[\|A(mx_n+y)-f(mx_n+y) \\ &\quad + f(mx_n+y)-f(y)+f(y)-A(y)\|] \\ \text{(by (6.3) and continuity of } f \text{ at } y) \quad &\le \frac{2\delta+\varepsilon}{m}\end{aligned}$$

for sufficiently large n and any integer m; that is, A is also continuous at y.

For a fixed $x \in X$, if $f(tx)$ is continuous in t, then it follows that $A(tx)$ is continuous in t and hence A is linear. This proves the theorem. □

Several remarks are in order. If f in the theorem is continuous at a point in X, then the corresponding A is continuous everywhere in X. This pioneering result of D.H. Hyers can be expressed in the following way. The additive Cauchy functional equation

$$f(x+y) = f(x)+f(y)$$

is stable for any pair of Banach spaces. The function

$$(x,y) \mapsto f(x+y)-f(x)-f(y)$$

is called the Cauchy difference of the function (see Chapter 13). Functions with a bounded Cauchy difference are called approximately additive mappings. The sequence $\left\{\frac{f(2^n x)}{2^n}\right\}$ is called the Hyers-Ulam sequence.

This theorem is an immediate consequence of the following result of Polya and Szegö [670] with $Y = \mathbb{R}$.

Theorem 6.2. *For any real sequence* (a_n) *satisfying the inequality*

$$|a_{n+m}-a_n-a_m| < 1, \quad n,m \in Z_+^*, \tag{6.1a}$$

there exists a finite limit

$$A := \lim_{n\to\infty} \frac{a_n}{n}$$

and

$$|a_n - nA| < 1, \quad n \in Z_+^*.$$

Theorem 6.1 is an immediate consequence of Theorem 6.2 (in the case where $(Y, \|\cdot\|) = (\mathbb{R}, |\cdot|)$. Actually, arbitrarily fix an $x \in X$ and put $a_n := \left(\frac{1}{\delta}\right) f(nx)$, $n \in Z_+^*$. Then (a_n) satisfies (6.1a) because of (6.1), whence, on setting

$$A(x) := \lim_{k\to\infty}\left(\frac{a_k}{k}\right) = \left(\frac{1}{\delta}\right)\lim_{k\to\infty}\left(\frac{f(kx)}{k}\right),$$

we get

$$\left|\frac{1}{\delta}f(nx) - nA(x)\right| < 1$$

for all $n \in Z_+^*$ and all $x \in X$ or

$$\left| f(x) - n\delta A\left(\frac{x}{n}\right) \right| < \delta.$$

Since, in view of (6.1), for every $x, y \in X$ one has

$$\begin{aligned} |\delta A(x+y) - \delta A(x) - \delta A(y)| &= \lim_{n\to\infty} \left| \frac{f(nx+ny)}{n} - \frac{f(nx)}{n} - \frac{f(ny)}{n} \right| \\ &\le \lim_{n\to\infty} \frac{\delta}{n} = 0, \end{aligned}$$

consequently

$$|f(x) - A(x)| < \delta \quad \text{for all } x \in X.$$

□

There are two possible ways to generalize Theorem 6.1. One of the natural ways is to generalize the domain; the other is to have different bounds on the right side of (6.1).

It is possible to prove a stability result similar to Hyers for functions that are not necessarily bounded Cauchy differences. Mainly using the technique of Hyers, Th.M. Rassias [678] gave the following generalization of Hyers's result.

Result 6.3. [678, 682]. *If $f : \mathbb{R} \to \mathbb{R}$ is a real map satisfying*

$$|f(x+y) - f(x) - f(y)| \le \delta(|x|^p + |y|^p) \tag{6.6}$$

for some $\delta > 0$, $p \in [0, 1)$, and for all $x, y \in \mathbb{R}$, then there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - A(x)| \le \frac{2\delta}{2 - 2^p}|x|^p$$

for all $x \in \mathbb{R}$.

Remark. $p = 0$ in Result 6.3 yields Theorem 6.1.

Remark 6.4. Result 6.3 is true for all real p except $p = 1$. Gajda in [302] proved the following result. Let X be a normed space and Y a Banach space. Assume that $p > 1$ is a real number, and let $f : X \to Y$ be a mapping for which there exists an $\varepsilon > 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p)$$

holds for all x, y in X. Then the limit

$$A(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$$

exists for all x in X and $A : X \to Y$ is the unique additive mapping that satisfies

$$\|f(x) - A(x)\| \leq \frac{2}{2^p - 2}\varepsilon$$

for all x in X.

That Result 6.3 does not hold for $p = 1$ is shown in [302] by the following counterexample. In [296], the following example of a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$|f(x+y) - f(x) - f(y)| \leq |x| + |y| \quad \text{for } x, y \in \mathbb{R}, \tag{6.6a}$$

with

$$\lim_{x \to 0} \frac{f(x)}{x} = \infty$$

behaving badly near 0, is constructed. For a fixed $\theta > 0$, let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \sum_{n=0}^{\infty} 2^{-n} \phi(2^n x), \quad x \in \mathbb{R},$$

where the function $\phi : \mathbb{R} \to \mathbb{R}$ is given by

$$\phi(x) = \begin{cases} \frac{1}{6}\theta & \text{if } 1 \leq x < \infty, \\ \frac{1}{6}\theta x & \text{if } -1 < x < 1, \\ -\frac{1}{6}\theta & \text{if } -\infty < x \leq -1, \end{cases}$$

and f satisfies the inequality

$$|f(x+y) - f(x) - f(y)| \leq \theta(|x| + |y|) \tag{6.6b}$$

for all $x, y \in \mathbb{R}$. But there is no constant $\delta \in \mathbb{R}_+$ and no additive function $A : \mathbb{R} \to \mathbb{R}$ satisfying the inequality

$$|f(x) - A(x)| \leq \delta |x|$$

for all $x \in \mathbb{R}$.

Rassias and Šemrl [682] also constructed a continuous function f given by

$$f(x) = \begin{cases} x \log(x+1) & \text{if } x > 0, \\ x \log|x-1| & \text{if } x < 0, \end{cases}$$

satisfying (6.6a) for all $x, y \in \mathbb{R}$ with

$$\lim_{x \to \infty} \frac{f(x)}{x} = \infty.$$

It follows that the set

$$\left\{ \frac{|f(x) - A(x)|}{|x|} : x \neq 0 \right\}$$

is unbounded for any linear mapping $A : \mathbb{R} \to \mathbb{R}$ defined by $A(x) = cx$ for a given $c \in \mathbb{R}$. In other words, an analogue of Result 6.3 cannot be obtained.

The following general result of the type above is given by Th.M. Rassias and P. Šemrl [686].

Result 6.5. *Let $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a positive homogeneous function of degree $p \neq 1$. Let X be a normed space and Y be a Banach space, and further assume that the function $f : X \to Y$ satisfies*

$$\|f(x + y) - f(x) - f(y)\| \leq \beta(\|x\|, \|y\|)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \leq \delta \|x\|^p$$

for all x in X, where

$$\delta = \begin{cases} \frac{\beta(1,1)}{2-2^p} & \text{for } p < 1, \\ \frac{\beta(1,1)}{2^p-2} & \text{for } p > 1. \end{cases}$$

The following theorem of J. Rätz [691] shows that a far-reaching generalization of the range is still possible. We remark that a nonempty set S with a binary operation $*$ is called a *power-associative groupoid* if $x^{m+n} = x^m * x^n$ holds for any natural numbers n and m and any element x in S.

Result 6.6. *Let S be a power-associative groupoid and let Y be a sequentially complete topological vector space over the rationals. Further, let V be a nonempty bounded convex subset of Y containing the origin, and let $f : S \to Y$ be a function satisfying*

$$f[(x * y)^{k^n}] = f(x^{k^n} * y^{k^n})$$

and

$$f(x * y) - f(x) - f(y) \in V$$

for all $x, y \in S$, for some integer $k \geq 2$, and for all n in Z_+^*. Then there exists an additive function $A : S \to Y$ with $f(x) - A(x) \in \overline{V}$ for all $x, y \in S$. If Y is a Hausdorff space, then A is unique. (Here $\overline{V}$ denotes the sequential closure of V.)

As J. Rätz [691] pointed out, power associativity does not imply commutativity, even if S is a semigroup.

There are several possible ways to generalize Theorem 6.1. A natural way is to generalize the domain X. The following generalization is a consequence of a more general result of J. Rätz [691].

Result 6.6a. *Let S be a semigroup with the property that for each x, y in S there exists an integer $n \geq 2$ with*

$$(x \cdot y)^n = x^n \cdot y^n.$$

Further, let Y be a Banach space. Then Cauchy's functional equation is stable for the pair (S, Y).

6.2 Stability—Multiplicative Equations

Theorem 6.7. (Baker [92]). *Let $\delta > 0$, let S be a semigroup, and let $f : S \to \mathbb{C}$ be such that*

$$|f(xy) - f(x)f(y)| \le \delta \quad \text{for all } x, y \in S. \tag{6.7}$$

Then either

$$|f(x)| \le \frac{1+\sqrt{1+4\delta}}{2} =: \varepsilon$$

for all $x \in S$ or f is multiplicative for all $x, y \in S$.

Proof. Notice that $\varepsilon^2 - \varepsilon = \delta$ and $\varepsilon > 1$. Suppose there exists $a \in S$ such that $|f(a)| > \varepsilon$, say $|f(a)| = \varepsilon + \rho$ for some $\rho > 0$. Then

$$\begin{aligned} |f(a^2)| &\ge |f(a)^2 - (f(a)^2 - f(a^2))| \\ &\ge |f(a)^2| - |f(a)^2 - f(a^2)| \\ &\ge |f(a)|^2 - \delta \\ &= (\varepsilon+\rho)^2 - \delta \\ &= (\varepsilon+\rho) + (2\varepsilon - 1)\rho + \rho^2 > \varepsilon + 2\rho. \end{aligned}$$

By induction, $|f(a^{2^n})| > \varepsilon + (n+1)\rho$ for all $n = 1, 2, \ldots$.

For every $x, y, z \in S$,

$$|f(xyz) - f(xy)f(z)| \le \delta \quad \text{and} \quad |f(xyz) - f(x)f(yz)| \le \delta,$$

so that

$$|f(xy)f(z) - f(x)f(yz)| = |f(xyz) - f(x)f(yz) + f(xy)f(z) - f(xyz)| \le 2\delta$$

and

$$\begin{aligned} |f(xy)f(z) - f(x)f(y)f(z)| &\le |f(xy)f(z) - f(x)f(yz)| \\ &\quad + |f(x)f(yz) - f(x)f(y)f(z)| \\ &\le 2\delta + |f(x)|\delta \end{aligned}$$

or

$$|f(xy) - f(x)f(y)| \cdot |f(z)| \le 2\delta + |f(x)|\delta.$$

In particular,

$$|f(xy) - f(x)f(y)| \le \frac{2\delta + |f(x)|\delta}{|f(a^{2^n})|}$$

for all $x, y \in S$ and all $n = 1, 2, \ldots$. Letting $n \to +\infty$ shows that $f(xy) = f(x)f(y)$ for all $x, y \in S$; that is, f is multiplicative. □

Remark. The crucial step in this proof is the multiplicative structure of complex numbers. Because the quaternions and Cayley numbers enjoy the same property, it is easy to see that Theorem 6.7 is also true for functions f with values in the quaternions or Cayley numbers. For real normed algebras for which the norm is not multiplicative, the situation is not so simple, as the following example shows. Let $\delta > 0$, and choose $\varepsilon > 0$ so that $|\varepsilon - \varepsilon^2| = \delta$. Then $\varepsilon \neq 1$, and let

$$f(x) = \begin{bmatrix} e^x & 0 \\ 0 & \varepsilon \end{bmatrix} \quad \text{for } x \text{ real.}$$

Then, with the usual matrix norm,

$$\| f(x+y) - f(x) f(y) \| = \delta$$

for all real x and y, and f is unbounded, but it is not true that $f(x+y) = f(x) f(y)$ for all real x and y.

Remark. Theorem 6.7 is true when the codomain is a normed algebra with the multiplicative norm instead of $\mathbb{C}$.

Result 6.8. (Shulman [738]). *A measurable complex-valued function f on a locally compact group G satisfies the condition*

$$\sup_{x \in G} \operatorname{ess\,sup}_{y \in G} | f(xy) - f(x) f(y) | < \infty$$

if and only if f is either essentially bounded or equal to a continuous multiplicative function almost everywhere.

6.3 Stability—Logarithmic Function

We note that a semigroup S is called *left* (*right*) *amenable* if there exists a left (right) invariant mean on the Banach space of all bounded complex-valued functions defined on S. A *left* (*right*) *invariant mean* is a bounded left (right) translation-invariant linear functional taking the value 1 on the constant 1 function.

Result 6.9. (Székelyhidi [793, 789]). *Let S be a left amenable semigroup and let M be a left invariant mean on the space of all bounded complex-valued functions defined on S. Let $f : S \to \mathbb{C}$ be a mapping satisfying*

$$\| f(xy) - f(x) - f(y) \| \leq \varepsilon$$

for all x, y in S. Then the function $L : S \to \mathbb{C}$ defined by

$$L(x) = M_y [f(xy) - f(y)]$$

for any x in S is the unique logarithmic function for which

$$\| f(x) - L(x) \| \leq \varepsilon$$

holds for any x in S.

From this result, we have immediately that Cauchy's functional equation is stable for the pair $(S, \mathbb{C})$, where S is a finite semigroup, or a commutative semigroup.

The following result explains the role of the Hyers-Ulam sequence (see [285]).

Result 6.9a. (Forti [285]). *Let S be a semigroup and Y a Banach space. Further, let $f : S \to Y$ be a mapping satisfying*

$$\|f(xy) - f(x) - f(y)\| \leq \varepsilon \tag{6.8}$$

for all x, y in S. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(x^{2^n})}{2^n}$$

exists for all x in S and $L : S \to Y$ is the unique mapping satisfying

$$\|f(x) - L(x)\| \leq \varepsilon$$

and

$$L(x^2) = 2L(x)$$

for all x in S. If S is commutative, then L is logarithmic.

Result 6.9b. (Ger [309]). *Let S be a left amenable semigroup, Y a reflexive Banach space, and $\rho : S \to [0, \infty[$ an arbitrary function. If the function $f : S \to Y$ satisfies*

$$\|f(xy) - f(x) - f(y)\| \leq \rho(y)$$

for all x, y in S, then there exists an additive function $A : S \to Y$ such that

$$\|f(x) - A(x)\| \leq \rho(x)$$

holds for all x in S.

Result 6.10. [738]. *A measurable complex-valued function f on an amenable locally compact group G satisfies the condition*

$$\sup_{x\in G} \operatorname{ess\,sup}_{y\in G} |f(xy) - f(x) - f(y)| < \infty$$

if and only if $f = \Phi + L$, where L is logarithmic for all x, y in G and Φ is essentially bounded.

The problem of stability of algebra homomorphisms also leads to the study of systems of functional equations. A result in this direction is given in [132].

Result 6.11. (Bourgin [132]). *Let X, Y be normed algebras, where Y is complete. If $f : X \to Y$ is surjective and the functions $(x, y) \mapsto f(x + y) - f(x) - f(y)$ and $(x, y) \mapsto f(xy) - f(x)f(y)$ are bounded, then f is an algebra isomorphism of X onto Y.*

The stability of the equation of multiplicative derivation of this functional equation has been studied in [731]. Let X be a Banach space and $\mathcal{B}(X)$ denote the algebra of all bounded linear operators on X.

Result 6.12. (Šemrl [731]). *Let X be a Banach space and let $\mathcal{A}$ be a standard operator algebra on X. Assume that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that*

$$\lim_{t\to+\infty} \frac{\psi(t)}{t} = 0.$$

If $f : \mathcal{A} \to \mathcal{B}(X)$ is a mapping satisfying

$$\| f(xy) - xf(y) - f(x)y\| \leq \psi(\|x\| \, \|y\|)$$

for all x, y in $\mathcal{A}$, then f satisfies

$$f(xy) = xf(y) + yf(x)$$

for all x, y in $\mathcal{A}$.

6.4 Stability—Trigonometric Functions

First we start off with the d'Alembert equation (C).

Theorem 6.13. (Baker [92]). *Let $\delta > 0$, let G be an Abelian group, and let f be a complex-valued function defined on G such that*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \delta \quad \text{for all } x, y \in G. \tag{6.9}$$

Then either

$$|f(x)| \leq \frac{1+\sqrt{1+2\delta}}{2} \quad \text{for all } x \in G$$

or there exists a complex-valued function E on G such that

$$f(x) = \frac{\{E(x) + E^*(x)\}}{2} \quad \text{for all } x \in G$$

and

$$|E(x+y) - E(x)E(y)| \leq \frac{\delta}{2} \quad \text{for all } x, y \in G.$$

Proof. We will use a few times the inequality

$$|z-w| \geq \big||z| - |w|\big| \quad \text{for } z, w \in \mathbb{C}.$$

Setting $x = 0 = y$ in (6.9), we have

$$|2f(0) - 2f(0)^2| \leq \delta$$

or

$$2|f(0)^2| - 2|f(0)| \leq \delta;$$

that is,

$$|f(0)| \le \frac{1+\sqrt{1+2\delta}}{2} = \varepsilon \quad \text{(say)}. \tag{6.10}$$

Note that $2(\varepsilon^2 - \varepsilon) = \delta$ and $2\varepsilon > 1$.

Put $y = x$ in (6.9) to get

$$|f(2x) + f(0) - f(x)^2| \le \delta.$$

Using this and (6.10), we have

$$\begin{aligned} |f(2x)| &= |2f(x)^2 - f(0) - \{2f(x)^2 - f(0) - f(2x)\}| \\ &\ge |2f(x)^2 - f(0)| - |f(2x) - f(0) - 2f(x)^2| \\ &> |2f(x)^2 - f(0)| - \delta \\ &> |2f(x)^2| - |f(0)| - \delta \\ &> 2|f(x)|^2 - \varepsilon - \delta. \end{aligned}$$

If there is a $b \in G$ with $|f(b)| > \varepsilon$, then

$$|f(2^n b)| \to \infty \quad \text{as } n \to \infty.$$

Let $|f(b)| = \varepsilon + \rho$ for some $\rho > 0$. Then

$$\begin{aligned} 2|f(b)|^2 - |f(b)| - (\varepsilon + \delta) &= 2(\varepsilon+\rho)^2 - (\varepsilon+\rho) - (\varepsilon+\delta) \\ &= 2(\varepsilon^2 - \varepsilon) + (4\varepsilon - 1)\rho + 2\rho^2 - \delta \\ &= (4\varepsilon - 1)\rho + 2\rho^2 > 3\rho. \end{aligned}$$

Hence

$$2|f(b)|^2 - (\varepsilon + \delta) > |f(b)| + 3\rho > \varepsilon + 2^2\rho.$$

So,

$$|f(2b)| \ge 2|f(b)|^2 - (\varepsilon + \delta) > \varepsilon + 2^2\rho.$$

By induction on n,

$$|f(2^n b)| > \varepsilon + 2^{n+1}\rho \quad \text{for all } n.$$

Hence f is unbounded and $|f(2^n b)| \to \infty$ as $n \to \infty$. With $x = b,\ y = 0$ in (6.9), we have $2|f(b)|\,|1 - f(0)| \le \delta$. Since f is unbounded, it follows that $f(0) = 1$.

For any $x, y \in G$,

$$\begin{aligned} 2|f(x)| \cdot |f(y) - f(-y)| &= |2f(x)f(y) - 2f(x)f(-y)| \\ &= |2f(x)f(y) - f(x+y) - f(x-y)| \\ &\quad + |f(x+y) + f(x-y) - 2f(x)f(-y)| \\ &\le 2\delta \end{aligned}$$

(as we see by replacing y by $-y$ in (6.9)), and since f is unbounded, it follows that $f(-y) = f(y)$ for all $y \in G$; that is, f is even.

Letting $y = x$, (6.9) yields

$$|f(2x) + 1 - f(x)^2| \leq \delta.$$

If $f(2x) = 1$ for all $x \in G$, then $|2 - f(x)^2| \leq \delta$ for all x or $|f(x)^2| \leq 2 + \delta$ for all $x \in G$, contradicting the unboundedness of f. So, there is an $a \in G$ with $f(2a) \neq 1$ and $\alpha \in \mathbb{C}$ such that $2\alpha^2(f(2a) - 1) = 1$.

Define

$$g(x) = f(x + a) - f(x - a)$$

and

$$E(x) = f(x) + \alpha g(x) \quad \text{for all } x \in G. \tag{6.11}$$

Since f is even,

$$\begin{aligned} g(-x) &= f(-x + a) - f(-x - a) \\ &= f(x - a) - f(x + a) \\ &= -g(x). \end{aligned}$$

That is, g is odd. Further,

$$\begin{aligned} E(-x) &= f(-x) + \alpha g(-x) \\ &= f(x) - \alpha g(x). \end{aligned} \tag{6.11a}$$

Adding (6.11) and (6.11a), we get

$$f(x) = \frac{E(x) + E(-x)}{2},$$

as asserted in the theorem.

Let

$$2f(x)f(y) = f(x + y) + f(x - y) + t(x, y), \tag{6.12}$$

so that

$$|t(x, y)| \leq \delta$$

for all $x, y \in G$. Next we compute (see Chapter 3)

$$\begin{aligned} 2f(x)g(y) + 2f(y)g(x) &= 2f(x)f(y + a) - 2f(x)f(y - a) \\ &\quad + 2f(y)f(x + a) - 2f(y)f(x - a) \\ &= f(x + y + a) + f(x - y - a) + t(x, y + a) \\ &\quad - f(x + y - a) - f(x - y + a) - t(x, y - a) \\ &\quad + f(x + y + a) + f(y - x - a) + t(x + a, y) \\ &\quad - f(y + x - a) - f(y - x + a) - t(x - a, y) \end{aligned}$$

$$
\begin{aligned}
&= 2f(x+y+a) - 2f(x+y-a) + f(x-y-a) \\
&\quad - f(y-x+a) + f(y-x-a) - f(y-x+a) \\
&\quad + t(x, y+a) - t(x, y-a) + t(x+a, y) - t(x-a, y) \\
&= 2g(x+y) + t(x, y+a) - t(x, y-a) \\
&\quad + t(x+a, y) - t(x-a, y)
\end{aligned}
$$

since f is even. Hence, we have

$$
\begin{aligned}
&2g(x+y) - 2f(x)g(y) - 2f(y)g(x) \\
&\quad = t(x, y-a) - t(x, y+a) + t(x-a, y) - t(x+a, y).
\end{aligned}
$$

Since $|t(x, y)| \le \delta$, we have from the equation above

$$
|g(x+y) - f(x)g(y) - f(y)g(x)| \le 2\delta \tag{6.13}
$$

for all $x, y \in G$.

Similarly, using (6.12), we obtain

$$
\begin{aligned}
2g(x)g(y) &= 2[f(x+a) - f(x-a)][f(y+a) - f(y-a)] \\
&= 2f(x+a)f(y+a) - 2f(x+a)f(y-a) - 2f(x-a)f(y+a) \\
&\quad + 2f(x-a)f(y-a) \\
&= f(x+y+2a) - f(x-y) + t(x+a, y+a) - f(x+y) \\
&\quad - f(x-y+2a) + t(x+a, y-a) - f(x+y) - f(x-y-2a) \\
&\quad - t(x-a, y+a) + f(x+y-2a) + f(x-y) + t(x-a, y-a) \\
&= 2f(x-y) - 2f(x+y) + f(x+y+2a) + f(x+y-2a) \\
&\quad - f(x-y+2a) - f(x-y-2a) + t(x+a, y+a) \\
&\quad - t(x+a, y-a) + t(x-a, y-a) - t(x-a, y+a) \\
&= 2f(x-y) - 2f(x+y) + 2f(x+y)f(2a) \\
&\quad - t(x+y+2a, x+y-2a) - 2f(x-y)f(2a) \\
&\quad + t(x-y+2a, y-x-2a) + t(x+a, y+a) \\
&\quad - t(x+a, y-a) + t(x-a, y-a) - t(x-a, y+a).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&2g(x)g(y) - 2(f(2a) - 1)[f(x+y) - f(x-y)] \\
&\quad = t(x-y+2a, x-y-2a) - t(x+y+2a, x+y-2a) \\
&\qquad + t(x+a, y+a) - t(x+a, y-a) \\
&\qquad + t(x-a, y-a) - t(x-a, y+a).
\end{aligned}
$$

Since $|t(x, y)| \le \delta$, we get

$$
|g(x)g(y) - (f(2a) - 1)(f(x+y) - f(x-y))| \le 3\delta. \tag{6.14}
$$

Further, we get

$$\left| f(x)\,f(y) - \frac{f(x+y)+f(x-y)}{2} \right| \le \frac{\delta}{2}. \tag{6.15}$$

Now, using (6.13), (6.14), and (6.15), we compute

$$\begin{aligned}
|E(x+y) - E(x)E(y)| &= |f(x+y) + \alpha g(x+y) - f(x)f(y) - \alpha^2 g(x)g(y) \\
&\quad - \alpha f(x)g(y) - \alpha f(y)g(x)| \\
&\le |\alpha| \cdot |g(x+y) - f(x)g(y) - f(y)g(x)| \\
&\quad + |f(x+y) - f(x)f(y) - \alpha^2 g(x)g(y)| \\
&\le 3|\alpha|\delta + \left| f(x+y) - \alpha^2 g(x)g(y) \right. \\
&\quad \left. - \frac{f(x+y)+f(x-y)}{2} \right| + \frac{\delta}{2} \\
&\le 3|\alpha|\delta + \frac{\delta}{2} + \left| \frac{f(x+y) - f(x-y)}{2} \right. \\
&\quad \left. - \alpha^2 (f(a) - 1)[f(x+y) - f(x-y)] \right| + 2|\alpha|^2\delta \\
&= \delta\left(3|\alpha| + |\alpha|^2 + \frac{1}{2}\right)
\end{aligned}$$

for all $x, y \in \mathbb{R}$. Since f is unbounded, a can be chosen so that

$$|f(a)| \quad \text{and} \quad |f(2a)|$$

can be as large as desired. Thus α can be chosen so that $|\alpha|$ is as small as we wish. Thus we must have

$$|m(x+y) - m(x)m(y)| \le \frac{\delta}{2}$$

for all $x, y \in \mathbb{R}$. This completes the proof of the theorem. □

From Theorems 6.7 and 6.13, we easily obtain the following.

Result 6.13a. *Let $\delta > 0$, let G be an Abelian group, and let f be a complex-valued function on G such that*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \le \delta \quad \textit{for all } x, y \in G. \tag{6.9}$$

Then either

$$|f(x)| \le \frac{1+\sqrt{1+2\delta}}{2} \quad \textit{for all } x \in G$$

or

$$f(x+y) + f(x-y) = 2f(x)f(y) \quad \textit{for all } x, y \in G. \tag{C}$$

If we let

$$f(x) = \frac{1+\sqrt{1+2\delta}}{2} \quad \text{for all } x \in G,$$

then (6.9) holds but (C) fails. Thus the estimate in Theorem 6.13a is optimal. Also notice that if f is sufficiently uniformly small or sufficiently uniformly close to 1, then (6.9) holds. □

Using well-known results concerning the cosine equation (see, e.g., [424]), we obtain the following corollary to Theorem 6.13a.

Corollary. *Let $\delta > 0$ and let f be a complex-valued, Lebesgue measurable function defined on $\mathbb{R}^n$ such that*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \le \delta \quad \textit{for all } x, y \in \mathbb{R}^n.$$

Then either

$$|f(x)| \le \frac{1+\sqrt{1+2\delta}}{2} \quad \textit{for all } x \in \mathbb{R}^n$$

or

$$f(x) - \cos(c \cdot x) \quad \textit{for all } x \in \mathbb{R}^n,$$

where c is an n-tuple of complex numbers and

$$c \cdot x = \sum_{i=1}^{n} c_i x_i$$

if $c = (c_1, \ldots, c_n)$ and $x = (x_1, \ldots, x_n)$.

Badora [82] gave a new, shorter proof of Baker's result, Theorem 6.13, which we will adopt in Theorems 6.14 and 6.15 investigating the stability of (6.16) and (6.17). The following result is proved in [82], and an example shows that the theorem is not valid in the case of vector-valued functions.

Result 6.13b. (Badora [82]). *Let $\varepsilon, \delta \ge 0$ and $f : G \to \mathbb{C}$ (G a group) satisfy the inequality* (6.9) *and*

$$|f(x+y+z) - f(x+z+y)| \le \delta \quad \textit{for } x, y, z \in G$$

(*similar to the* (K) *condition*). *Then f is either bounded or satisfies d'Alembert's equation* (C).

Counterexample. [82]. *Let G be a group and $f : G \to M_2(\mathbb{C})$ be defined by*

$$f(x) = \begin{pmatrix} g(x) & 0 \\ 0 & c \end{pmatrix}, \quad x \in G,$$

where $g : G \to \mathbb{C}$ is an unbounded function fulfilling equation (C) *and c $(\neq 1) \in \mathbb{R}_+^*$. Then*

$$\| f(x+y) + f(x-y) - 2f(x)f(y)\| = \text{constant} > 0 \quad \text{for } x, y \in G.$$

This difference is bounded, but f is neither bounded nor satisfies (C)*.*

We now investigate the stability problem of the functional equations

$$f(x+y) + f(x-y) = 2g(x)f(y), \quad x, y \in G, \tag{6.16}$$

and

$$f(x+y) + f(x-y) = 2f(x)g(y), \quad x, y \in G, \tag{6.17}$$

which are generalizations of (C) (refer to (3.27)).

Stability of (6.16) and (6.17) for Complex Functions

In this section, we will consider the stability of the equations above. First we will take up (6.16) and prove the following theorem.

Theorem 6.14. (Kannappan and Kim [484]). *Let $\varepsilon \geq 0$ and $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + f(x-y) - 2g(x)f(y)| \leq \varepsilon, \tag{6.16a}$$

with f satisfying the (K) *condition, where $G(+)$ is a group. Then either f and g are bounded or g satisfies* (C) *and f and g satisfy* (6.16) *and* (6.17)*. Further, in the latter case there exists a* (*homomorphism*) *exponential $E : G \to \mathbb{C}^*$ satisfying* (E) *such that*

$$f(x) = \frac{b}{2}(E(x) + E(-x)) \quad \text{and} \quad g(x) = \frac{1}{2}(E(x) + E(-x)) \tag{6.18}$$

for $x \in G$, where b is a constant.

Proof. The proof is modelled after [82]. We will consider only the nontrivial f (that is, $f \neq 0$).

Put $y = 0$ in (6.16a) to get

$$|f(x) - g(x)f(0)| < \frac{\varepsilon}{2} \quad \text{for } x \in G.$$

If g is bounded, then using the above we have

$$\begin{aligned} |f(x)| &= |f(x) - g(x)f(0) + g(x)f(0)| \\ &\leq \frac{\varepsilon}{2} + |g(x)f(0)|, \end{aligned}$$

which shows that f is also bounded. On the other hand, if f is bounded, choose y_0 such that $f(y_0) \neq 0$ and then use (6.16a)

$$|g(x)| - \left| \frac{f(x+y_0) + f(x-y_0)}{2f(y_0)} \right| \leq \left| \frac{f(x+y_0) + f(x-y_0)}{2f(y_0)} - g(x) \right| \leq \frac{\varepsilon}{2|f(y_0)|}$$

to get that g is also bounded on G. It follows easily now that if f (or g) is unbounded, then so is g (or f).

Let f and g be unbounded. Then there are sequences $\{x_n\}$ and $\{y_n\}$ in G such that $g(x_n) \neq 0$, $|g(x_n)| \to \infty$ as $n \to \infty$ and $f(y_n) \neq 0$, $\lim_n |f(y_n)| = \infty$.

First we will show that g indeed satisfies (C). From (6.16a) with $y = y_n$, we obtain

$$\left| \frac{f(x+y_n) + f(x-y_n)}{2f(y_n)} - g(x) \right| \leq \frac{\varepsilon}{2|f(y_n)|};$$

that is,

$$\lim_n \frac{f(x+y_n) + f(x-y_n)}{2f(y_n)} = g(x). \tag{6.19}$$

Using (6.16a) again and (K), we have

$$\begin{aligned} |f(x+(y+y_n)) + f(x-(y+y_n)) \\ - 2g(x)f(y+y_n) + f(x+(y-y_n)) \\ + f(x-(y-y_n)) - 2g(x)f(y-y_n)| \leq 2\varepsilon, \end{aligned}$$

so that

$$\begin{aligned} \left| \frac{f((x+y)+y_n) + f((x+y)-y_n)}{2f(y_n)} + \frac{f((x-y)+y_n) + f((x-y)-y_n)}{2f(y_n)} \right. \\ \left. -2g(x)\frac{f(y+y_n) + f(y-y_n)}{2f(y_n)} \right| \leq \frac{\varepsilon}{|f(y_n)|} \quad \text{for } x, y \in G, \end{aligned}$$

which with the use of (6.19) implies that

$$|g(x+y) + g(x-y) - 2g(x)g(y)| \leq 0;$$

that is, g is a solution of (C).

As before, applying (6.16a) twice and the (K) condition, first we have

$$\lim_n \frac{f(x_n+y) + f(x_n-y)}{2g(x_n)} = f(y) \tag{6.19a}$$

and then

$$\begin{aligned} |f((x_n+x)+y) + f((x_n+x)-y) - 2g(x_n+x)f(y) \\ + f((x_n-x)+y) + f((x_n-x)-y) - 2g(x_n-x)f(y)| \leq 2\varepsilon, \end{aligned}$$

so that

$$\left| \frac{f(x_n + (x+y)) + f(x_n - (x+y))}{2g(x_n)} + \frac{f(x_n + (x-y)) + f(x_n - (x-y))}{2g(x_n)} \right.$$
$$\left. -2 \cdot \frac{g(x_n + x) + g(x_n - x)}{2g(x_n)} f(y) \right| \leq \frac{\varepsilon}{|g(x_n)|}.$$

From (6.19a) and g satisfying (C), it follows that

$$|f(x+y) + f(x-y) - 2g(x)f(y)| \leq 0;$$

that is, f and g are solutions (6.16).

Choose y_0 such that $f(y_0) \neq 0$. Then (6.16) gives

$$g(x) = \frac{f(x+y_0) + f(x-y_0)}{2f(y_0)}$$

so that g also satisfies the condition (K). Since g satisfies (K), from [424] and Theorem 3.21, we see that there exists a homomorphism (exponential) $E : G \to \mathbb{C}^*$ satisfying the second part of (6.18).

Finally, applying (6.16), (6.19), and (K), we get

$$|f((x_n + y) + x) + f((x_n + y) - x) - 2g(x_n + y)f(x)$$
$$+ f((x_n - y) + x) + f((x_n - y) - x) - 2g(x_n - y)f(x)| \leq 2\varepsilon$$

and that

$$\left| \frac{f(x_n + (x+y)) + f(x_n - (x+y))}{2g(x_n)} + \frac{f(x_n + (x-y)) + f(x_n - (x-y))}{2g(x_n)} \right.$$
$$\left. -2f(x) \cdot \frac{g(x_n + y) + g(x_n - y)}{2g(x_n)} \right| \leq \frac{\varepsilon}{|g(x_n)|},$$

resulting in (6.17). From (6.16) and (6.17), it is easy to see that $f(x) = bg(x)$ for some constant b. This proves the theorem. □

Now we take up the stability of (6.17). We prove the following theorems.

Theorem 6.15. *Let $\varepsilon \geq 0$ and G be a group. Suppose $f, g : G \to \mathbb{C}$ satisfy the inequality*

(6.17a) $$|f(x+y) + f(x-y) - 2f(x)g(y)| \leq \varepsilon \quad \text{for } x, y \in G,$$

with f even (that is, $f(-x) = f(x)$), and f satisfies (K)*. Then f and g are either bounded or unbounded, and g satisfies* (C) *and f and g are solutions of* (6.17) *and* (6.16)*.*

Proof. We consider only nontrivial f; that is, $f \neq 0$. When f is bounded, choose x_0 such that $f(x_0) \neq 0$ and use (6.17a) to get

$$|g(y)| - \frac{|f(x_0+y)+f(x_0-y)|}{2|f(x_0)|} \le \left|\frac{f(x_0+y)+f(x_0-y)}{2f(x_0)} - g(y)\right| \le \frac{\varepsilon}{2|f(x_0)|},$$

which shows that g is also bounded.

Suppose f is unbounded. Choose $x = 0$ in (6.17a) to have

$$|f(y)+f(-y)-2f(0)g(y)| \le \varepsilon;$$

that is,

$$|f(y)-f(0)g(y)| \le \frac{\varepsilon}{2}$$

(this is the only place where we use that f is even). Since f is unbounded, $f(0) \neq 0$. Hence g is also unbounded.

Let f and thus g be unbounded. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in G such that $f(x_n) \neq 0,\ |f(x_n)| \to \infty,\ g(y_n) \neq 0,\ |g(y_n)| \to \infty$.

Twice applying the inequality (6.17) and using (K) for f twice, first we get

$$\left|\frac{f(x_n+y)+f(x_n-y)}{2f(x_n)} - g(y)\right| \le \frac{\varepsilon}{2|f(x_n)|};$$

that is,

$$\lim_n \frac{f(x_n+y)+f(x_n-y)}{2f(x_n)} = g(y) \quad \text{for } y \in G. \tag{6.20}$$

Then we obtain

$$\begin{aligned}|f((x_n+x)+y)+f((x_n+x)-y)-2f(x_n+x)g(y)+f((x_n-x)+y)\\ +f((x_n-x)-y)-2f(x_n-x)g(y)| \le 2\varepsilon;\end{aligned}$$

that is,

$$\left|\frac{f(x_n+(x+y))+f(x_n-(x+y))}{2f(x_n)} + \frac{f(x_n+(x-y))+f(x_n-(x-y))}{2f(x_n)} - 2g(y)\frac{f(x_n+x)+f(x_n-x)}{2f(x_n)}\right| \le \frac{\varepsilon}{|f(x_n)|},$$

which by (6.20) leads to

$$|g(x+y)+g(x-y)-2g(y)g(x)| \le 0,$$

so that g satisfies (C).

Again applying the inequality (6.17a) twice and using the (K) condition for f twice, first we have

$$\left|\frac{f(x+y_n)+f(x-y_n)}{2g(y_n)} - f(x)\right| \le \frac{\varepsilon}{2|g(y_n)|}, \tag{6.21}$$

and then we get

$$|f(x+(y_n+y))+f(x-(y_n+y))-2f(x)g(y_n+y)+f(x+(y_n-y))\\ +f(x-(y_n-y))-2f(x)g(y_n-y)|\le 2\varepsilon;$$

that is,

$$\left|\frac{f((x+y)+y_n)+f((x+y)-y_n)}{2g(y_n)}+\frac{f((x-y)+y_n)+f((x-y)-y_n)}{2g(y_n)}\right.\\ \left.-2f(x)\frac{g(y_n+y)+g(y_n-y)}{2g(y_n)}\right|\le\frac{\varepsilon}{|g(y_n)|},$$

which by (6.21) and (C) yields

$$|f(x+y)+f(x-y)-2f(x)g(y)|\le 0,$$

so that f and g are solutions of (6.17).

Consider the inequality

$$|f((y_n+x)+y)+f((y_n+x)-y)-2g(y_n+x)f(y)+f((y_n-x)+y)\\ +f((y_n-x)-y)-2g(y_n-x)f(y)|\le 2\varepsilon.$$

As before, using (K), (6.20), the evenness of f and (C), and the division by $2g(y_n)$ yields

$$\frac{f((x+y)+y_n)+f((x+y)-y_n)}{2g(y_n)}+\frac{f((x-y)+y_n)+f((x-y)-y_n)}{2g(y_n)}\\ -2f(y)\frac{g(y_n+x)+g(y_n-x)}{2g(y_n)}\to 0\quad\text{as } n\to\infty,$$

so that f and g are solutions of (6.16). This completes the proof of the theorem. □

Note that the evenness of f is used to prove that g is unbounded when f is and nowhere else.

6.4.1 Stability for Vector-Valued Functions

Badora [82] gave a counterexample to illustrate the failure of the superstability of the cosine functional equation (C) in the case of the vector-valued mapping (see the counterexample above). Here consider the following example. Let f and g be the unbounded solution of (6.16) (or (6.17)), where $f, g : G \to \mathbb{C}$. Define $f_1, g_1 : G \to M_2(\mathbb{C})$ by

$$f_1(x)=\begin{pmatrix} f(x) & 0\\ 0 & c_1\end{pmatrix},\quad g_1(x)=\begin{pmatrix} g(x) & 0\\ 0 & c_2\end{pmatrix},$$

for $x \in G$, where $c_1 \neq 0,\ c_2 \neq 1$. Then

$$\|f_1(x+y)+f_1(x-y)-2f_1(y)g_1(x)\|=\text{constant}\ >0$$

(or $\| f_1(x+y) + f_1(x-y) - 2f_1(x)g_1(y)\| = \text{constant} > 0$) for $x, y \in G$. This f_1 and g_1 are neither bounded nor satisfy (C). Therefore there is a need to consider the vector-valued functions separately. We prove the following two theorems in this section. Let G be a group and $\mathcal{A}$ be a complex normed algebra with identity.

Theorem 6.16. [484]. *Suppose $f, g : G \to \mathcal{A}$ satisfy the inequality*

$$\| f(x+y) + f(x-y) - 2g(x)f(y)\| \le \varepsilon \tag{6.16b}$$

for $x, y \in G$, with f satisfying (K) *and*

$$\| f(x) - f(-x)\| \le \eta, \quad \text{for } x \in G, \tag{6.22}$$

for some $\varepsilon, \eta \ge 0$. Suppose there is a $z_0 \in G$ such that $g(z_0)^{-1}$ exists and $\| f(x)g(z_0)\|$ is bounded for $x \in G$. Then there is an $E : G \to \mathcal{A}$ such that

$$\|E(x+y) - E(x)E(y)\| \le a_1, \quad \text{for } x, y \in G, \tag{6.23}$$

and

$$\left\| f(x) - \frac{1}{2}(E(x) + E(-x))\right\| \le a_2, \quad \text{for } x \in G, \tag{6.23a}$$

for some constants a_1 and a_2.

Proof. Let $m := \sup_{x\in G} \| f(x)g(z_0)\|$. Then, using (6.16b) and (6.22), we get by using (K)

$$\begin{aligned}
\| f(x)g(-z_0)\| &\le \| f(-x)g(z_0)\| + \| f(x)g(-z_0) - f(-x)g(z_0)\| \\
&\le M + \frac{1}{2}\| f(z_0 - x) + f(z_0 + x) - 2g(z_0)f(-x) \\
&\quad - (f(-z_0 + x) + f(-z_0 - x) - 2g(-z_0)f(x)) \\
&\quad - (f(z_0 + x) - f(-z_0 - x) + f(z_0 - x) - f(-z_0 + x))\| \\
&\le m + \varepsilon + \eta.
\end{aligned}$$

Define a function $h : G \to \mathcal{A}$ by the formula

$$h(x) = \frac{1}{2}(f(x) + f(-x)) \quad \text{for } x \in G.$$

Then h is even (that is, $h(-x) = h(x)$) and

$$\|h(x) - f(x)\| \le \frac{\eta}{2}, \quad \text{for } x \in G, \quad \|h(x)g(z_0)\| \le m. \tag{6.24}$$

Define a function $E : G \to \mathcal{A}$ by

$$E(x) = h(x) + ig(z_0) \quad \text{for } x \in G.$$

Utilizing (6.24), we get (first using commutativity in $\mathcal{A}$)

$$\begin{aligned}
&\|E(x+y)-E(x)E(y)\| \\
&\quad = \|h(x+y)+ig(z_0)-h(x)h(y)+i(h(x)+h(y))g(z_0)+g(z_0)^2\| \\
&\quad \le \|h(x+y)\|+\|h(x)h(y)\|+\|(h(x)+h(y))g(z_0)\|+\|g(z_0)\|+\|g(z_0)\|^2 \\
&\quad \le \|h(x+y)-f(x+y)\|+\|f(x+y)\|+\|h(x)h(y)g(z_0)^2\cdot g(z_0)^{-2}\| \\
&\qquad +\|h(x)g(z_0)\|+\|h(y)g(z_0)\|+\|g(z_0)\|+\|g(z_0)\|^2 \\
&\quad \le \frac{\eta}{2}+m\|g(z_0)\|^{-1}+m^2\|g(z_0)\|^{-2}+2m+\|g(z_0)\|+\|g(z_0)\|^2 \\
&\quad = a_1
\end{aligned}$$

(say), which is (6.23). Finally, by (6.24), we have

$$\begin{aligned}
&\left\| f(x)-\frac{1}{2}(E(x)+E(-x))\right\| \\
&\quad = \left\| f(x)-h(x)+h(x)-\frac{1}{2}(h(x)+h(-x))-ig(z_0)\right\| \\
&\quad \le \frac{\eta}{2}+\|g(z_0)\| = a_2
\end{aligned}$$

(say), which is (6.23a). This proves the theorem. □

Lastly, we prove the following result.

Result 6.16a. *Let $f, g : G \to \mathcal{A}$ satisfy the inequality*

$$\|f(x+y)+f(x-y)-2f(x)g(y)\| \le \varepsilon, \quad x, y \in G, \tag{6.17b}$$

with f satisfying (K) *and*

$$\|f(x)-f(-x)\| \le \eta, \quad \text{for } x \in G,$$

for some nonnegative ε and η. Suppose there exists a $z_0 \in G$ such that $g(z_0)^{-1}$ exists and $\|f(x)g(z_0)\|$ is bounded over G. Then there exists a mapping $E : G \to \mathcal{A}$ such that

$$\|E(x+y)-E(x)E(y)\| \le a_1, \quad \text{for } x, y \in G,$$

and

$$\left\| f(x)-\frac{1}{2}(E(x)+E(-x))\right\| \le a_2, \quad \text{for } x \in G,$$

for some constants a_1 and a_2.

The proof runs parallel to that of Theorem 6.16. The following result, a slight variation of Theorem 6.13, also holds. Let $\varepsilon \ge 0$ and $f_n : G \to \mathbb{C}$ be a sequence of functions converging uniformly to f on G. Suppose $f, g, f_n : G \to \mathbb{C}$ are such that

$$|f(x+y)+f(x-y)-2g(x)f_n(y)| \le \varepsilon, \quad x, y \in G, \tag{6.16b}$$

holds with f satisfying (K). Then either f is bounded or g satisfies (C) and f and g satisfy (6.16) and (6.17).

The following extension of Theorem 6.13a is true (Székelyhidi [788]).

Result 6.17. *Let G be a group, F a field, and V a vector space of functions from G into F, invariant by right translation. Let $f, g : G \to F$ be functions for which* (K) *holds for all x, y, z in G and the function $x \mapsto f(xy) + f(xy^{-1}) - 2f(x)g(y)$ belongs to V for all y in G. Then either f belongs to V or g satisfies* (C) *for all x, y in G.*

If here we take V as the space of bounded functions and $f = g$, then we get Theorem 6.13a. We remark that if S is a semigroup, F is a field, and V is a linear space of F-valued functions on S, then we say that the functions $f, g : S \to F$ are linearly independent modulo V if $\lambda f + \mu g \in V$ implies $\lambda = \mu = 0$ for any $\lambda, \mu \in F$. We say that V is *two-sided invariant* if the functions $x \mapsto f(xy)$ and $x \mapsto f(yz)$ belong to V for any f in V.

Result 6.17a. *Let S be a semigroup and V a two-sided invariant linear space of F-valued functions on S, and suppose that $f, g : S \to F$ are linearly independent modulo V. If the function $x \mapsto f(xy) - f(x)g(y) - f(y)g(x)$ belongs to V for all y in S, then*

$$f(xy) = f(x)g(y) + f(y)g(x)$$

holds for all x, y in S.

Result 6.17b. *Let S be a semigroup and V a two-sided invariant linear space of F-valued functions on S, and suppose that $f, g : S \to F$ are linearly independent modulo V. If the functions $x \mapsto f(xy) - f(x)f(y) + g(x)g(y)$ and $x \mapsto f(xy) - f(yx)$ belong to V for all y in S, then*

$$f(xy) = f(x)f(y) - g(x)g(y)$$

holds for all x, y in S.

6.5 Stability of the Equation $f(x + y) + g(x - y) = h(x)k(y)$

$$f(x + y) + g(x - y) = h(x)k(y). \tag{6.18}$$

Result 6.18. (Székelyhidi [791]). *Let $f, g, h, k : G \to \mathbb{C}$ (G a group) be functions for which the function $(x, y) \to f(x + y) + g(x - y) - h(x)k(y)$ is bounded.Then there are an exponential $E : G \to \mathbb{C}$, an additive function $A : G \to \mathbb{C}$, bounded functions $a, b, c : G \to \mathbb{C}$, and constants $\alpha, \beta, \gamma, \delta$ such that we have the following possibilities:*

(i) *f, g, h, k are bounded;*
(ii) *f, g are bounded, $h = 0$, k is arbitrary;*

(iii) f, g *are bounded,* h *is arbitrary,* $k = 0$;
(iv) f *is bounded,* $g = \alpha\beta E + b,\ h = \alpha E,\ k = \beta E^{-1}$;
(v) $f = \alpha\beta E + a,\ g$ *is bounded,* $h = \alpha E,\ k = \beta E$;
(vi) $f = \frac{1}{2}\alpha A + a,\ g = -\frac{1}{2}\alpha A + b,\ h = \alpha,\ k = A + c$;
(vii) $f = \frac{1}{2}\beta A + a,\ g = \frac{1}{2}\beta A + b,\ h = A + c,\ k = \beta$;
(viii) $f = \frac{1}{4}\alpha\beta A^2 + \frac{1}{2}(\alpha\delta + \beta\gamma)A + a,\ g = -\frac{1}{4}\alpha\beta A^2 + \frac{1}{2}(\alpha\delta - \beta\gamma)A + b,$
$h = \alpha A + \gamma,\ k = \beta A + \delta$;
(ix) $f = \frac{1}{2}(\alpha\gamma + \beta\delta)E_e + \frac{1}{2}(\alpha\delta + \beta\gamma)E_o + a,\ h = \alpha E_e + \beta E_o,$
$g = \frac{1}{2}(\alpha\gamma - \beta\delta)E_e - \frac{1}{2}(\alpha\delta - \beta\gamma)E_o + b,\ k = \gamma E_e + \delta E_o$;

where E_e *and* E_o *are even and odd parts of* E, *respectively.*

6.6 Stability of the Sine Functional Equation

Result 6.19. (Cholewa [157]). *Let* $\delta > 0$, *let* $(G, +)$ *be an Abelian group that is* 2*-divisible, and let a function* $f : G \to \mathbb{C}$ *satisfy the inequality*

$$|f(x+y)f(x-y) - f(x)^2 + f(y)^2| \le \delta \tag{6.19}$$

unbounded. Then f *satisfies the equation*

$$f(x+y) + f(x-y) = 2f(x)g(y) \tag{6.17}$$

for all $x, y \in G$, *where*

$$g(x) = \frac{f(x+a) - f(x-a)}{2f(a)}$$

for all $x \in G$ *and for some* $a \in G$ *with* $|f(a)| \ge 4$, *and further* f *is a solution of the sine functional equation* (S).

6.7 Stability—Alternative Cauchy Equation

We are now going to deal with the stability of the alternative Cauchy equation

$$|f(x+y)| = |f(x) + f(y)| \quad \text{for } x, y \in G, \tag{6.25}$$

$$\|f(x+y)\| = \|f(x) + f(y)\| \quad \text{for } x, y \in G, \tag{6.26}$$

where G is a commutative semigroup.

We say that a partially ordered vector space V is a *Riesz space* if $\sup\{x, y\}$ exists for all $x, y \in V$. We define the absolute value of x by the formula

$$|x| := \sup(x, -x) \ge 0.$$

V is called *Archimedean* if $x \leq 0$ holds whenever the set $\{nx \mid n \in Z_+^*\}$ is bounded from above. The lexicographically ordered plane is an example of a non-Archimedean Riesz space.

Definition. The sequence $\{f_n\}$ in V is said to *converge e-uniformly* to the element $f \in V$ whenever, for every $\varepsilon > 0$, there exists a natural number n_0 such that $|f - f_n| \leq \varepsilon e$ holds for all $n \geq n_0$. The sequence $\{f_n\}$ in V is called an *e-uniform Cauchy sequence* whenever, for every $\varepsilon > 0$, there exists a natural number n_1 such that $|f_m - f_n| \leq \varepsilon e$ holds for all $m, n \geq n_1$.

Let us point out that if V is Archimedean, the e-uniform limit of a sequence, if it exists, is unique.

We say that the set $W \subset G$ is *weakly bounded* if and only if for arbitrary $x \in G\backslash\{0\}$ there exists an $n \in Z_+^*$ such that

$$kx \notin W \quad \text{for } k \geq n.$$

Clearly every bounded set in a normed space is weakly bounded.

Result 6.20. [106]. *Let G be a commutative semigroup and let $W \subset G$ be a weakly bounded set. If a function $f : G \to \mathbb{R}$ satisfies the inequality*

$$\big||f(x+y)| - |f(x) + f(y)|\big| \leq \delta, \tag{6.26a}$$

for some $\delta > 0$ and every $x, y \in G\backslash W$, then there exists a unique additive function $A : G \to \mathbb{R}$ such that

$$|f(x) - A(x)| \leq 3\delta \quad \text{for } x \in G.$$

Result 6.20a. (Batko and Tabor [106]). *Let G be a commutative semigroup and let $W \subset G$ be a weakly bounded set. Let K be an Archimedean, e-uniformly complete Riesz space. If a mapping $F : G \to K$ satisfies the inequality*

$$\big||F(x+y)| - |F(x) + F(y)|\big| \leq e \tag{6.25a}$$

for some $e \in K_+$ and every $x, y \in G\backslash W$, then there exists a unique additive mapping $A : G \to K$ such that

$$|F(x) - A(x)| \leq 3e \quad \text{for } x \in G.$$

Result 6.21. [106]. *Let S be a real interval with $0 \in S$, $\varepsilon \in \left(0, \frac{1}{4}\right)$, and let $f : S \to \mathbb{R}$ be a function satisfying*

$$f(x+y) - f(x) - f(y) \in Z + [-\varepsilon, \varepsilon] \quad \text{for } x, y \in S,$$

with $x + y \in S$. If f is Lebesgue or Baire measurable, then there exists a $c \in \mathbb{R}$ such that

$$f(x) - cx \in Z + [-\varepsilon, \varepsilon] \quad \text{for } x \in S.$$

Result 6.22. (Bean and Baker [107]). *Let $f : \mathbb{R} \to \mathbb{R}$ and f be Lebesgue measurable. The Cauchy difference of f, $f(x+y) - f(x) - f(y)$, takes integer values only if there exists an additive $A : \mathbb{R} \to \mathbb{R}$ such that $f - A$ is Lebesgue measurable and takes integer values only.*

6.8 Stability—Wave Equation

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is such that

$$f(x+h, y) + f(x-h, y) - f(x, y+h) - f(x, y-h) = 0. \tag{6.27}$$

If $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous solution of (6.27), then

$$f(x, y) = \alpha(x, y) + \beta(x - y) \quad \text{for all } x, y \in \mathbb{R},$$

where $\alpha, \beta \in \mathbb{R} \to \mathbb{R}$ are continuous (Brzdek and Tabor [138]).

A simpler proof of this fact was given by Haruki [353]. In [132] it is shown that if $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are arbitrary functions and $B : \mathbb{R}^2 \to \mathbb{R}$ is biadditive and skew-symmetric, and if $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x, y) = \alpha(x + y) + \beta(x - y) + B(x, y) \quad \text{for all } x, y \in \mathbb{R},$$

then (6.27) holds. McKiernan [629] showed that this is indeed the general solution of (6.27) by transforming it into an equivalent equation as follows. For $f : \mathbb{R}^2 \to \mathbb{R}$, define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x, y) = f(x + y, x - y)$ for all $x, y \in \mathbb{R}$. Then f satisfies (6.27) if and only if g satisfies

$$g(x + h, y + h) - g(x + h, y) - g(x, y + h) + g(x, y) = 0 \quad \text{for all } x, y, h \in \mathbb{R}.$$

Result 6.23. [107]. *Suppose G is an Abelian group, X is a Banach space, $\delta > 0$, and $f : G \times G \to X$ such that*

$$\| f(x + h, y + h) - f(x + h, y) - f(x, y + h) + f(x, y) \| \leq \delta \tag{6.27a}$$

for all $x, y, h \in G$. Then there exist functions $\alpha, \beta : G \to X$ and $B : G \times G \to X$ such that B is biadditive and skew-symmetric and

$$\| f(x, y) - [\alpha(x) + \beta(y) + B(x, y)] \| \leq 20\delta \quad \textit{for all } x, y \in G.$$

Result 6.23a. [107]. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is Lebesgue measurable on $\mathbb{R}^2$, $\delta \geq 0$, and*

$$| f(x + h, y + h) - f(x + h, y) - f(x, y + h) + f(x, y) | \geq \delta \quad \textit{for all } x, y, h \in \mathbb{R}.$$

Then there exist Lebesgue measurable functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ such that

$$| f(x, y) - \{\varphi(x) + \psi(y)\} | \leq 60\delta$$

for all $x, y \in \mathbb{R}$.

6.9 Stability—Polynomial Equation

Now we consider stability results for the polynomial equation and for Frèchet's equation. We will use the standard notation for the difference operators. If S is a semigroup, V is a vector space, and $f : S \to V$ is a function, then the *difference* $\Delta_y f$ of f by $y \in V$ is defined by the formula

$$\Delta_y f(x) = f(xy) - f(x)$$

for all x in S. As S is not necessarily commutative, we should rather call $\Delta_y f$ the *right difference* of f by y. The composite operator $\Delta_{y_1}\Delta_{y_2}\cdots\Delta_{y_n}$ is denoted by $\Delta_{y_1,y_2,\ldots,y_n}$. If $y_1 = y_2 = \cdots = y_n$, then we use the notation Δ_y^n for $\Delta_{y_1,y_2,\ldots,y_n}$. A function $p : S \to V$ is called a *polynomial* of degree $n-1$ if

$$\Delta_y^n p(x) = 0 \tag{6.27b}$$

holds for all x, y in S. For instance, any solution $p : S \to V$ of the *Frèchet-functional equation*

$$\Delta_{y_1,y_2,\ldots,y_n} p(x) = 0 \tag{6.27c}$$

is obviously a polynomial of degree $n-1$ (see Chapter 13).

If S is a semigroup and V is a topological vector space, then, as above, we can say that equation (6.27b) is *stable* for the pair (S, V) if, for any function $f : S \to V$ for which the function $(x, y) \mapsto \Delta_y^n f(x)$ is bounded, there exists a polynomial $p : S \to V$ of degree at most $n-1$ for which $f - p$ is bounded. Similarly, we say that Frèchet's equation is stable for the pair (S, V) if, for any function $f : S \to V$ for which the function $(x, y_1, y_2, \ldots, y_n) \mapsto \Delta_{y_1,y_2,\ldots,y_n} f(x)$ is bounded, there exists a solution $p : S \to V$ of Frèchet's equation for which $f - p$ is bounded. The main result of D.H. Hyers's paper [384] is the following.

Result. *If X is a rational vector space and Y is a Banach space, then Frèchet's functional equation is stable for the pair (X, Y).*

Actually Hyers requires the boundedness of the mapping

$$(x, y_1, y_2, \ldots, y_n) \mapsto \Delta_{y_1,y_2,\ldots,y_n} f(x)$$

only on a convex cone in X.

A generalization of Hyers's theorem is the following.

Result. *If S is a commutative semigroup and Y is a Banach space, then Frèchet's equation is stable for the pair (S, V).*

The method of proof of this theorem is similar to that of Hyers applied in [384]. Székelyhidi [788] gives a short proof of the theorem above in the case $Y = \mathbb{C}$ using invariant means. By the same technique, he proves the following theorem in [776].

Result. *If S is an amenable semigroup with identity, then Frèchet's equation is stable for the pair $(S, \mathbb{C})$.*

Together with equation (6.27b), one can consider the *monomial equation*

$$\Delta_y^n f(x) = n!\, f(y) \tag{6.27d}$$

for functions $f : S \to Y$, where S is a semigroup and Y is a linear space. Solutions of (6.27d) are called *monomials* of degree n. Obviously, any monomial of degree n is a polynomial of degree at most n.

Some of the stability results for different functional equations we have considered so far are special cases of the following general result for linear functional equations [776].

Result. *Let G be an Abelian group and let $\Phi_i, \Psi_i : G \to G$ be homomorphisms for which the range of Φ_i is contained in the range of Ψ_i $(i = 1, 2, \ldots, n+1)$. Let f, $f_i : G \to \mathbb{C}$ be functions for which the function*

$$(x, y) \mapsto f(x) + \sum_{i=1}^{n+1} f_i(\Phi_i(x) + \Psi_i(y))$$

is bounded. Then there exists a polynomial $p : G \to \mathbb{C}$ for which $f - p$ is bounded.

Among others, Cauchy's functional equation, Jensen's functional equation, Pexider's functional equation, the quadratic equation, the polynomial equation, and the monomial equation belong to the scope of this theorem.

6.10 Stability—Quadratic Equation

The first result is due to Skof [750]. Thereafter many authors investigated the stability problem of the quadratic equation (Q) in various settings.

Theorem 6.24. *Let G be an Abelian group and let X be a Banach space and $f : G \to X$. If the quadratic difference*

$$(x, y) \to f(x+y) - f(x-y) - 2f(x) - 2f(y), \quad \text{for } x, y \in G,$$

is bounded, there exists a quadratic function $q : G \to X$ for which $f - q$ is bounded; that is, for a fixed $\delta > 0$, if

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \delta, \quad \text{for } x, y \in G, \tag{6.28}$$

there exists a unique quadratic mapping $q : G \to X$ such that

$$\|f(x) - q(x)\| \le \frac{\delta}{2} \quad \text{for all } x \in G. \tag{6.28a}$$

Moreover, the function q is given by

$$q(x) = \lim_{n\to\infty} \left\| \frac{f(2^n x)}{4^n} \right\| \quad \text{for all } x, \tag{6.29}$$

and if G is a real normed space, f is measurable, or $f(tx)$ is continuous in t for each fixed $x \in G$, then $q(tx) = t^2 q(x)$ for all $x \in G,\ t \in \mathbb{R}$.

Proof. The proof runs parallel to that of Theorem 6.1 (see [750, 186, 158]).

Set $x = 0 = y$ in (6.28) to have

$$\|f(0)\| \le \frac{\delta}{2}.$$

Letting $x = y$ in (6.28), we obtain

$$\|f(2x) - 4f(x) - f(0)\| \le \delta;$$

that is,

$$\|f(2x) - 4f(x)\| - \|f(0)\| \le \delta$$

or

$$\left\| \frac{1}{4} f(2x) - f(x) \right\| \le \frac{3}{8}\delta \le \frac{\delta}{2} \quad \text{for } x \in G. \tag{6.30}$$

Replace x by $2x$ in (6.30) to get

$$\left\| \frac{1}{4} f(2^2 x) - f(2x) \right\| \le \frac{3}{8}\delta;$$

that is,

$$\left\| \frac{1}{4} 2 f(2^2 x) - f(x) + f(x) - \frac{1}{4} f(2x) \right\| \le \frac{3}{32}\delta$$

or

$$\begin{aligned} \left\| \frac{1}{4^2} f(2^2 x) - f(x) \right\| &\le \frac{3}{32}\delta + \frac{3}{8}\delta \quad \text{(by (6.30))} \\ &= \frac{3}{8}\delta\left(1 + \frac{1}{4}\right) < \frac{\delta}{2}. \end{aligned}$$

By induction,

$$\begin{aligned} \left\| \frac{1}{4^n} f(2^n x) - f(x) \right\| &\le \frac{3}{8}\delta\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^n}\right) \\ &= \frac{1}{2}\delta\left(1 - \frac{1}{4^n}\right) < \frac{\delta}{2}. \end{aligned} \tag{6.31}$$

Now we prove that

$$\left\{\frac{f(2^n x)}{4^n}\right\}$$

is a Cauchy sequence for each $x \in G$. Choose $m > n$. Then

$$\begin{aligned}\left\|\frac{1}{4^n} f(2^n x) - \frac{1}{4^m} f(2^m x)\right\| &= \frac{1}{4^n}\left\|\frac{1}{4^{m-n}} f(2^{m-n} \cdot 2^n x) - f(2^n x)\right\| \\ &\le \frac{1}{4^n} \cdot \frac{\delta}{2}\left(1 - \frac{1}{4^{m-n}}\right) \quad \text{(by (6.31))} \\ &= \frac{\delta}{2}\left(\frac{1}{4^n} - \frac{1}{4^m}\right).\end{aligned}$$

Thus the sequence is Cauchy and, since X is a Banach space, converges to a limit function, say $q : G \to X$, which proves (6.29).

The next step is to show that $q(x)$ is quadratic. Now

$$\begin{aligned}&\|q(x+y) + q(x-y) - 2q(x) - 2q(y)\| \\ &\qquad = \lim_{n\to\infty} \frac{1}{4^n} \| f(2^n x + 2^n y) \\ &\qquad\quad + f(2^n x - 2^n y) - 2 f(2^n x) - 2 f(2^n y)\| \\ &\qquad \le \lim_{n\to\infty} \frac{\delta}{4^n} \to 0 \quad \text{as } n \to \infty \quad \text{(by (6.29))}.\end{aligned}$$

Hence q is quadratic.

The next goal is to show that (6.28a) holds.

$$\begin{aligned}\|q(x) - f(x)\| &= \left\|\lim_{n\to\infty} \frac{f(2^n x)}{4^n} - f(x)\right\| \\ &= \lim_{n\to\infty} \left\|\frac{f(2^n x)}{4^n} - f(x)\right\| \\ &\le \lim_{n\to\infty} \frac{\delta}{2} \quad \text{(by (6.31))} \\ &= \frac{\delta}{2}.\end{aligned}$$

Hence (6.28a) holds.

Finally, we prove the uniqueness of q. Suppose $q : G \to X$ is not unique. Then there exists another quadratic function $t : G \to X$ such that

$$\|t(x) - f(x)\| \le \frac{\delta}{2} \quad \text{for all } x \in G.$$

Note that

$$\begin{aligned}\|t(x) - q(x)\| &\le \|t(x) - f(x)\| + \|f(x) - q(x)\| \\ &\le \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta.\end{aligned}$$

Therefore

$$\|t(x) - q(x)\| \leq \delta \quad \text{for all } x \in G.$$

Since a quadratic function is rationally homogeneous of degree 2, we have

$$\begin{aligned}\|t(x) - q(x)\| &= \left\| \frac{n^2 t(x)}{n^2} - \frac{n^2 q(x)}{n^2} \right\| \\ &= \left\| \frac{t(nx)}{n^2} - \frac{q(nx)}{n^2} \right\| \\ &= \frac{1}{n^2}\|t(nx) - q(nx)\| \\ &\leq \frac{\delta}{n^2} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus, we get

$$t(x) = q(x) \quad \text{for all } x \in G.$$

Therefore q is unique. □

Result 6.25. (Czerwik [186]). *Let X be a normed space and let Y be a Banach space. If a function $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for $p > 2$, some $\theta \geq 0$, and $x, y \in X$, then there exists a unique quadratic function q such that

$$\|f(x) - q(x)\| \leq \frac{2\theta}{2^p - 4}\|x\|^p, \quad x \in X.$$

If $p = 2$, then the result is no longer valid. In [186], the following result was proved.

Result 6.25a. [186]. *Suppose the function $f : \mathbb{R} \to \mathbb{R}$ is defined by*

$$f(x) = \sum_{i=Q}^{\infty} 4^{-n}\phi(2^n x),$$

where the function $\phi : \mathbb{R} \to \mathbb{R}$ is given by

$$\phi(x) = \begin{cases} \theta & \text{if } |x| \geq 1, \\ \theta x^2 & \text{if } |x| < 1, \end{cases}$$

with a constant $\theta > 1$. Then the function f satisfies the inequality

$$|f(x+y) + f(x-y) - 2f(x) - 2f(y)| \leq 32\theta(x^2 + y^2)$$

for all $x, y \in \mathbb{R}$. Moreover, there exists no quadratic function $q : \mathbb{R} \to \mathbb{R}$ such that the image set of $x^{-2}|f(x) - q(x)|$ (for $x \neq 0$) is bounded.

It is true that a mapping $q : X \to Y$ (X a real normed space, Y a real Banach space) is quadratic if and only if

$$\|q(x+y)+q(x-y)-2q(x)-2q(y)\| \to 0 \quad \text{as } \|x\|+\|y\| \to 0.$$

7

Characterization of Polynomials

In this chapter, polynomials are characterized, mostly of second and *n*th order. Divided difference, Rudin's problem and generalization, Rudin's problem on groups, Frechet's result, and polynomials in several variables are treated.

7.1 Polynomials

Polynomials are essential tools for studying continuous functions. Several (results) characterizations of polynomials are known. In this chapter, starting with Lagrange's mean value theorem, we show that polynomials can be (defined) characterized by using many tools and through functional equations. The study of polynomials on semigroups is based on the notion of multiadditive functions. We show that functional equations are very useful tools to study polynomials (see Aczél [31], Sh. Haruki [359], Sahoo and Riedel [722], Aczél and Kuczma [54], Kannappan and Crstici [483], Kannappan [462], and Cross and Kannappan [180]). With the help of many functional equations, we give several characterizations in general of polynomials of degree n and in particular quadratic polynomials or polynomials of degree 2.

Lagrange's mean value theorem is one of the important results in differential calculus. It is known that (refer to Aczél [31] and Sh. Haruki [359]) if $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f satisfying the functional equation

$$\frac{f(x+h) - f(x)}{h} = f'(x + \theta h), \quad 0 < \theta < 1, \tag{7.1}$$

holding for $x \in \mathbb{R}$ and $h \in \mathbb{R}^*$, has the solution

$$f(x) = ax^2 + bx + c \quad \text{for } x \in \mathbb{R}, \tag{7.2}$$

a polynomial of degree at most 2, if and only if $\theta = \frac{1}{2}$.

(It is easy to see that if f has the form (7.2), then f is a solution of (7.1), provided $\theta = \frac{1}{2}$. Conversely, suppose $\theta = \frac{1}{2}$ in (7.1). Since f has a derivative, differentiating

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the left side of (7.1) with respect to x, we see that from the right side of (7.1), f has a second derivative and so on. Thus f has derivatives of all orders. With $\theta = \frac{1}{2}$, (7.1) becomes

$$f(x+h) = f(x) + hf'\left(x + \frac{1}{2}h\right).$$

Differentiate with respect to h and replace x by $x - \frac{h}{2}$ to obtain

$$f'\left(x + \frac{h}{2}\right) = \frac{h}{2} f''(x) + f'(x).$$

Then set $x = 0$ to have $f'(h) = 2ah + b$. Thus f has the form (7.2).

Soon we will see that differentiability can be eliminated to obtain the same conclusion (7.2).

It is easy to see that (7.1) with $\theta = \frac{1}{2}$ can be rewritten as

$$\frac{f(x) - f(y)}{x - y} = f'\left(\frac{x+y}{2}\right) \quad \text{for } x, y \in \mathbb{R},\ x \neq y. \tag{7.3}$$

Lagrange's mean value theorem for differential calculus says that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $[x, y]$ is any interval, then there exists a mean value $\eta \in]x, y[$ (that is, η is a function of x and y) such that

$$\frac{f(x) - f(y)}{x - y} = f'(\eta(x, y)) \tag{7.4}$$

for $x, y \in \mathbb{R}$ with $x \neq y$. The first statement of this result appeared in a paper by the physicist Ampére (1775–1836) [74]. Geometrically, it means that the chord joining $(x,\ f(x))$ and $(y,\ f(y))$ is parallel to the tangent line at $(\eta,\ f(\eta))$.

One may ask for what f does the mean value $\eta(x, y)$ in (7.4) depend on x and y in a given manner. From this point of view, (7.4) can be treated as a functional equation with unknown function f and given $\eta(x, y)$. The function $\eta(x, y)$ can be any linear or nonlinear combination of x and y—for example, $\eta(x, y) = \frac{x+y}{2}$ or $\sqrt{xy}$ or $sx + ty$, etc. With $\eta(x, y) = \frac{x+y}{2}$, (7.4) becomes (7.3). In connection with (7.3), Shigeru HarukiauthorciteH12 and Aczél [31] proved theorems (7.1) and (7.2). We present a modified proof from Kannappan and Laird [487]. Note that these results are obtained without assuming any regularity conditions on the functions involved. The most general form of the functional equation (7.4) is the functional equation

$$\frac{f(x) - g(y)}{x - y} = \phi\left(\frac{x+y}{2}\right) \quad \text{for } x, y \in \mathbb{R},\ x \neq y. \tag{7.5}$$

Theorem 7.1. [487, 359]. *Functions $f, g, \phi : \mathbb{R} \to \mathbb{R}$ satisfy equation* (7.5) *for all $x, y \in \mathbb{R}$ with $x \neq y$ if, and only if, there exist constants a, b, c such that*

$$f(x) = ax^2 + bx + c = g(x) \quad \textit{and} \quad \phi(x) = 2ax + b, \quad x \in \mathbb{R}. \tag{7.6}$$

Proof. To prove Theorem 7.1, we obtain, on interchanging x and y in (7.5), $f(x) - g(x) = g(y) - f(y)$, so that $f = g$. Putting $y = 0$ in (7.5) yields

$$f(x) = x\phi\left(\frac{x}{2}\right) + f(0) \quad \text{for all } x \text{ (valid for } x = 0 \text{ also).} \tag{7.7}$$

Substitution of (7.7) into (7.5) gives

$$x\phi(x) - y\phi(y) = (x - y)\phi(x + y) \quad \text{for all } x, y, \text{ including } x = y. \tag{7.8}$$

First $y = -x$ in (7.8) shows that

$$\phi(x) + \phi(-x) = 2\phi(0) \quad \text{for all } x. \tag{7.9}$$

Now use (7.8) and (7.9) to get

$$x\phi(x) = -y\phi(-y) + (x + y)\phi(x - y) = y\phi(y) + (x - y)\phi(x + y),$$

so that

$$(x + y)[\phi(x - y) - \phi(0)] = (x - y)[\phi(x + y) - \phi(0)] \quad \text{for all } x, y;$$

that is,

$$u(\phi(v) - \phi(0)) = v(\phi(u) - \phi(0)] \quad \text{for all } u, v.$$

So, we see that ϕ is linear (say, $\phi(x) = ax + b$), and by (7.7), $f(x) = \frac{1}{2}ax^2 + bx + c$ for all x, so f is quadratic. This proves the theorem. □

Theorem 7.2. [359, 487, 31]. *In* $\mathbb{R}$, *the functional equations*

$$\frac{f(x) - f(y)}{x - y} = \frac{\phi(x) + \phi(y)}{2}, \quad \textit{for } x, y \in \mathbb{R},\ x \neq y, \tag{7.10}$$

and

$$\frac{f(x) - f(y)}{x - y} = \phi\left(\frac{x + y}{2}\right), \quad \textit{for } x, y \in \mathbb{R},\ x \neq y, \tag{7.11}$$

are equivalent to each other, where $f, \phi : \mathbb{R} \to \mathbb{R}$.

Proof. It is enough to prove that ϕ satisfies (J) since (7.10) with (J) gives (7.11) and (7.11) with (J) gives (7.10).

Suppose that (7.10) holds. With $y = 0$, $f(0) = c$, $\phi(0) = b$, (7.10) gives

$$f(x) = \frac{x}{2}(\phi(x) + b) + c \quad \text{for all } x \in \mathbb{R}. \tag{7.12}$$

Consequently, equation (7.10) becomes

$$x\phi(y) - y\phi(x) = b(x - y) \quad \text{for all } x, y \in \mathbb{R}. \tag{7.13}$$

Now, applying (7.13) twice, we have

$$\begin{aligned} 2x\phi(x+y) &= (x+y)\phi(2x) + b(x-y) \\ &= x\phi(2x) + \frac{1}{2}[2x\phi(2y) + b(2y-2x)] + b(x-y). \end{aligned} \tag{7.14}$$

This is $x[(2\phi(x+y) - \phi(2x) - \phi(2y)] = 0$. Thus ϕ satisfies (J) for $x \neq 0$. Now, to show that $\phi\left(\frac{y}{2}\right) = \frac{\phi(y)+b}{2}$ (i.e., it results from (J) when $x = 0$) for all y, $y = 0$ in (7.14) yields $2x\phi(x) = x\phi(2x) + bx$; that is, $2\phi(x) = \phi(2x) + b$ for all $x \in \mathbb{R}$. Thus (J) holds for all x, y, so that (7.11) is true.

Remark 7.3. It follows from (7.13) that ϕ is linear (take y a nonzero constant and from (7.12) that f is quadratic and that $\phi = f'$). This result is obtained without assuming any regularity conditions on f and ϕ.

Finally, let us assume that (7.11) holds. Now (7.8) also holds. Making use of (7.8) twice, we get

$$\begin{aligned} (x-y)\phi(x-y) &= y\phi(y) + (x-2y)\phi(x) \\ &= [x\phi(x) + (y-x)\phi(x+y)] + (x-2y)\phi(x). \end{aligned}$$

Hence $\phi(x-y) + \phi(x+y) = 2\phi(x)$ for $x \neq y$, which indeed implies that (J) holds for all x, y in $\mathbb{R}$. Hence (7.10) holds. This proves the theorem. □

Remark 7.4. Moreover, since ϕ satisfies (J), $\phi(x) = A(x) + b$, where A satisfies (A), and (7.8) becomes $xA(x) - yA(y) = (x-y)A(x+y) = (x-y)(A(x) + A(y))$, producing $xA(y) = yA(x)$. So, $A(x) = ax$ for some constant a. Then ϕ is linear and f is quadratic.

Now we present more characterizations of quadratic polynomials.

7.2 More Characterizations of Polynomials of Degree Two

Theorem 7.5. [359, 487]. *Functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation*

$$\frac{f(x) - g(y)}{x - y} = \frac{f'(x) + g'(y)}{2}, \quad \text{for } x, y \in \mathbb{R},\ x \neq y, \tag{7.15}$$

if, and only if, $f(x) = ax^2 + bx + c = g(x)$ for some constants a, b, and c.

Proof. Suppose f and g satisfy (7.15). First, we present a short proof using derivatives.

It is easy to see that f and g have derivatives of all orders. Rewrite (7.15) as $f(x) - g(y) = \frac{(x-y)}{2}(f'(x) + g'(y))$ and differentiate with respect to x to have

$$f'(x) = \frac{1}{2}(f'(x) + g'(y)) + \frac{x-y}{2} f''(x), \tag{7.16}$$

which, by differentiating with respect to y, yields

$$f''(x) = g''(y) = \text{constant} = 2a \text{ (say)}.$$

Then $f(x) = ax^2 + b_1x + c_1$ and $g(y) = ay^2 + b_2y + c_2$. Substitution of these f and g in (7.16) gives $b_2 = b_1$ and in (7.15) gives $c_2 = c_1$.

Now we present a proof without resorting to derivatives.

Second Proof. Letting $y = 0$ and $x = 0$ separately in (7.15), we get

$$\begin{aligned} f(x) &= \frac{x}{2}(f'(x) + g'(0)) + g(0) \quad \text{for } x \neq 0, \\ g(y) &= \frac{y}{2}(g'(y) + f'(0)) + f(0) \quad \text{for } y \neq 0. \end{aligned} \tag{7.17}$$

Use (7.17) and (7.15) to get

$$\frac{y}{2}(f'(x) - f'(0)) = \frac{x}{2}(g'(y) - g'(0)) + f(0) - g(0)$$

for $x, y \neq 0,\ x \neq y$.

From the above, first we obtain (fixing x and y separately)

$$f'(x) = a_1x + b_1,\ g'(y) = a_2y + b_2,$$

and then putting these back in the equation above, we have

$$f'(x) = ax + b_1,\ g'(x) = ax + b_2, \tag{7.18}$$

where $b_1 = f'(0),\ b_2 = g'(0)$, with $f(0) = g(0)$. Utilizing (7.18) and (7.17), we get

$$f(x) = g(x) = \frac{a}{2}x^2 + b_1 + b_2x + c(b_1 - b_2).$$

The converse is easy to verify. This completes the proof of this theorem. □

Now we consider a generalization of (7.15) and prove the following theorem.

Theorem 7.6. [462]. *Let $f, g, h : \mathbb{R} \to \mathbb{R}$. They satisfy the functional equation*

$$\frac{f(x) - f(y)}{x - y} = g(x) + h(y), \quad \textit{for } x, y \in \mathbb{R},\ x \neq y, \tag{7.19}$$

if, and only if, $f(x) = ax^2 + (c_1 + c_2)x + c,\ g(x) = ax + c_1,\ h(x) = ax + c_2$ for some constants a, c, c_1, and c_2.

Proof. Suppose f, g, h satisfy (7.19). Set $x = 0$ and $y = 0$ separately in (7.19) to get

$$\begin{aligned} f(y) &= yh(y) + c_2y + c \quad \text{for } y \neq 0, \\ f(x) &= xg(x) + c_1x + c \quad \text{for } x \neq 0, \end{aligned} \tag{7.20}$$

where $g(0) = c_2,\ h(0) = c_1$. These values of f in (7.19) yield

$$x(h(y) - c_1) = y(g(x) - c_2), \quad x \neq y,\ x, y \neq 0.$$

Fix x and y separately to have

$$h(y) = a_1 y + c_1,\ g(x) = a_2 x + c_2 \quad \text{for all } x, y \in \mathbb{R}.$$

Substituting these h and g in the equation above gives $a_1 = a_2 = a$ (say), and then (7.20) yields

$$f(x) = ax^2 + (c_1 + c_2)x + c.$$

This proves the theorem. □

It is noted (Laird and Mills [596, p. 68]) that a nontrivial system

$$\sum_{j=1}^{m} [f(x - taj) - f(x - tbj)] = 0, \quad \text{for all } x, t \in \mathbb{R},$$

has solutions that are polynomials of degree m. It also follows that these polynomials will be of degree 2 when $\sum_{j=1}^{m} (a_j - b_j) = 0,\ \sum_{j=1}^{m} (a_j^2 - b_j^2) = 0$, and $\sum_{j=1}^{m} (a_j^3 - b_j^3) = 0$.

Remark 7.7. (Kannappan [462]). Suppose $f, g, h, k : F \to F$, where F is a field with more than three elements and char $F \neq 2$, satisfy the generalized divided difference equation

$$\frac{f(x) - g(y)}{x - y} = h(x) + k(y), \quad x \neq y,\ x, y \in \mathbb{R}. \tag{7.19a}$$

Then the general solution of (7.19a) is given by

$$\begin{cases} f(x) = ax^2 + (c_1 + c_2)x + c = g(x), \\ h(x) = ax + c_1,\ k(x) = ax + c_2, \end{cases} \tag{7.19b}$$

where a, c, c_1, c_2 are constants.

The proof follows as in Theorem 7.6, or see [462].

Remark 7.8. For fields of characteristic 2, Remark 7.7 is not true. There are 16 sets of solutions that $F = Z_2$. Some of the solutions that are not of the form (7.19b) are: $f(x) = x,\ g(x) = x + 1,\ h(x) = x,\ k(x) = x + 1;\ f(x) = 0,\ g(x) = 1,\ h(x) = x,\ k(x) = x$; and $f(x) = 0,\ g(x) = x + 1,\ h(x) = 1$, and $k(x) = x$.

Remark 7.9. [462]. When F has three elements in Remark 7.7, say $F = Z_3$, the general solution of (7.19a) is not of the form (7.19b). As a matter of fact, the solution is given by $f(x) = a_1 x^2 + (a_2 + d)x + b,\ g(x) = b_1 x^2 + (b_2 + c)x + a,\ h(x) = a_1 x + a_2,\ k(x) = b_1 x + b_2$, for $x \neq 0$, where $a = f(0),\ b = g(0),\ c = h(0),\ d = k(0)$, with $a_1 - b_1 = b - a$ and $d + c = a_2 + b_2$.

Indeed, from (7.19a) we have

$$y = 0 \text{ yielding } f(x) = b + (h(x) + d)x, \quad x \neq 0,$$
$$x = 0 \text{ yielding } g(y) = a + (k(y) + c)y, \quad y \neq 0,$$

and these f and g in (7.19a) yield

$$x(k(y) - d) + a = y(h(x) - c) + b, \quad x, y \neq 0,\ x \neq y,\ x, y \in Z_3.$$

By fixing $x \neq 0$ and separately fixing $y \neq 0$ in the above, we get

$$k(y) = b_1 y + b_2,\ h(x) = a_1 x + a_2$$

and $x(b_1 y + b_2 - d) + a = y(a_1 x + a_2 - c) + b$. Since $\mathbb{Z}_3$ has only three elements, 0, 1, 2, and x, y are different from 0 and $x \neq y$, the possibilities are $x = 1,\ y = 2$ or $x = 2,\ y = 1$, so that we obtain $2b_1 + b_2 - d + a = 2(a_1 + a_2 - c) + b$ and $2(b_1 + b_2 - d) + a = 2a_1 + a_2 - c + b$. This leads to the conditions $a_1 - b_1 = b - a$ and $d + c = a_2 + b_2$.

Thus it is possible to have solutions of (7.19a) with $f \neq g$ when $F = \mathbb{Z}_3$.

We will later need the following result, which is slightly different from Theorem 7.6.

Theorem 7.9a. *Functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation*

$$\frac{f(x) - f(y)}{x - y} = g(x) + g(y), \quad \text{for } x, y \in \mathbb{R},\ x \neq y,\ x, y \neq 0, \tag{7.19c}$$

if, and only if, f is quadratic; that is, $f(x) = ax^2 + bx + c$ and $g(x) = ax + \frac{b}{2}$ (linear), where a, b, c are real constants.

Proof. Rewrite (7.19c) as

$$f(x) - f(y) = (x - y)(g(x) + g(y)) \quad \text{for } x \neq y,\ x, y \neq 0,\ x, y \in \mathbb{R}. \tag{7.21}$$

Let f, g be solutions of (7.19c). First replacing y by $-y$ in (7.21) and subtracting the resultant equation, we have

$$f(y) - f(-y) = 2yg(x) + (x + y)g(-y) - (x - y)g(y), \quad x \neq y,\ x, y \neq 0.$$

Then set $y = -x$ in (7.21) to get

$$f(x) - f(-x) = 2x(g(x) + g(-x)) \quad \text{for } x \in \mathbb{R}.$$

These two equations yield

$$2yg(x) + (x - y)g(-y) - (x + y)g(y) = 0 \tag{7.22}$$

for $x \neq y,\ x, y \neq 0$.

First replacing x by $-x$ in the equation above and adding to it the resultant equation, we obtain

$$g(x) + g(-x) = g(y) + g(-y) = b \quad \text{(say)}.$$

Substituting this value of $g(-y)$ in (7.22) gives

$$y\left(g(x) - \frac{b}{2}\right) = x\left(g(y) - \frac{b}{2}\right);$$

that is, $g(x) = ax + \frac{b}{2}$ (as asserted). This value of g in (7.22) results in

$$f(x) - ax^2 - bx = f(y) - ay^2 - by = c \quad \text{(say)}.$$

This proves the theorem. □

Remark 7.10. If the right side of (7.19c) is replaced by $g(x) + h(y)$ for $g, h : \mathbb{R} \to \mathbb{R}$, exchanging x and y in (7.19c), we can conclude that g and h differ by a constant and then we can use Theorem 7.9a to solve this equation.

7.3 Generalization

After Theorem 7.5, (7.15) can be written as

$$\frac{f(x) - f(y)}{x - y} = \frac{f'(x) + f'(y)}{2}, \quad x, y \in \mathbb{R},\ x \neq y. \tag{7.23}$$

It is not immediately clear what the generalization of (7.23) should be for higher-order derivatives. The geometric interpretation of (7.23) is that the slope of the chord on an interval is equal to the average of the values of the derivatives at the end points, but this, too, offers no clues as to what the generalizations should be. Revelation came from an unlikely source—Peano derivatives (see [180]).

7.3.1 First Generalization Using Derivatives

Result 7.11. (Cross and Kannappan [180]). *If $f : \mathbb{R} \to \mathbb{R}$ has derivatives of order up to n, then the functional equation*

$$\frac{f(x) - \sum_{k=0}^{n-1} \frac{(x-y)^k}{k!} f^{(k)}(y)}{(x-y)^n} = \frac{f^{(n)}(x) + nf^{(n)}(y)}{(n+1)!}$$

holds for all $x, y \in \mathbb{R}$, $x \neq y$, if, and only if, f is a polynomial of degree $(n+1)$.

7.3.2 Second Generalization Without Using Any Regularity Condition

Result 7.12. [180]. *The functions* $f, g_k, h : \mathbb{R} \to \mathbb{R},\ k = 0, 1, \ldots, n$, *satisfy the functional equation*

$$f(x) = \sum_{k=0}^{n} (x-y)^k g_k(y) + (x-y)^n h(x),$$

for $x, y \in \mathbb{R},\ x \neq y,\ n \geq 1$, *if, and only if,* f *is a polynomial of degree at most* $(n+1)$,

$$g_k(y) = \frac{f^{(k)}(y)}{k!} \quad \text{for } k = 0, 1, \ldots, n-1,$$
$$g_n(y) = \frac{k^{(n)}(y)}{n!} - \frac{y}{(n+1)!} f^{(n+1)}(y) - c,$$
$$h(x) = \frac{x}{(n+1)!} f^{(n+1)}(x) + c,$$

where c *is an arbitrary constant.*

7.4 Another Generalization—Divided Difference

For distinct $x, y \in \mathbb{R},\ f : \mathbb{R} \to \mathbb{R}$, define

$$[x, y] := \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = y - x,$$
$$[x, y;\ f] := \begin{vmatrix} 1 & f(x) \\ 1 & f(y) \end{vmatrix} = f(y) - f(x).$$

Then (7.23) and (7.19) can be rewritten as

$$\frac{[x, y;\ f]}{[x, y]} = \frac{f'(x) + f'(y)}{2},$$
$$\frac{[x, y;\ f]}{[x, y]} = g(x) + h(y);$$

that is, the left side can be regarded as the divided difference of f on the distinct points x and y. This gives a clue to generalization by divided difference.

For distinct points $x_1, x_2, \ldots, x_n$ in $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, define

$$[x_1, x_2, \ldots, x_n] := \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{i>j} (x_i - x_j), \quad (i, j = 1, 2, \ldots, n)$$

(Vandermonde's determinant),

$$[x_1, x_2, \ldots, x_n : f] := \begin{vmatrix} 1 & x_1 & x_1^2 \cdots x_1^{n-2} & f(x_1) \\ 1 & x_2 & x_2^2 \cdots x_2^{n-2} & f(x_2) \\ \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 \cdots x_n^{n-2} & f(x_n) \end{vmatrix}.$$

The *divided difference* of the function $f : \mathbb{R} \to \mathbb{R}$ on the distinct points $x_1, x_2, \ldots, x_n$ in $\mathbb{R}$ is defined by (Kannappan and Crstici [483])

$$\frac{[x_1, x_2, \ldots, x_n; f]}{[x_1, x_2, \ldots, x_n]}. \tag{7.24}$$

The divided difference denoted by $f[x_1, x_2, \ldots, x_n]$ and defined recursively by

$$\begin{aligned} f[x_0] &= f(x_0), \\ f[x_0, x_1] &= \frac{f(x_0) - f(x_1)}{x_0 - x_1}, \ldots, \\ f[x_0, x_1, \ldots, x_n] &= \frac{f[x_0, x_1, \ldots, x_{n-1}] - f[x_1, \ldots, x_n]}{x_0 - x_n} \end{aligned}$$

in Bailey [84], Bailey and Fix [85], and Sahoo and Riedel [722] is the same as the divided difference defined in (7.24) [483].

Now characterization of polynomials of degree n can be obtained by using the divided difference. For example, consider the functional equations

$$\frac{[x_1, \ldots, x_n; f]}{[x_1, \ldots, x_n]} = g(x_1) + \cdots + g(x_n), \tag{7.25}$$

$$\frac{[x_1, \ldots, x_n, f]}{[x_1, \ldots, x_n]} = g(x_1 + x_2 + \cdots + x_n), \tag{7.26}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ and $x_1, \ldots, x_n$ are distinct reals.

It is easy to see that

$$\frac{[x_1, \ldots, x_n; f]}{[x_1, \ldots, x_n]} = \sum_{i=1}^{n} \frac{f(x_i)}{\prod\limits_{j \neq 1} (x_j - x_i)}, \quad j = 1, 2, \ldots, n. \tag{7.24}$$

Now we will prove some results and state some others (see [84, 722, 182, 462, 483, 511]).

Theorem 7.13. *Suppose $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation* (7.25)*, where $x_1, x_2, \ldots, x_n$ are distinct points. Then, and only then, f is a polynomial of degree at most n and g is linear; that is, a polynomial of degree 1.*

Proof. If f is a solution of (7.25), so is $f(x) + \sum\limits_{k=0}^{n-2} a_k x^k$. So, without loss of generality, choose distinct points $y_1, y_2, \ldots, y_{n-2}$ different from 0 such that $f(0) = 0 = f(y_1) = \cdots = f(y_{n-2})$. Clearly there are plenty of choices for $y_1, \ldots, y_{n-2}$.

Letting $(x, 0, y, y_1, \ldots, y_{n-3})$ for $(x_1, x_2, \ldots, x_n)$ in (7.25), we obtain

$$-\frac{f(x)}{x(y-x)(y_1-x)\cdots(y_{n-3}-x)} - \frac{f(y)}{(x-y)y(y_1-y)\cdots(y_{n-3}-y)} = g(x) + g(y) + c_1,$$

where $c_1 = g(0) + g(y_1) + \cdots + g(y_{n-3})$.

Define $h(x) = \frac{f(x)}{x(y_1-x)\cdots(y_{n-3}-x)}$, so that the equation above can be written as

$$\frac{h(x) - h(y)}{x - y} = g(x) + g(y) + c_1 \quad \text{for } x \neq y,\ x, y \neq 0, \tag{7.27}$$

which is (7.19a) and (7.19c). Then, from Theorem 7.9a, it follows that g is linear and h is a polynomial of degree 2; that is, f is a polynomial of degree n. This proves the theorem. □

Corollary 7.13a. (Kannappan and Crstici [483]). *If* $f, f', \ldots, f^{(n)} : \mathbb{R} \to \mathbb{R}$, *then*

$$\frac{[x_1, x_n \ldots, x_n;\ f]}{[x_1, x_2, \ldots, x_n]} = \frac{1}{n!}\left[f^{(n-1)}(x_1) + \cdots + f^{(n-1)}(x_n)\right] \tag{7.25a}$$

for all distinct $x_1, x_2, \ldots, x_n$ *in* $\mathbb{R}$ *if, and only if,* f *is a polynomial of degree at most* n.

Take $g = \frac{1}{n!} f^{(n-1)}$ *in* Theorem 7.13. *Note that* Theorem 7.13 *was proved without any regularity condition on* f.

Corollary 7.13b. [483]. *If* $f, g_1, \ldots, g_n : \mathbb{R} \to \mathbb{R}$ *satisfy the functional equation*

$$\frac{[x_1, \ldots, x_n;\ f]}{[x_1, \ldots, x_n]} = \sum_{k=1}^{n} g_k(x_k) \tag{7.25b}$$

for all distinct $x_1, \ldots, x_n$ *in* $\mathbb{R}$, *then* f *is a polynomial of degree at most* n *and the* g_k*'s are linear and differ from each other by constants.*

Proof. By following the steps of the proof of Theorem 7.13, we arrive at

$$\frac{h(x) - h(y)}{x - y} = g_1(x) + g_2(y) + \text{constant}. \tag{7.27b}$$

Now, by using Remark 7.10, we obtain the result sought.

Result 7.14. (Crstici and Neagu [182], Kannappan and Sahoo [511]). *If functions* $f, g : \mathbb{R} \to \mathbb{R}$ *satisfy the functional equation* (7.26) *for a set of* n *distinct points* $x_1, x_2, \ldots, x_n$ *in* $\mathbb{R}$, *then* f *is a polynomial of degree at most* n *and* g *is linear.*

7.5 Generalization of Divided Difference

For distinct points $x_1, \ldots, x_n$ in $\mathbb{R}$ and functions $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$, we define

$$[x_1, x_2, \ldots, x_n;\ f_1, \ldots, f_n] = \begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-2} & f_1(x_1) \\ 1 & x_2 & \cdots & x_2^{n-2} & f_2(x_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & \cdots & x_n^{n-2} & f_n(x_n) \end{vmatrix}$$

and the *generalized divided difference* [462] by

$$\frac{[x_1, x_2, \ldots, x_n;\ f_1,\ f_2, \ldots,\ f_n]}{[x_1, x_2, \ldots, x_n]}, \tag{7.24a}$$

which is a generalization of (7.24).

It is easy to check that the generalized divided difference (7.24a) is given by

$$\frac{[x_1, \ldots, x_n;\ f_1, \ldots,\ f_n]}{[x_1, \ldots, x_n]} = \sum_{k=1}^{n} \frac{f_k(x_k)}{(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}.$$

The following results hold [462], which are generalizations to Theorem 7.13 and Result 7.14.

Result 7.15. (Kannappan [462]). *Let $f_i, g_k : \mathbb{R} \to \mathbb{R}$ $(i, k = 1, 2, \ldots, n)$ satisfy the functional equation*

$$\frac{[x_1, \ldots, x_n;\ f_1, \ldots,\ f_n]}{[x_1, \ldots, x_n]} = \sum_{k=1}^{n} g_k(x_k)$$

for distinct $x_1, x_2, \ldots, x_n$ in $\mathbb{R}$. Then all f_i's are equal, f_1 is a polynomial of degree at most n, and the g_k's are linear functions differing by constants.

Result 7.16. [462]. *Suppose $f_i, g : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, \ldots, n)$ satisfy the functional equation*

$$\frac{[x_1, \ldots, x_n;\ f_1, \ldots,\ f_n]}{[x_1, \ldots, x_n]} = g(x_1 + \cdots + x_n).$$

Then all f_i's are equal, f_1 is a polynomial of degree at most n, and g is linear.

Remark 7.17. Note that no regularity condition was used in the two theorems above.

7.6 Problem of W. Rudin and a Generalization

In the *American Mathematical Monthly*, W. Rudin [716] posed the following problem: Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$\frac{f(x) - f(y)}{x - y} = f'(sx + ty), \quad x \neq y,\ x, y \in \mathbb{R}, \tag{7.28}$$

where s and t are given real numbers. In this connection, we prove the following theorem (see also E3338 [235] and Sahoo and Riedel [722]).

Theorem 7.18. *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies equation* (7.28) *for all $x, y \in \mathbb{R}$, $x \neq y$, and s and t are given reals, if, and only if,*

$$f(x) = \begin{cases} ax^2 + bx + c, & \text{if } s = t = \frac{1}{2}, \\ bx + c, & \text{otherwise}, \end{cases} \tag{7.29}$$

where a, b, and c are real constants.

Proof. First we rewrite (7.28) as

$$f(x) - f(y) = (x - y) f'(sx + ty), \quad x \neq y,\ x, y \in \mathbb{R}. \tag{7.30}$$

It is easy to see that f has derivatives of all orders. (We do not really need this fact.) If f is a solution of (7.30), so is $f(x) - f(0)$. So, without loss of generality, assume $f(0) = 0$. Now we dispose of some special cases of s and t.

With $s = 0 = t$, (7.30) becomes $f(x) - xf'(0) = f(y) - yf'(0)$, implying that f is linear.

Next assume $s = 0,\ t \neq 0$ ($s \neq 0,\ t = 0$ is similar). Then (7.30) is $f(x) - f(y) = (x - y) f'(ty)$. Setting $y = 0$ (or interchanging x and y) shows that f is linear. Now, we assume that $s \neq 0,\ t \neq 0$.

Letting $y = 0$ and $x = 0$ separately in (7.30), we have $f(x) = xf'(sx),\ x \neq 0$ and $f(y) = yf'(ty),\ y \neq 0$, so that (7.30) can be written as

$$xf'(sx) - yf'(ty) = (x - y) f'(sx + ty), \quad x \neq y,\ x, y \neq 0. \tag{7.31}$$

Replace x by $\frac{x}{s}$ and y by $\frac{y}{t}$ in (7.31) to get

$$\frac{x}{s} f'(x) - \frac{y}{t} f'(y) = \left(\frac{x}{s} - \frac{y}{t}\right) f'(x + y), \quad x \neq y,\ x, y \neq 0. \tag{7.32}$$

Interchanging x and y in the equation above and subtracting the resultant, we obtain

$$\left(\frac{1}{s} + \frac{1}{t}\right)[xf'(x) - yf'(y)] = \left(\frac{1}{s} + \frac{1}{t}\right)(x - y) f'(x + y), \quad x \neq y,\ x, y \neq 0. \tag{7.33}$$

Suppose $\frac{1}{s} + \frac{1}{t} = 0$; that is, $t = -s$. Then (7.32) becomes

$$xf'(x) + yf'(y) = (x + y) f'(x + y), \quad x \neq y,\ x, y \neq 0.$$

Defining $A(x) = xf'(x)$ for $x \in \mathbb{R}$, we see that A is additive and $A(x) = cx$ (here we use that f' is continuous), so that $f'(x) = c$; that is, f is linear.

Finally, we consider the case $\frac{1}{s} + \frac{1}{t} \neq 0$. Then (7.33) yields

$$xf'(x) - yf'(y) = (x - y) f'(x + y), \quad x \neq y,\ x, y \neq 0. \tag{7.34}$$

Now we will show that f is quadratic (that is, a polynomial of degree 2). This can be achieved with and without using that f has higher derivatives.

First assume that f has higher derivatives, say up to 3. Differentiating (7.34) first with respect to x and then with respect to y, we get

$$(x-y)f'''(x+y)=0;$$

that is, $f'''(x)=0$ or f is quadratic. This f in (7.29) shows that $s=t=\frac{1}{2}$.

Now we prove this result without assuming higher derivatives of f. First $y=-x$ in (7.34) gives

$$f'(x)+f'(-x)=2f'(0) \quad \text{for all } x.$$

Then change y to $-y$ in (7.34) and subtract the resultant to obtain

$$\begin{aligned}(x+y)f'(x-y)-(x-y)f'(x+y)&=y(f'(y)+f'(-y))=2yf'(0)\\&=[(x+y)-(x-y)]f'(0)\end{aligned}$$

(that is,

$$(x+y)(f'(x-y)-f'(0))=(x-y)(f'(x+y))-f'(0)),$$

showing thereby that $f'(x)=ax+b$ or f is quadratic. This proves the theorem. □

7.7 Generalization of Rudin's Problem

Result 7.19. (Kannappan, Sahoo, and Jacobson [519]). *Let s and t be real parameters. Functions $f,g,h:\mathbb{R}\to\mathbb{R}$ satisfy the functional equation*

$$\frac{f(x)-g(y)}{x-y}=h(sx+ty) \quad \text{for } x,y\in\mathbb{R},\ x\neq y, \tag{7.35}$$

if, and only if,

$$f(x)=\begin{cases}ax+b & \text{if } s=0=t\\ ax+b & \text{if } s=0,\ t\neq 0\\ atx^2+ax+b & \text{if } s=t\neq 0\\ \frac{A(tx)}{t}+b & \text{if } s=-t\neq 0\\ \beta x+b & \text{if } s^2\neq t^2,\end{cases}$$

$$g(x)=\begin{cases}ax+b & \text{if } s=0=t\\ ax+b & \text{if } s=0,\ t\neq 0\\ atx^2+ax+b & \text{if } s=t\neq 0\\ \frac{A(tx)}{t}+c & \text{if } s=-t\neq 0\\ \beta x+b & \text{if } s^2\neq t^2,\end{cases}$$

$$h(x)=\begin{cases}\text{arbitrary with } h(0)=a & \text{if } s=0=t\\ a & \text{if } s=0,\ t\neq 0\\ \alpha x+b & \text{if } s=t\neq 0\\ \frac{A(x)}{x}+\frac{(c-b)t}{x} & \text{if } s=-t\neq 0,\ x\neq 0\\ \beta & \text{if } s^2\neq t^2,\end{cases}$$

where $A:\mathbb{R}\to\mathbb{R}$ is an additive function and a,b,c,α,β are arbitrary real constants.

7.8 Fréchet's Result

Let E be a normed vector space over $\mathbb{R}$. The general *differences* of $g : E \to \mathbb{R}$ are given by

$$\Delta'_{x_0} g(x) = g(x + x_0) - g(x), \dots$$
$$\Delta^{n+1}_{x_0,x_1,\dots,x_n} g(x) = \Delta^{n}_{x_0,x_1,\dots,x_{n-1}}[g(x + x_n) - g(x)]$$

(called the $(n+1)$th-order difference) and the equidistant differences by

$$\Delta'_v g(u) = g(u + v) - g(u), \dots$$
$$\Delta^n_v g(u) = \Delta^{n-1}_v [g(u + v) - g(u)]$$
$$= \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} g(u + jv) \tag{7.36}$$

(nth-order difference).

Fréchet's result. (Fréchet [288, 287], Amir [73]). If a continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfies

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} g(u + jv) = 0 \quad \text{for } u, v \in \mathbb{R} \tag{7.36a}$$

(that is, the nth-order difference of g is zero), then g is a polynomial of degree $< n$. In Theorem 7.20, we obtained the general solution of (7.36a) for $n = 3$ without assuming any regularity condition on g.

First we consider the definition of polynomials due to Fréchet [290] and others (Mazur and Orlicz [627], Van der Lyn [809]).

Let $A : E \to \mathbb{R}$ be additive. We call $A_n : E^n \to \mathbb{R}$ *n-additive* if A_n is a homomorphism in each variable; that is, if A_n is additive componentwise ($n = 1$ additive, $n = 2$ biadditive). The diagonalization of the n-additive function A_n is denoted by A^*_n; that is,

$$A^*_n(x) = A_n([x]_n) \quad \text{for } x \in E$$

(here $[x]_n$ denotes the element of E^n all components of which are equal to x). Evidently, $A^*_n = A$ for $n = 0$ or 1.

Definition. (Fréchet [290], Székelyhidi [796]). A continuous function $f : E \to \mathbb{R}$ will be called a *polynomial* if and only if f has a representation as the sum of diagonalizations of multiadditive functions from E into $\mathbb{R}$; that is, f has a representation

$$f = \sum_{k=0}^{n} A^*_k, \tag{7.37}$$

where n is a nonnegative integer and $A_k : E^k \to \mathbb{R}$ is a k-additive function ($k = 0, 1, \ldots, n$). (We say that f is a *polynomial* of degree at most n.)

Now we turn our attention to (7.36a). First note that the case $n = 2$ reduces to the Jensen equation (J) and that $g(x) = d + A(x)$, where A is additive (cf. (7.37)); that is, f is a polynomial of degree at most 1 (with $n = 1$).

For $n = 3$, (7.36a) yields

$$g(u + 3v) - 3g(u + 2v) + 3g(u + v) - g(u) = 0 \quad \text{for } u, v \in \mathbb{R}. \tag{7.38}$$

Now we prove the following theorem.

Theorem 7.20. (Kannappan [470]). *Suppose $g : E \to \mathbb{R}$ has all its third-order differences zero; that is, g satisfies*

$$g(u + 3v) - 3g(u + 2v) + 3g(u + v) - g(u) = 0 \tag{7.38A}$$

for $u, v \in E$. Then g has the form

$$g(u) = d + A(u) + q(u) = d + A(u) + B(u, u) \quad \textit{for } u \in E, \tag{7.39}$$

where d is a constant, A is additive, q satisfies (Q)*, and B is symmetric and biadditive; that is, f is a polynomial of degree at most* 2 (*see* (7.37)).

Proof. Replace u by $u - v$ in (7.38A) to get

$$g(u + 2v) - 3g(u + v) + 3g(u) - g(u - v) = 0 \quad \text{for } u, v \in E. \tag{7.40}$$

First, put $u = 0$ in (7.40) to have

$$g(2v) = 3g(v) + g(-v) - 3g(0) \quad \text{for } u \in E. \tag{7.41}$$

Second, interchange u and v in (7.40) to obtain

$$g(2u + v) = 3g(u + v) + g(v - u) - 3g(v) \quad \text{for } u, v \in E. \tag{7.42}$$

Third, replace u by $2u$ in (7.40) to get

$$g(2u + 2v) = 3g(2u + v) + g(2u - v) - 3g(2u) \quad \text{for } u, v \in E. \tag{7.43}$$

Using (7.42) and (7.41) in (7.43), we have

$$\begin{aligned} 2g(u + v) + g(u - v) + g(v - u) &= 3g(u) \\ &\quad + g(-u) + 3g(v) + g(-v) - 4g(0) \quad \text{for } u, v \in E. \end{aligned} \tag{7.44}$$

Replacing v by $-v$ in (7.44) and subtracting the resultant equation from (7.44), we get

$$J(v + u) + J(v - u) = 2J(v) \quad \text{for } u, v \in E \tag{J}$$

(which is Jensen), where $J(x) = g(x) - g(-x)$ for $x \in E$.

Since J is odd, $J(0) = 0$ and

$$J(x) = g(x) - g(-x) = A(x) \quad \text{for } x \in E, \tag{7.45}$$

where $A : E \to \mathbb{R}$ is additive.

Substituting of (7.45) into (7.44) yields

$$\begin{aligned} &2g(u+v) + 2g(u-v) - A(u-v) \\ &\quad = 4g(u) - A(u) + 4g(v) - A(v) - 4g(0) \quad \text{for } u, v \in E; \end{aligned}$$

that is,

$$q(u+v) + q(u-v) = 2q(u) + 2q(v) \quad \text{for } u, v \in E,$$

where

$$q(u) = g(u) - \frac{1}{2}A(u) - g(0) \quad \text{for } u \in E,$$

meaning q satisfies (Q). Now (7.39) follows from Theorem 4.1. This completes the proof. □

Remark 7.21. If, in Theorem 7.20, $E = \mathbb{R}$, and g is continuous, then f is a polynomial of degree at most 2 (see Fréchet [288, 287], Amir [73]). In the general case, compare (7.37) for $n = 2$.

The following most general result is true.

Result 7.22. *Suppose $g : E \to \mathbb{R}$ has its $(n+1)$th-order difference be zero; that is, g satisfies* (7.36a) *for $u, v \in E$. Then g is given by* (7.37)*, where A_k are symmetric, k-additive functions (for $k = 0$ to n); that is, g is a "polynomial" of degree at most n. In the special case $E = \mathbb{R}$, g is continuous and satisfies* (7.36a)*. Then g is a polynomial of degree at most n.*

The equation (7.36a) is known as the *Fréchet equation*. For proof, see [796, Theorem 9.1] or use [53, p. 66, 4.2.3, etc.].

7.9 Polynomials in Several Variables

For functions of several real variables, conditions similar to (7.36a) can be given. Let $\mathbb{C}(\mathbb{R}^n)$ denote the set of all complex-valued functions defined on $\mathbb{R}^n$, $D_j = \frac{\partial}{\partial x_j}$. For $f \in \mathbb{C}(\mathbb{R}^n)$, let $\Delta_{i,h} f(x) = f(x + he_i) - f(x)$ for $x \in \mathbb{R}^n, h$ real, where $\{e_1, e_2, \ldots, e_n\}$ denotes the standard basis for $\mathbb{R}^n$. Characterization of polynomials of several variables can be given by way of translation- and dilation-invariant subspaces. For $a \in \mathbb{R}^n$, $t \in \mathbb{R}$, $f \in \mathbb{C}(\mathbb{R}^n)$, let $T_a f$ and $I_t f$ be defined by $T_a f(x) = f(x-a)$ and $I_t f(x) = f(tx)$ for all $x \in \mathbb{R}^n$. Let X_f denote the subspace of $\mathbb{C}(\mathbb{R}^n)$ spanned by all translations $T_a f$ (for $a \in \mathbb{R}^n$) and $T_t f$ (for $t \in \mathbb{R}$). Let R_j denote the subspace of $\mathbb{C}(\mathbb{R}^n)$ spanned by all translations $T_a f$ of f and all orthogonal transformations $o_p f : x \to f(P, x)$, where P is any orthogonal real matrix. The following theorems are true (see Laird and McCann [595]).

Result 7.23. *Let $f \in \mathbb{C}(\mathbb{R}^n)$. The following conditions are equivalent:*

(a) *f is a polynomial.*
(b) *For some positive integer m, $D_j^m f = 0$ for $j = 1$ to n.*
(c) *For some positive integer m, $D_{i,h}^m f = 0$ for all real h and $i = 1$ to n.*

Result 7.23A. *Let $f \in \mathbb{C}(\mathbb{R}^n)$. A necessary and sufficient condition that f be a polynomial is that X_j be finite-dimensional.*

Result 7.23B. *For $f \in \mathbb{C}(\mathbb{R}^n)$, a necessary and sufficient condition that f be a polynomial is that R_j be finite-dimensional.*

7.10 Quadratic Polynomials in Two Variables

By the mean value theorem for functions of two variables $f : \mathbb{R}^2 \to \mathbb{R}^-$ (see Courant [176, p. 80]), it is well known that

$$f(x+h, y+k) - f(x,y) = h f_x(x+\theta h, y+\theta k) + k f_y(x+\theta h), y+\theta k \tag{7.46}$$

holds, where $0 < \theta < 1$, f_x, and f_x and f_y are partial derivatives.

Setting $x + h = u$, $y + k = v$, $\theta = \frac{1}{2}$ in (7.46), one obtains the functional differential equation

$$f(u,v) - f(x,y) = (u-x) f_x\left(\frac{x+u}{2}, \frac{y+v}{2}\right) + (v-y) f_y\left(\frac{x+u}{2}, \frac{y+v}{2}\right). \tag{7.47}$$

If we replace the partial derivative by unknown functions g and h, (7.47) becomes

$$f(u,v) - f(x,y) = (u-x)g(x+u, y+v) + (v-y)h(x+u, y+v) \tag{7.48}$$

for $x, y, u, v \in \mathbb{R}$, $f, g, h : \mathbb{R}^2 \to \mathbb{R}$.

The following results are true [513].

Result 7.24. *If a quadratic polynomial $f(x,y) = ax^2+bx+cy^2+dy+\alpha xy+\beta$, with $a \neq 0$, $c \neq 0$, is a solution of the functional equation* (7.48) *assuming $x, y, h, k \in \mathbb{R}$ with $h^2+k^2 \neq 0$, then $\theta = \frac{1}{2}$. Conversely, if a function f satisfies* (7.48) *with $\theta = \frac{1}{2}$, then the only solution is a quadratic polynomial.*

Result 7.25. *Suppose $f, g, h : \mathbb{R}^2 \to \mathbb{R}$ satisfy* (7.48) *for all $x, y, u, v \in \mathbb{R}$ with $(u-x)^2+(v-y)^2 \neq 0$. Then f is of the form $f(x,y) = ax^2+bx+cy^2+dy+\alpha+B(x,y)$, where $B : \mathbb{R}^2 \to \mathbb{R}$ is biadditive and a, b, c, d, α are arbitrary constants.*

7.11 Functional Equations on Groups

In this section, first we consider two functional equations treated in Sablik [719],

$$f(x) - g(y) = h(x+y)(x-y), \quad x \neq y, \tag{7.49}$$

$$f(x) - g(y) = (k(x) + \ell(y))(x-y), \tag{7.50}$$

for $x, y \in G$, where $f, g : G \to H$, $h, k, \ell : G \to \mathrm{Hom}(G, H)$, and G, H are Abelian groups. These equations on $\mathbb{R}$ were treated in [359, 31, 487, 462], etc.; cf. (7.10), (7.11), (7.15), (7.19a), etc.

Equations (7.49) and (7.50) are solved in Theorems 7.26 and 7.27, where x, y are in Abelian groups. It is necessary to explain what the multiplication on the right sides means. Note that, in the real case, $h(x)$ and $k(x) + \ell(y)$ may be identified with linear mappings "$t \to h(x)t$" and "$t \to (k(x) + \ell(y))t$". That is what is required of $h(x)$ and $(k(x) + \ell(y))$ for $x, y \in G$; that is, we assume $h(x)$ and $k(x) + \ell(y)$ are homomorphisms from G to H for $x, y \in G$.

First we treat (7.49).

Theorem 7.26. [719]. *Let $G(+)$ and $H(+)$ be Abelian, 2-divisible groups. The functions $f, g : G \to H$, $h : G \to \mathrm{Hom}(G, H)$ satisfy the functional equation* (7.49) *for all $x, y \in G$, $x \neq y$, if and only if*

$$\begin{aligned} f(x) &= g(x) = b + A(x) + B(x, x), \\ h(x) &= A + B(x, \cdot), \end{aligned} \tag{7.51}$$

for $x \in G$, where b is a constant in H, $A : G \to H$ is additive, and $B : G \times G \to H$ is a symmetric, biadditive function.

Proof. Interchanging x and y in (7.49), we get

$$f(y) - g(x) = h(x+y)(y-x) = -(f(x) - g(y));$$

that is, $f(x) - g(x) = -(f(y) - g(y)) = c$ (say). Then $2c = 0$, so $c = 0$ since H is 2-divisible (this is the only place we use 2-divisibility of H). Thus $f(x) = g(x)$. Set $y = 0$ in (7.49) to have

$$f(x) = g(0) + h(x)x \quad \text{for } x \in G,\ x \neq 0. \tag{7.52}$$

Putting (7.52) into (7.49), we obtain

$$h(x)x - h(y)y = h(x+y)(x-y) \quad \text{for } x, y \in G,\ x \neq y,\ x, y \neq 0. \tag{7.53}$$

Put $y = -x$ in (7.53) to have (note that $h(-x)$ and $h(0)$ are homomorphisms)

$$(h(x) + h(-x))(x) = 2h(0)(x) \quad \text{for } x \in G,\ x \neq 0. \tag{7.54}$$

Replace y by $-y$ in (7.53) and subtract the resultant from (7.53) to get

$$\begin{aligned} h(x-y)(x+y) - h(x+y)(x-y) &= (h(y) + h(-y))(y) \\ &= 2h(0)(y) \quad \text{by (7.54)} \\ &= h(0)[x+y-(x-y)] \end{aligned}$$

or

$$[h(x-y) - h(0)](x+y) = [h(x+y) - h(0)](x-y).$$

Letting $x - y = u,\ x + y = v$ for $u, v \in G,\ v \neq 0$ (possible since G is 2-divisible), we have

$$(h(u) - h(0))(v) = (h(v) - h(0))u,\ u, v \in G,\ v \neq 0. \tag{7.55}$$

Define $B : G \times G \to H$ by

$$B(u, v) = (h(u) - h(0))(v). \tag{7.56}$$

Then B is additive in the second variable and (7.55) shows that B is symmetric. Thus B is a symmetric biadditive function. Equation (7.56) gives $h(u)(v) = A(v) + B(u, v)$, where $A = h(0) \in \mathrm{Hom}(G, H)$. Equation (7.52) now gives (7.51). This proves the theorem. □

Now we consider (7.50).

Theorem 7.27. [719]. *Suppose $f, g : G \to H, k, \ell : G \to \mathrm{Hom}(G, H)$, where $G(+)$ is an Abelian group and H is a 2-divisible Abelian group. Then f, g, k, ℓ satisfy equation* (7.50) *if and only if*

$$\begin{cases} f(x) = g(x) = b + 2A_2(x) + \frac{1}{2}B_2(x, x), \\ k(x) = (A_1 + A_2) + B_1(x, \cdot) + B_2(x, \cdot), \\ \ell(x) = (A_2 - A_1) + B_2(x, \cdot) - B_1(x, \cdot), \end{cases} \tag{7.57}$$

where $A_1, A_2 \in \mathrm{Hom}(G, H)$: $B_2 : G \times G \to H$ is symmetric and biadditive, $B_1 : G \times G \to H$ is skew-symmetric and biadditive, and $b \in H$ is a constant.

Proof. Suppose f, g, k, ℓ satisfy (7.50). Then $x = y$ in (7.50) gives $f(x) = g(x)$ for $x \in G$.

First $y = 0$ in (7.50) yields $f(x) = f(0) + (k(x) + \ell(0))(x),\ x \in G$, and then $x = 0$ in (7.50) yields $f(y) = f(0) + (\ell(y) + k(0))(y)$ for $y \in G$.

From the last two equations, first we get

$$(k(x) - \ell(x))(x) = (k(0) - \ell(0))(x) = A_1(x), \quad \text{say,} \tag{7.58}$$

where $A_1 \in \mathrm{Hom}(G, H)$; that is, A_1 is additive and then we have

$$f(x) = f(0) + \frac{1}{2}(k(x) + \ell(x) + A_2)(x), \tag{7.59}$$

where $k(0) + \ell(0) = A_2 \in \mathrm{Hom}(G, H)$.

Interchange x and y in (7.50) to have

$$f(y) - f(x) = (k(y) + \ell(x))(y - x) \quad \text{for } x, y \in G. \tag{7.60}$$

Adding (7.50) and (7.60) and using (7.58), we have

$$(k(x) - \ell(x))(y) + (k(y) - \ell(y))(x) = A_1(x) + A_1(y) \quad \text{for } x, y \in G;$$

that is,

$$(k(x) - \ell(x) - A_1)(y) = -(k(y) - \ell(y) - A_1)(x).$$

Define $B_1 : G \times G \to H$ by

$$B_1(x, y) = (k(x) - \ell(x) - A_1)(y) \quad \text{for } x, y \in G.$$

Then B_1 is a skew-symmetric, biadditive function and

$$(k(x) - \ell(x))(y) = A_1(y) + B_1(x, y) \quad \text{for } x, y \in G. \tag{7.61}$$

Subtracting (7.60) from (7.50), we obtain

$$2(f(x) - f(y)) = (k(x) + \ell(x) + k(y) + \ell(y))(x - y),$$

which by the substitution of $f(x)$ from (7.59) results in

$$(k(x) + \ell(x) - A_2)(y) = (k(y) + \ell(y) - A_2)(x) \quad \text{for } x, y \in G.$$

Defining $B_2 : G \times G \to H$ by

$$B_2(x, y) = (k(x) + \ell(x) - A_2)(y),$$

we see that B_2 is symmetric and biadditive and

$$(k(x) + \ell(x))(y) = A_2(y) + B_2(x, y) \quad \text{for } x, y \in G. \tag{7.62}$$

From (7.61), (7.62), and (7.59), follow (7.57). This completes the proof. □

7.12 Rudin's Problem on Groups

Now we treat the functional equation

$$f(x) - f(y) = h(sx + ty)(x - y) \quad \text{for } x, y \in G,\ x \neq y, \tag{7.63}$$

where $f : G \to F$ (G a divisible Abelian group, F a field of characteristic zero), $h : G \to \text{Hom}(G, F)$ (an element $A \in \text{Hom}(G, F)$ means $A : G \to F$ is such that $A(x + y) = A(x) + A(y)$ for $x, y \in G$). Recall that $x \in G$ is *n-divisible* if there is a $y \in G$ with $ny = x$ and G is divisible when each x in G is divisible by every $n > 0$ [713]. In order to make sense in the argument $sx + ty$ in G, we assume that s and t are integers. We prove the following theorem (see Theorem 7.18 and Result 7.19, and refer to E3338 [235], Sahoo and Riedel [722], and Kac [408]).

Theorem 7.28. *The functions* $f : G \to F$, $h : G \to \mathrm{Hom}(G, F)$ *satisfy the functional equation* (7.63) *for* $x, y \in G$, $x \neq y$, *and* s *and* t *integers if and only if*

$$f(x) = A(x) + b,\ h(x) = \begin{cases} \text{arbitrary} & \text{for } x \neq 0, \\ A & \text{for } x = 0, \end{cases} \quad x \in G,\ s = 0 = t,$$

$$f(x) = A(x) + b,\ h(x) = A,\ \text{for } x \in G,\ s \neq 0,\ t = 0,\ s = 0,\ t \neq 0,$$

$$f(x) = A(x) + B(x, x) + b,\ h(sx) = A + B(x, \cdot),\ x \in G,\ s = t \neq 0,$$

$$f(x) = A(x) + b,\ h(sx)(x) = A(x), \quad \text{for } x \in G,\ t = -s \neq 0,$$

$$f(x) = A(x) + b,\ h(x) = A, \quad \text{for } x \in G,\ s^2 \neq t^2 \neq 0,$$

where $A \in \mathrm{Hom}(G, F)$, $B : G \times G \to F$ *is biadditive and symmetric, and* $b \in F$ *is a constant.*

Proof. It is easy to verify that f and h given above are solutions of (7.63). We prove the converse in three cases according to whether s and t are zeros or not.

Case 1. Suppose $s = 0 = t$.

Then (7.63) can be rewritten as

$$f(x) - h(0)(x) = f(y) - h(0)(y) \quad \text{for } x, y \in G,\ x \neq y.$$

Now $y = 0$ yields $f(x) = A(x) + b$, where $b = f(0)$ and $h(0) = A \in \mathrm{Hom}(G, F)$ for $x \neq 0$. But this equation holds for $x = 0$ also. $h(x)$ is arbitrary for $x \neq 0$.

Case 2. Suppose $s = 0,\ t \neq 0$.

Equation (7.63) becomes

$$f(x) - f(y) = h(ty)(x - y) \quad \text{for } x, y \in G,\ x \neq y.$$

First $y = 0$ yields

$$f(x) = h(0)(x) + f(0) = A(x) + b \quad \text{for all } x \in G,$$

where A is additive. Then $x = 0$ gives

$$f(y) = h(ty)(y) + f(0) \quad \text{for all } y \in G.$$

Thus $h(ty) = A$ for all y; that is, $h(x) = A$ for all $x \in G$. Here we use that G is divisible.

Case 3. Suppose $s \neq 0,\ t \neq 0$. This case is subdivided into three cases according to whether $s = t$ or not.

First of all, $x = 0$ and $y = 0$ separately in (7.63) give

$$f(x) = h(sx)(x) + b \quad \text{for all } x \in G,$$
$$f(y) = h(ty)(y) + b \quad \text{for all } y \in G,$$

where $b = f(0)$ so that (7.63) becomes

$$h(sx)(x) - h(ty)(y) = h(sx + ty)(x - y) \quad \text{for } x \neq y,\ x, y \in G. \tag{7.64}$$

Subcase 1. Let $s = t$.

Then (7.64) becomes

$$h(sx)(x) - h(sy)(y) = h(s(x + y))(x - y) \quad \text{for } x, y \in G,\ x \neq y.$$

Define $g : G \to F,\ H : G \to \text{Hom}(G, F)$ by

$$g(x) = h(sx)(x) \quad \text{and} \quad H(x) = h(sx) \quad \text{for } x \in G,$$

so that we can rewrite (7.64) as

$$g(x) - g(y) = H(x + y)(x - y) \quad \text{for } x, y \in G,\ x \neq y.$$

Then, from Theorem 7.26 (refer to [722]),

$$\begin{aligned} h(sx)(x) = g(x) &= a_0 + A(x) + B(x, x), \\ h(sx) = H(x) &= A + B(x, \cdot) \quad \text{for } x \in G, \end{aligned}$$

where $A \in \text{Hom}(G, F)$ and $B : G \times G \to F$ is a symmetric, biadditive function, $a_0 \in F$.

It is easy to see that $a_0 = 0$ and

$$f(x) = h(sx)(x) + b = A(x) + B(x, x) + b \quad \text{for } x \in G.$$

Subcase 2. Let $s = -t$.

Substitute $t = -s$ in (7.64) to get

$$h(sx)(x) - h(-sy)(y) = h(s(x - y))(x - y) \quad \text{for } x, y \in G,\ x \neq y.$$

Changing y to $-y$, we have

$$h(sx)(x) + h(sy)(y) = h(s(x + y))(x + y) \quad \text{for } x, y \in G,\ x \neq y.$$

Define $A : G \to F$ by

$$A(x) = h(sx)(x) \quad \text{for } x \in G.$$

Then A is additive on G and

$$f(x) = h(sx)(x) + b = A(x) + b \quad \text{for } x \in G.$$

Subcase 3. Suppose $s^2 \neq t^2$.

Replace x by tx and y by sy in (7.64) to have

$$h(stx)(tx) - h(sty)(sy) = h(st(x + y))(tx - sy) \quad \text{for } x, y \in G,\ x \neq y.$$

Interchanging x and y in the equation above and adding the resultant equation, we obtain

$$(t-s)[h(stx)(x)+h(sty)(y)]=(t-s)h(st(x+y))(x+y);$$

that is,

$$h(stx)(x)+h(sty)(y)=h(st(x+y))(x+y) \quad \text{for } x,y\in G,\ x\neq y.$$

Here we make use of F as a field with characteristic zero, $s-t\neq 0$, and $h(sx)$ is a homomorphism.

Define $A: G\to F$ by

$$A(x)=h(stx)(x) \quad \text{for } x\in G.$$

Then A is a homomorphism and

$$f(x)=h(sx)(x)+b \quad \text{for all } x\in G.$$

Now

$$\begin{aligned} f(tx)&=h(stx)(tx)+b=tA(x)+b\\ &=A(tx)+b \quad \text{for } x\in G, \end{aligned}$$

so that

$$f(y)=A(y)+b \quad \text{for all } y\in G.$$

Here we use that G is divisible. This f in (7.63) yields

$$A(x-y)=h(sx+ty)(x-y) \quad \text{for } x,y\in G,\ x\neq y.$$

Replacing x by $x+y$, we get

$$A(x)=h(sx+(s+t)y)(x) \quad \text{for } x,y\in G.$$

Let $u\in G$. Then there is a $y\in G$ with $(s+t)y=u$ (also use $s+t\neq 0$) and

$$A(x)=h(sx+u)(x).$$

Given $x,v\in G$, there is a $u\in G$ such that $sx+u=v$ and $A(x)=h(v)(x)$ so that $h(v)=A$ for all $v\in G$. This completes the proof. □

7.13 Generalization

We consider a functional equation that is a generalization of (7.5), (7.10), (7.19), and (7.19a) and arises in connection with Simpson's Rule [722], namely

(7.65) $$f(x)-g(y)=(x-y)(h(x+y)+k(x)+\ell(y)) \quad \text{for } x,y\in\mathbb{R},$$

where $f,g,h,k,\ell:\mathbb{R}\to\mathbb{R}$ and prove the following theorem (see [722]).

Theorem 7.29. *The functions* $f, g, h, k, \ell : \mathbb{R} \to \mathbb{R}$ *satisfy the functional equation* (7.65) *for* $x, y \in \mathbb{R}$ *if and only if*

$$\text{(7.65s)} \qquad \begin{cases} f(x) = g(x) = & b_1x^4 + b_2x^3 + b_3x^2 + dx + b, \\ h(x) = & \frac{1}{3}b_1x^3 + \frac{1}{2}b_2x^2 + A(x) + c_1, \\ k(x) = & \frac{2}{3}b_1x^3 + \frac{1}{2}b_2x^2 + b_3x - A(x) + c_2, \\ \ell(x) = & \frac{2}{3}b_1x^3 + \frac{1}{2}b_2x^2 + b_3x - A(x) + c_3. \end{cases}$$

where $A : \mathbb{R} \to \mathbb{R}$ *satisfies* (A) *and* $b, b_1, b_2, b_3, c_1, c_2, c_3$ *are constants with* $d = c_2 + c_3 + c_1$.

Proof. This result appears in Sahoo and Riedel [722] with a very lengthy proof. Here we present a simple, direct proof. Let f, g, h, k, ℓ be solutions of (7.65). Setting $y = x$ in (7.65), we have $f(x) = g(x)$ for $x \in G$.

If $f(x)$ is a solution of (7.65), so is $f(x) - f(0)$. So let us assume that $f(0) = 0$. Interchanging x and y in (7.65) and adding the resultant to (7.65), we have

$$(x - y)(k(x) - \ell(x) + \ell(y) - k(y)) = 0 \quad \text{for } x, y \in \mathbb{R};$$

that is,

$$\text{(7.66)} \qquad k(x) - \ell(x) = k(y) - \ell(y) = c = k(0) - \ell(0) \quad \text{for } x \neq y.$$

Putting this value of $k(x)$ in (7.65), we get

$$\text{(7.67)} \qquad f(x) - f(y) = (x - y)(h(x + y) + \ell(x) + \ell(y) + c) \quad \text{for } x, y \in \mathbb{R}.$$

Set $y = 0$ in (7.67) to get

$$\text{(7.68)} \qquad f(x) = x(h(x) + \ell(x) + k(0)) \quad \text{for } x \in \mathbb{R}.$$

Substituting for $f(x)$ from (7.68) into (7.67), we obtain

$$(x - y)h(x + y) + x(\ell(y) - \ell(0)) - y(\ell(x) - \ell(0)) = xh(x) - yh(y) \quad \text{for } x, y \in \mathbb{R},$$

which, after adding $xh(y) - yh(x)$ to both sides, becomes

$$\text{(7.69)} \qquad (x - y)(h(x + y) - h(x) - h(y)) = y\phi(x) - x\phi(y), \quad x, y \in \mathbb{R},$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is given by

$$\text{(7.70)} \qquad \phi(x) = \ell(x) + h(x) - \ell(0) \quad \text{for } x \in \mathbb{R}.$$

Let $y = -x$ in (7.69) to have

$$\text{(7.71)} \qquad 2(h(0) - h(x) - h(-x)) = -(\phi(x) + \phi(-x)) \quad \text{for } x \in \mathbb{R}.$$

Changing y into $-y$ in (7.69) and adding the resultant equation to (7.69) and utilizing (7.71), we obtain

$$\begin{aligned}(7.72)\qquad &(x+y)(h(x-y)-h(x)+h(y)-h(0))\\ &+(x-y)(h(x+y)-h(x)+h(-y)-h(0))=0, \quad x,y\in\mathbb{R}.\end{aligned}$$

Exchange x and y in (7.72) and add the resultant to (7.72) to have

$$(x+y)(h(x-y)+h(y-x)-2h(0))=(x-y)(h(x)+h(-x)-h(y)-h(-y));$$

that is,

$$(7.73)\qquad (x+y)t(x-y)=(x-y)(t(x)-t(y)) \quad \text{for } x,y\in\mathbb{R},$$

where $t:\mathbb{R}\to\mathbb{R}$ is defined by

$$(7.74)\qquad t(x)=h(x)+h(-x)-2h(0) \quad \text{for } x\in\mathbb{R}.$$

Note that $t(0)=0$ and t is even.

Replace y by $-y$ in (7.73) to get

$$(7.75)\qquad (x-y)t(x+y)=(x+y)(t(x)-t(y)) \quad \text{for } x,y\in\mathbb{R}.$$

From (7.73) and (7.75), we obtain ((7.73) $\times (x+y)-$ (7.75) $\times (x-y)$)

$$(x+y)^2t(x-y)=(x-y)^2t(x+y),$$

so that

$$(7.76)\qquad t(x)=b_2x^2 \quad \text{for } x\in\mathbb{R},$$

where b_2 is a constant.

Change x to $-x$ in (7.72) and add the resultant to (7.72) to have

$$(7.77)\qquad 2xs(x)-2ys(y)=(x+y)s(x-y)+(x-y)s(x+y) \quad \text{for } x,y\in\mathbb{R},$$

where $s:\mathbb{R}\to\mathbb{R}$ is defined by

$$(7.78)\qquad s(x)=h(x)-h(-x) \quad \text{for } x\in\mathbb{R}.$$

Note that $s(0)=0$ and s is odd. If we determine s, then (7.74) and (7.76) give h. Using this in (7.69) will yield ϕ and hence ℓ in (7.70). Then we can obtain f from (7.68). We now proceed to find s.

Applying (7.77) twice, we get

$$\begin{aligned}2xs(x)&=2(x+y)s(x+y)+(2x+y)s(-y)-ys(2x+y)\\ &=2(x-y)s(x-y)+(2x-y)s(y)+ys(2x-y),\end{aligned}$$

which (by using (7.77) again and that s is odd) results in

$$\begin{aligned} 4xs(y) + ys(2x - y) + ys(2x + y) &= 2(x + y)s(x + y) - 2(x - y)s(x - y) \\ &= 2xs(2y) + 2ys(2x) \quad \text{for } x, y \in \mathbb{R}. \end{aligned}$$

Replace $2x$ by x in the equation above to obtain

$$y(s(x + y) + s(x - y)) = x(s(2y) - 2s(y)) + 2ys(x) \quad \text{for } x, y \in \mathbb{R}. \tag{7.79}$$

Interchanging x and y in (7.79), we have

$$x(s(x + y) - s(x - y)) = y(s(2x) - 2s(x)) + 2xs(y) \quad \text{for } x, y \in \mathbb{R}. \tag{7.80}$$

Now (7.79) $\times\, x$ + (7.80) $\times\, y$ yields

$$2xy(s(x + y) - s(x) - s(y)) = x^2(s(2y) - 2s(y)) + y^2(s(2x) - 2s(x));$$

that is,

$$s(x + y) = s(x) + s(y) + xp(y) + yp(x) \quad \text{for } x, y \in \mathbb{R},\ x, y \neq 0, \tag{7.81}$$

where $p : \mathbb{R} \to \mathbb{R}$ is defined by

$$p(x) = \begin{cases} \frac{s(2x) - 2s(x)}{2x} & \text{for } x \in \mathbb{R},\ x \neq 0 \\ 0, & \text{for } x = 0. \end{cases}$$

Note that p is even and that (7.81) holds for all $x, y \in \mathbb{R}$. Applying (7.81) four times, we get

$$\begin{aligned} s(x + y + z) &= s((x + y) + z) \\ &= s(x) + s(y) + s(z) + xp(y) + yp(x) + (x + y)p(z) + zp(x + y) \\ &= s(x + (y + z)) \\ &= s(x) + s(y) + s(z) + yp(z) + zp(y) + xp(y + z) + (y + z)p(x), \end{aligned}$$

which results in

$$x(p(y + z) - p(y) - p(z)) = z(p(x + y) - p(x) - p(y)) \quad \text{for } x, y, z \in \mathbb{R}. \tag{7.82}$$

Changing z to $-z$ in (7.82) and adding the resultant to (7.82), we have

$$p(y + z) + p(y - z) = 2p(y) + 2p(z) \quad \text{for } y, z \in \mathbb{R}. \tag{Q}$$

Thus $p(x) = B(x, x)$, where $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is symmetric and biadditive (see Theorem 4.1).

Putting this value of $p(x)$ in (7.82) yields

$$xB(y, z) = zB(x, y) \quad \text{for } x, y, z \in \mathbb{R}.$$

With $z = y$, we get $xB(y, y) = yB(x, y)$ for $x, y \in \mathbb{R}$. Exchange x and y to obtain $yB(x, x) = xB(x, y)$, so that finally we have $x^2B(y, y) = y^2B(x, x)$ and

$$p(x) = B(x, x) = b_1x^2 \quad \text{for } x \in \mathbb{R}, \tag{7.83}$$

where b_1 is a constant.

From (7.81) and (7.83) there results

$$\begin{aligned} s(x + y) &= s(x) + s(y) + b_1xy^2 + b_1x^2y \\ &= s(x) + s(y) + \frac{1}{3}b_1((x + y)^3 - x^3 - y^3); \end{aligned}$$

that is,

$$A(x + y) = A(x) + A(y) \quad \text{for } x, y \in \mathbb{R},$$

where

$$A(x) = s(x) - \frac{1}{3}b_1x^3 \quad \text{for } x \in \mathbb{R}.$$

Thus

$$h(x) - h(-x) = s(x) = \frac{1}{3}b_1x^3 + A(x) \quad \text{for } x \in \mathbb{R}. \tag{7.84}$$

Now (7.84), (7.76), and (7.74) yield

$$h(x) = \frac{1}{6}b_1x^3 + \frac{1}{2}b_2x^2 + \frac{1}{2}A(x) + c_1 \quad \text{for } x \in \mathbb{R}. \tag{7.85}$$

$h(x)$ from (7.85) in (7.69) gives

$$(x - y)\left(b_2xy + \frac{1}{2}b_1xy(x + y) - c_1\right) = y\phi(x) - x\phi(y);$$

that is,

$$x\left(\phi(y) - \frac{1}{2}b_1y^3 - b_2y^2 - c_1\right) = y\left(\phi(x) - \frac{1}{2}b_1x^3 - b_2x^2 - c_1\right),$$

so that

$$\ell(x) + h(x) - \ell(0) = \phi(x) = \frac{1}{2}b_1x^3 + b_2x^2 + b_3x + c_1 \quad \text{for } x \in \mathbb{R},$$

and (using (7.85))

$$\ell(x) = \frac{1}{3}b_1x^3 + \frac{1}{2}b_2x^2 + b_3x - \frac{1}{2}A(x) + c_2 \quad \text{for } x \in \mathbb{R}, \tag{7.86}$$

where b_3 is a constant.

From (7.86), (7.85), (7.68), and (7.66), we obtain

$$f(x) = \frac{1}{2}b_1x^4 + b_2x^3 + b_3x^2 + dx + b,$$
$$k(x) = \frac{2}{3}b_1x^3 + \frac{1}{2}b_2x^2 + b_3x - \frac{1}{2}A(x) + c_3 \quad \text{for } x \in \mathbb{R}.$$

This proves the theorem. □

Corollary 7.30. (Sahoo and Riedel [722]). *The general solutions* $f, s : \mathbb{R} \to \mathbb{R}$ *satisfying the functional equation*

$$f(x) - f(y) = (x-y)s(x+y) + (x+y)s(x-y) \quad \text{for } x, y \in \mathbb{R} \tag{7.87}$$

are given by $s(x) = b_1x^3 + A(x)$, $f(x) = 2b_1x^4 + 2xA(x) + b$, *where* $A : \mathbb{R} \to \mathbb{R}$ *satisfies* (A) *and* b, b_1 *are constants.*

Proof. Setting $x = y$ in (7.87) shows that $s(0) = 0$. Put $x = 0$ and $y = 0$ separately in (7.87) to obtain

$$f(y) = ys(y) - ys(-y) + b, \quad f(x) = 2xs(x) + b,$$

where $b = f(0)$. Clearly we see that s is odd and (7.87) becomes (7.77). The result now follows from Theorem 7.29.

Corollary 7.31. [722]. *The functions* $h, \phi : \mathbb{R} \to \mathbb{R}$ *satisfy the functional equation*

$$(x-y)(h(x+y) - h(x) - h(y)) = y\phi(x) - x\phi(y), \tag{7.69}$$

for $x, y \in \mathbb{R}$, *if, and only if,*

$$h(x) = b_1x^3 + b_2x^2 + A(x) + b, \quad \phi(x) = 3b_1x^3 + 2b_2x^2 + b_3x + c,$$

where $A : \mathbb{R} \to \mathbb{R}$ *satisfies* (A) *and* b, b_1, b_2, c *are constants.*

The result follows from Theorem 7.29.

8

Nondifferentiable Functions

Nondifferential functions, Weierstrass functions, Vander Waerden type functions, and generalizations are considered in this chapter.

In classical analysis, one of the problems that has fascinated mathematicians since the end of the nineteenth century is 'Does there exist a continuous function that is not differentiable?' It is an interesting question. Motivated by this exciting question, many well-known mathematicians, starting with Weierstrass (1872) [827], started to work in this area to produce such a function. It is well known that the answer is affirmative. This chapter is devoted to listing several continuous non-(nowhere) differentiable functions (c.n.d.f.s). What is of interest to us and is the primary motive of this chapter is to show that most of the well-known examples can be obtained as solutions of functional equations, highlighting the functional equation connection. Kairies's [412] report is an excellent survey article and a main source for this chapter.

First we give many well-known examples due to Weierstrass, Takagi, van der Waerden, Knopp, Wünderlich, etc., along with the associated functional equations. These functions can be characterized as solutions of the associated functional equations. The properties of these functions, especially their nondifferentiability, can be deduced directly from the functional equations without making use of their explicit analytic representations. Then we give a general system of functional equations from which all the above can be obtained as special cases.

8.1 Weierstrass Functions

The first widely known example of a c.n.d.f. was given by Weierstrass in 1872, when he reported it to the Königliche Akademie der Wissenschaften in Berlin. Even though it appeared in his *Mathematische Werke II* [827] in 1895, it had been discussed in a paper by du Bois-Reymond (1875) [233]. Weierstrass proved that

$$W(x) = \sum_{n=0}^{\infty} a^n \sin\left(b^n \cdot b\pi x + \frac{\pi}{2}\right), \quad \text{for } x \in \mathbb{R},$$

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has nowhere a finite or infinite derivative if b is an odd integer and $ab > 1 + \frac{3\pi}{2}$; that is, W is a c.n.d.f.

Several authors then gave slight improvements, and Hardy [340] in 1916 presented the optimal result. In [214, 220] the *general Weierstrass function* $W_{a,b,\theta} : \mathbb{R} \to \mathbb{R}$ was introduced by

$$W_{a,b,\theta}(x) = \sum_{n=0}^{\infty} a^n \sin(b^n \cdot b\pi x + \theta) \quad \text{for } x \in \mathbb{R},$$

where $a \in]0, 1[$, $b \in \mathbb{R}_+$, $\theta \in [0, 2\pi[$, and it was proved that $W_{a,b,\theta}$ is a c.n.d.f. for $ab \geq 1$, $b \in Z_+^*$.

Note that W is a special case of $W_{a,b,\theta}$. Now comes the associated functional equations. $W_{a,b,\theta}$ satisfies the functional equations

$$f(x) = af(bx) + \sin(b\pi x + \theta), \quad \text{for } x \in \mathbb{R}, \tag{8.1}$$

$$f\left(\frac{x+k}{b}\right) = (-1)^{kb} af(x) + (-1)^k \sin(\pi x + \theta), \tag{8.1a}$$

for $x \in [0, 1]$, $0 \leq k \leq b - 1$, $b \in Z_+^*$.

Here is the functional equation characterization of $W_{a,b,\theta}$.

Result 8.1. *If $f : \mathbb{R} \to \mathbb{R}$ is bounded and satisfies* (8.1) *for some $b \in \mathbb{R}_+ \backslash \{1\}$, then $f = W_{a,b,\theta}$.*

Result 8.1a. *If $f : \mathbb{R} \to \mathbb{R}$ is bounded, of period* 1, *and satisfies* (8.1a) *for some $b \in Z_+^* \backslash \{1\}$, then $f = W_{a,b,\theta}$.*

Another c.n.d.f., given by Kairies in [414], is

$$S_p(x) := \sum_{n=0}^{\infty} p^{-n} \sin(2\pi p^n x),$$

where p is a fixed prime. For a fixed prime p, the function $S_p : \mathbb{R} \to \mathbb{R}$ is a c.n.d.f. S_p satisfies many functional equations:

$$f(x+1) = f(x), \quad f(-x) = -f(x), \quad \text{for } x \in \mathbb{R}.$$

If $f : \mathbb{R} \to \mathbb{R}$ is bounded and satisfies the equation

$$f(x) = \frac{1}{p} f(px) + \sin 2\pi x \quad \text{for } x \in \mathbb{R},$$

then $f = S_p$.

If $f : \mathbb{R} \to \mathbb{R}$ is bounded with period 1 and satisfies the equation

$$f\left(\frac{x+k}{p}\right) = \frac{1}{p} f(x) + \sin 2\pi \frac{x+k}{p}, \quad 0 \leq k \leq p-1,\ x \in [0, 1],$$

then $f = S_p$.

Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$f(x+1) = f(x), \quad f(-x) = -f(x), \quad x \in \mathbb{R},$$
$$\frac{1}{p}\sum_{k=0}^{p-1} f\left(\frac{x+k}{p}\right) = u(p)\, f(x), \qquad x \in \mathbb{R},$$

where $u(p) = \frac{1}{p}$ if $p = q^n$, $n \in \{0, 1, 2, \ldots\}$, $u(p) = 0$ otherwise, and $f\left(\frac{1}{2q}\right) = \sin\left(\frac{\pi}{q}\right)$. Then $f = S_q$.

8.2 Wünderlich's Function

Wünderlich [834] defined a function $W : [0, 1] \to \mathbb{R}$ as follows. If $x \in [0, 1]$ has 3-adic expansion $0, x_1\, x_2\, x_3\, \ldots$, then $W(x) = 0, y_1\, y_2\, y_3\, \ldots$ is the 2-adic expansion, where $x_0 = 0 = y_0$ and $y_k = y_{k-1}$ if and only if $x_k = x_{k-1}$.

Wünderlich [835] attributed this example of a c.n.d.f. to P. Finsler. However, he was the first to present a system of functional equations satisfied by W and gives a functional equation characterization of W [834].

W satisfies the functional equations

$$f(x) = \frac{1}{2} f(3x) \qquad \text{for } x \in \left[0, \frac{1}{3}\right],$$
$$f(x) = \frac{1}{2} + f\left(x - \frac{1}{3}\right) \qquad \text{for } x \in \left[\frac{1}{3}, \frac{1}{2}\right],$$
$$f(x) = f(1-x) \qquad \text{for } x \in \left[\frac{1}{2}, \frac{2}{3}\right],$$

and

$$f(x) = 1 - f(1-x) \qquad \text{for } x \in \left[\frac{2}{3}, 1\right].$$

If $f : [0, 1] \to \mathbb{R}$ is continuous and satisfies these equations, then $f = W$.

8.3 Takagi Functions

The Takagi function $T : \mathbb{R} \to \mathbb{R}$ is defined by

$$T(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} d(2^n x),$$

where $d(y) = \text{dist}(y, \mathbb{Z})$. T was introduced in 1903 by T. Takagi [801] as an example of a function whose c.n.d. property can be proved much more easily than that of the Weierstrass function. T satisfies the functional equations [206, 414]

$$f(x+1) = f(x), \quad x \in \mathbb{R},$$

$$\text{(8.2)} \qquad \begin{cases} f\left(\dfrac{x}{2}\right) = \dfrac{1}{2} f(x) + \dfrac{x}{2}, \\ f\left(\dfrac{x+1}{2}\right) = \dfrac{1}{2} f(x) - \dfrac{x}{2} + \dfrac{1}{2}, \\ f\left(\dfrac{x}{2}\right) = f\left(\dfrac{x+1}{2}\right) + x - \dfrac{1}{2}, \\ f(x) = f\left(\dfrac{x}{2}\right) + f\left(\dfrac{x+1}{2}\right) - \dfrac{1}{2}, \quad x \in [0,1], \end{cases}$$

$$\text{(8.2}') \qquad f(x) = \frac{1}{2} f(2x) + d(x), \quad x \in \mathbb{R}.$$

In [206], the authors showed that any two of the equations (8.2) imply the other two and that every solution of (8.2) must be nowhere differentiable.

We have the following characterizations by functional equations. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and satisfy (8.2′). Then $f = T$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded, of period 1, and satisfies two of the equations in (8.2). Then $f = T$.

Remark. Let $g(x) := 2d(x)$ and denote by $g^{(n)}$ the nth iterate of g. Then we can generate T by iterations of g:

$$T(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} g^{(n)}(x).$$

8.4 van der Waerden Type Function

The van der Waerden function $w : [0,1] \to \mathbb{R}$ is given by

$$\text{(8.3)} \qquad w(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} d(2^n x) \quad \text{for } x \in [0,1]$$

[824, 141, 206], where $d(u)$ is the distance from u to the nearest integer. This function is the simplest and best-known example of a c.n.d.f. (compare it with the Takagi function T). That w does not possess a finite one-sided derivative of any point is proved in [141]. The properties of w, notably its nondifferentiability, can be obtained directly from functional equations satisfied by w, without making use of the explicit representation of w (8.3). The proof of nondifferentiability of w is given in Darsow, Frank, and Kairies [206]. To start with, we derive some functional equations satisfied by w (compare them with (8.1) and (8.1′)).

Since $d(y) = y$ when $0 \le y \le \frac{1}{2}$ and $d(y) = 1 - y$ for $\frac{1}{2} \le y \le 1$, we have for $0 \le x \le 1$

$$\begin{aligned} w(x) &= 2\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} d\left(2^{n+1}\frac{x}{2}\right) \\ &= 2\sum_{n=1}^{\infty} \frac{1}{2^n} d\left(2^n \frac{x}{2}\right) \\ &= 2\left[w\left(\frac{x}{2}\right) - d\left(\frac{x}{2}\right)\right] \\ &= 2w\left(\frac{x}{2}\right) - x \end{aligned}$$

and

$$\begin{aligned} w\left(\frac{x+1}{2}\right) &= \sum_{n=0}^{\infty} \frac{1}{2^n} d\left(2^n \frac{x+1}{2}\right) \\ &= d\left(\frac{x+1}{2}\right) + \sum_{n=1}^{\infty} \frac{1}{2^n} d\left(2^n \frac{x}{2}\right) \\ &= w\left(\frac{x}{2}\right) + d\left(\frac{x+1}{2}\right) - d\left(\frac{x}{2}\right) \\ &= w\left(\frac{x}{2}\right) - x + \frac{1}{2}. \end{aligned}$$

Thus w is a solution of the system of functional equations

$$(8.1') \qquad f(x) - 2f\left(\frac{x}{2}\right) = -x, \qquad 0 \le x \le 1,$$

$$(8.1'\text{a}) \qquad f\left(\frac{x+1}{2}\right) - f\left(\frac{x}{2}\right) = -x + \frac{1}{2}, \qquad 0 \le x \le 1.$$

It is easy to check that these two equations imply

$$(8.1'\text{b}) \qquad f(x) - f\left(\frac{x}{2}\right) - f\left(\frac{x+1}{2}\right) = -\frac{1}{2}, \quad 0 \le x \le 1,$$

and

$$(8.1'\text{c}) \qquad 2f\left(\frac{x+1}{2}\right) - f(x) = -x + 1, \qquad 0 \le x \le 1,$$

and w also satisfies the symmetry condition

$$(8.1'\text{d}) \qquad f(x) = f(1-x), \qquad 0 \le x \le 1.$$

Remark. (Compare this with (8.1)). It can be readily checked, as mentioned earlier, that any two of the equations (8.1′), (8.1′a), (8.1′b), (8.1′c) imply the other two. In what follows, we tacitly make use of the fact that any solution of a system of two of these equations satisfies all four of them.

8.5 Functional Equation Characterization of w

Theorem 8.2. [206]. *Let* $f : [0,1] \to \mathbb{R}$ *be bounded and satisfy two of* (8.1′), (8.1′a), (8.1′b), *and* (8.1′c). *Then* $f = w$.

Proof. An additional functional equation is derived that leads to the representation (8.3).

Noting that $f(0) = f(1) = 0$, we can extend f to a function defined on $\mathbb{R}$ by

$$f(x+1) = f(x). \tag{8.4}$$

Rewrite (8.1′a) and (8.1′c) as

$$f(2x) = 2f(x) - 2x, \quad 0 \le x \le \frac{1}{2},$$

and

$$f(2x-1) = 2f(x) - 2(1-x), \quad \frac{1}{2} \le x \le 1. \tag{8.5}$$

These equations can now be unified to yield the single equation

$$f(2x) = 2f(x) - 2d(x) \tag{8.5a}$$

for all $x \in [0,1]$, where d is the distance function.

Indeed, when $x \in \left[0, \frac{1}{2}\right]$, (8.5a) is obvious. For $x \in \left[\frac{1}{2}, 1\right]$, use $f(x+1) = f(x)$ for all $x \in \mathbb{R}$ on the left side of (8.5) to get (8.5a). Indeed, (8.5a) holds for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$, write $x = n + t$, with $n \in Z$, $t \in [0,1[$, and use f and d with period 1 to have

$$\begin{aligned} f(2x) &= f(2n+2t) = f(2t) = 2f(t) - 2d(t) \\ \text{(by (8.5a))} \quad &= 2f(n+t) - 2d(n+t) = 2f(x) - 2d(x), \end{aligned}$$

which is (8.5a) for $x \in \mathbb{R}$.

But it is well known (see [554, pp. 81–82]) that any bounded $f : \mathbb{R} \to \mathbb{R}$ that satisfies (8.5a) for all $x \in \mathbb{R}$ must be of the form (8.3). Indeed, iterating (8.3) gives

$$f(x) = \frac{1}{2^m} f(2^m x) + \sum_{n=0}^{m-1} \frac{1}{2^n} d(2^n x)$$

for all $m \in Z_+$ and $x \in \mathbb{R}$, whence the boundedness of f yields

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} d(2^n x), \quad x \in \mathbb{R}. \tag{8.3}$$

The extension procedure and the treatment on the larger domain $\mathbb{R}$ did not affect the original f on $[0,1]$. Thus $f(x) = w(x)$ for $x \in [0,1]$. □

It is remarkable that the function given in (8.3) satisfies equation (8.1′b) for all $x \in \mathbb{R}$. In fact, any function f of period 1 that satisfies (8.1′b) for $x \in [0,1]$ necessarily satisfies (8.1′b) for all $x \in \mathbb{R}$. By contrast, none of the equations (8.1′), (8.1′a), (8.1′c) has a solution on $\mathbb{R}$ that is bounded. On the other hand, there do exist solutions of (8.1′), (8.1′a), (8.1′b), (8.1′c) on $[0,1]$ that are unbounded.

8.6 Nondifferentiability of w

Theorem 8.3. [206]. *Suppose* $f : [0,1] \to \mathbb{R}$ *satisfies two of the equations* (8.1′), (8.1′a), (8.1′b), (8.1′c). *Then* f *is nowhere differentiable.*

Proof. We will show that f has no right derivative. Now $x = 0$ in (8.1′) gives $f(0) = 0$, and in (8.1′a) $f\left(\frac{1}{2}\right) = \frac{1}{2}$. From (8.1′a) with $x = 1$, we get $f(1) = 0$. Rewrite (8.1′) to obtain

$$f(2x) = 2f(x) - 2x \quad \text{for } 0 \le x \le \frac{1}{2}, \tag{8.1′a′}$$

which by iteration gives

$$f(2^m x) = 2^m f(x) - m2^m x \quad \text{for } 0 \le x \le \frac{1}{2^m}$$

or

$$f\left(\frac{x}{2^m}\right) = \frac{1}{2^m} f(x) + \frac{m}{2^m} x \quad \text{for } 0 \le x \le 1. \tag{8.1′a″}$$

Thus $f\left(\frac{1}{2^m}\right) = \frac{m}{2^m}$.

Now $\frac{f(\frac{1}{2}m) - f(0)}{\frac{1}{2}m - 0} = m$, showing thereby that f has no right derivative at 0.

Rewrite (8.1′a) to have

$$f\left(x + \frac{1}{2}\right) = f(x) - 2x + \frac{1}{2}, \quad 0 \le x \le \frac{1}{2}. \tag{8.1′a‴}$$

Equations (8.1′a″) and (8.1′a‴) yield

$$\begin{aligned} f\left(x + \frac{1}{2^m}\right) &= f\left[\frac{1}{2^{m-1}}\left(2^{m-1}x + \frac{1}{2}\right)\right] \\ &= \frac{1}{2^{m-1}}\left[f\left(2^{m-1}x + \frac{1}{2}\right) + (m-1)\left(2^{m-1}x + \frac{1}{2}\right)\right] \quad \text{(by (8.1′a″))} \\ &= \frac{1}{2^{m-1}}\left[f(2^{m-1}x) + \frac{1}{2} - 2^m x\right] + (m-1)\left(x + \frac{1}{2^m}\right) \quad \text{(by (8.1′a‴))} \\ &= f(x) - 2x + \frac{m}{2^m} \quad \text{(by (8.1′a‴)).} \end{aligned} \tag{8.6}$$

Fix a number $x_1 \in (0,1]$, and consider its nonterminating dyadic expansion

$$x_1 = \sum_{\lambda=1}^{\infty} \frac{1}{2^{n_\lambda}}, \quad 1 \le n_1 < n_2 < \cdots,\ n_\lambda \in Z_+^*.$$

Let $x_k := \sum_{\lambda=k}^{\infty} \frac{1}{2^{n_\lambda}}$, so that

$$x_k = \frac{1}{2^{n_k}} + x_{k+1} \quad \text{and} \quad 0 < x_{k+1} \le \frac{1}{2^{n_k}}.$$

From (8.6), we obtain

$$f(x_k) = f(x_{k+1}) + \frac{n_k}{2^{n_k}} - 2x_{k+1},$$

which can be applied recursively to yield

$$f(x_1) = f(x_{k+1}) + \frac{n_1}{2^{n_1}} + \cdots + \frac{n_k}{2^{n_k}} - 2(x_2 + \cdots + x_{k+1}). \tag{8.7}$$

Next, let $y_k := x - \frac{1}{2^{n_k}}$. Formula (8.7), applied to y_k, gives

$$\begin{aligned} f(y_k) = f(x_{k+1}) &+ \frac{n_1}{2^{n_1}} + \cdots + \frac{n_{k-1}}{2^{n_{k-1}}} \\ &- 2\left[\left(x_2 - \frac{1}{2^{n_k}}\right) + \cdots + \left(x_{k-1} - \frac{1}{2^{n_k}}\right) + x_{k+1}\right]. \end{aligned}$$

Thus

$$\frac{f(x_1) - f(y_k)}{\frac{1}{2^{n_k}}} = n_k - 2k + 2(1 - 2^{n_k} x_{k+1}) \tag{8.8}$$

holds for $0 < u_k \le 1$, where

$$k := 2^{n_k} x_{k+1} = 2^{n_k} \sum_{\lambda=k+1}^{\infty} \frac{1}{2^{n_\lambda}} \le 2^{n_k} \cdot \frac{2}{2^{n_{k+1}}} \le 1.$$

Suppose that x_1 is a dyadic rational. Then there is a $p \in Z_+^*$ such that $n_k = k - p + n_p$ for all $k \ge p$. It follows from (8.8) that

$$\frac{f(x_1) - f(y_k)}{\frac{1}{2} n_k} = -k + n_p - p + 2(1 - u_k) \to -\infty \quad \text{as } k \to \infty;$$

thus f cannot be left differentiable at x_1. Similarly, it is proved in [206] that f is not left differentiable at x_1 when x_1 is not a dyadic rational. Thus f has no left derivative at any $x \in]0, 1]$. □

Remark. The preceding argument actually proves a stronger statement than the conclusion of Theorem 8.3: For each $x \in]0, 1]$, the left-hand derivative of f at x does not exist. Therefore, since w satisfies (8.1′d), it follows at once that both the left- and right-hand derivatives of w fail to exist at each $x \in [0, 1]$.

8.7 Riemann's Function

The celebrated Riemann function $r : \mathbb{R} \to \mathbb{R}$ is defined by

$$r(x) := \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n^2 x).$$

Its first appearance in the literature was in 1875 in P. du Bois-Reymond's paper [233]. This function r had already been discussed in 1872 by K. Weierstrass in his Akademie lecture, which was published in 1895 [827]. In both papers, r is discussed in connection with the c.n.d. property, but neither of the two authors give a clear statement about the nondifferentiability properties of r.

In 1916, G.H. Hardy [340], in his paper on the Weierstrass c.n.d. function, was the first to prove substantial facts about the nondifferentiability of r, but the final result was first obtained in 1971 by J. Gerver in his papers [311] and [312]: r is differentiable exactly in the points $z_{\lambda,\mu} = \pi \frac{2\lambda+1}{2\mu+1}$, $\lambda, \mu \in \mathbb{Z}$, with $r'(z_{\lambda,\mu}) = -\frac{1}{2}$ for all $\lambda, \mu \in \mathbb{Z}$.

Since then, many papers on the history and the nondifferentiability, as well as other properties, of r have appeared.

In the papers on r, only the following functional equations occur:

$$r(x + 2\pi) = r(x), \quad r(-x) = -r(x), \quad r(x + \pi) = \frac{1}{2} r(4x) - r(x), \quad x \in \mathbb{R}.$$

Now we consider a modified r, the function $R : \mathbb{R} \to \mathbb{R}$, given by

$$R(x) := r(2\pi x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(2\pi n^2 x).$$

The functional equations for R are simpler than those for r. R satisfies the following functional equations:

$$\begin{cases} f(x+1) = f(x), \\ f(-x) = -f(x), \\ f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) = \frac{1}{2} f(2x), \quad x \in \mathbb{R} \end{cases} \tag{8.9}$$

(compare these with the Takagi equations). However, R cannot be characterized as the unique continuous solution of (8.9). A counterexample is given by

$$R^*(x) := \sum_{n=0}^{\infty} \frac{1}{4^n} \sin(2\pi 4^n x).$$

R^* is a c.n.d.f. of Weierstrass type, which satisfies all the equations of (8.9).

The function R satisfies the following functional equations:

$$\sum_{k=0}^{m^2q-1} f\left(\frac{x+k}{m^2q}\right) = \frac{1}{q} f(qx), \quad x \in \mathbb{R},\ m, q \in Z_+^* \text{ square-free.} \tag{8.10}$$

Particular cases of (8.10) are

$$\sum_{k=0}^{m^2-1} f\left(\frac{x+k}{m^2}\right) = f(x), \qquad x \in \mathbb{R},\ m \in Z_+^*\ (q=1), \tag{8.10a}$$

$$\sum_{k=0}^{q-1} f\left(\frac{x+k}{q}\right) = \frac{1}{q} f(qx), \quad x \in \mathbb{R},\ q \in Z_+^* \text{ square-free } (m=1). \tag{8.10b}$$

The following characterization of R is in some way analogous with S_p in [414]. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous, of period 1, odd, and satisfies (8.10) for every prime q with $f\left(\frac{1}{4}\right) = \frac{\pi^2}{8}$. Then $f = R$.

The proof is similar to that of Theorem 8.2 but more involved. The functional equations (8.10b) for primes q imply the following restrictions on the complex Fourier coefficients of f: $kc_k = c_1, kc_{-k} = c_{-1}$ if $k \in \mathbb{N}$ is a square and $c_k = c_{-k} = 0$ otherwise. Then $f(x+1) = f(x)$ and $f(-x) = -f(x)$ imply that the Fourier series of f coincides with R. The conclusion is obtained by Fejér's Theorem.

Remark. Both systems (8.10a) and (8.10b) have nontrivial real analytic solutions.

There are generalizations of Riemann's function: In 1916, G.H. Hardy [340] investigated

$$H_\alpha(x) := \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \sin(n^2 \pi x) \quad \text{for } 1 \le \alpha < 3.$$

8.8 Knopp Functions

The general Knopp function $K_{a,p,g} : \mathbb{R} \to \mathbb{R}$ is defined by

$$K_{a,p,g}(x) := \sum_{n=0}^{\infty} a^n g(p^n x),$$

where $|a| < 1$, $p \in Z_+^* \backslash \{1\}$, and $g : \mathbb{R} \to \mathbb{R}$ is of period 1.

Many results on the c.n.d. property in special cases have been obtained by K. Knopp (1918) [541, 542] and others. The most general ones are due to R. Girgensohn (1994) [319]. He showed that:

$|a|p > 1$ and g' bounded implies that $K_{a,p,g}$ is c.n.d. or $K'_{a,p,g}$ is bounded,

$|a|p = 1$ and g'' bounded implies that $K_{a,p,g}$ is c.n.d. or $K''_{a,p,g}$ is bounded, $g'(0) \neq 0$ and g'' bounded implies that $K_{1/p,p,g}$ is c.n.d.

$K_{a,p,g}$ satisfies the following functions equations (a, p, g as above):

$$f(x) = af(px + gx), \qquad x \in \mathbb{R}, \tag{8.11}$$

$$f\left(\frac{x+k}{p}\right) = af(x) + g\left(\frac{x+k}{p}\right), \quad 0 \leq k \leq p-1,\ x \in [0,1]. \tag{8.11a}$$

We have characterizations according to Results 8.5 and 8.6.

Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and satisfy (8.11) with a continuous g. Then $f = K_{a,p,g}$.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is bounded, of period 1, and satisfies (8.11a) with a continuous g. Then $f = K_{a,p,g}$.

Remark. $K_{a,p,g}$ is another generalization of Takagi's T. $K_{1/10,10,d}$ is the famous van der Waerden function [824], which played a prominent role in the history of c.n.d..

8.9 Generalization

Most of the equations studied so far fall under a system of functional equations in a single variable of the form

$$\sum_{\nu=1}^{n} f[h_\nu(x)] = \alpha f(x) + g(x), \quad x \in I \subseteq \mathbb{R},\ f : I \to \mathbb{R}, \tag{8.12}$$

where $h_\nu : I \to \mathbb{R}$ is a given rational function having a very simple structure, $g : I \to \mathbb{R}$ is a given function, and $\alpha \in \mathbb{R}$ is a given real number. Some important particular cases are

$$f(x) = af(bx) + g(x), \tag{8.13}$$

$$f\left(\frac{x+k}{p}\right) = a_k f(x) + g_k(x), \quad k \in \{0, 1, \ldots, p-1\}, \tag{8.14}$$

$$\frac{1}{p}\sum_{k=0}^{p-1} f\left(\frac{x+k}{p}\right) = u(p) f(x), \quad p \in S \subset \mathbb{N}. \tag{8.15}$$

Equation (8.13) is referred to as an iterative equation.

These equations were applied in various characterizations of c.n.d.f.s along with the following result.

Theorem 8.4. (Rathore [690]). *Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and bounded, $|a| < 1$, and $b \in \mathbb{R}$. Then there exists exactly one bounded function $f : \mathbb{R} \to \mathbb{R}$ that satisfies the iterative equation* (8.13)*. The function f is continuous and is given by $f(x) = \sum_{n=0}^{\infty} a^n g(b^n x)$, $x \in \mathbb{R}$.*

Proof. Assume that there is a bounded solution $f : \mathbb{R} \to \mathbb{R}$ of (8.13). Iteration gives

$$f(x) = a^m f(b^m x) + \sum_{n=0}^{m-1} a^n g(b^n x)$$

for every $m \in Z_+^*$ and $x \in \mathbb{R}$. As f is bounded and $|a| < 1$, necessarily

$$f(x) = \sum_{n=0}^{\infty} a^n g(b^n x)$$

for every $x \in \mathbb{R}$. The assumptions on g and a imply that the function given by the series is continuous, and it is easily checked that it satisfies (8.13).

Result 8.5. (Girgensohn [317]). *For fixed $p \in \{2, 3, 4, \dots\}$, let $g_k : [0, 1] \to \mathbb{R}$ be continuous, $|a_k| < 1$ for $0 \le k \le p - 1$, and assume*

$$\frac{a_{k-1}}{1 - a_{p-1}} g_{p-1}(1) + g_{k-1}(1) = \frac{a_k}{1 - a_0} g_0(0) + g_k(0), \quad 1 \le k \le p - 1.$$

Then there exists exactly one bounded $f : [0, 1] \to \mathbb{R}$ that satisfies the system

$$f\left(\frac{x + k}{p}\right) = a_k f(x) + g_k(x), \quad x \in [0, 1],\ 0 \le k \le p - 1. \tag{8.14}$$

The function f is continuous and is given (in terms of the p-adic expansion of x) by

$$f\left(\sum_{n=1}^{\infty} \frac{\xi_n}{p^n}\right) = \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} a_{\xi_k}\right) g_{\xi_n}\left(\sum_{k=1}^{\infty} \frac{\xi_{k+n}}{p^k}\right).$$

Result 8.6. (Kairies [414]). *Let $u \in \ell^1$ be completely multiplicative; i.e., $u(1) = 1$, $u(mn) = u(m)u(n)$, for every $m, n \in Z_+^*$. Then there is exactly one continuous $f : \mathbb{R} \to \mathbb{R}$ that satisfies*

$$\frac{1}{p} \sum_{k=0}^{p-1} f\left(\frac{x + k}{p}\right) = u(p) f(x), \quad x \in \mathbb{R}, \tag{8.15}$$

for every prime p, is of period 1, is odd, and is normalized by $b_1 = 1$ (the first Fourier sine coefficient of f). The function f is given by $f(x) = \sum_{k=1}^{\infty} u(k) \sin 2\pi kx$.

Most of the well-known examples of c.n.d.f.s are obtained as solutions of functional equations, and the nondifferentiability is deduced from the elementary functional equations, without making use of the explicit representations. Further, there are several c.n.d.f.s, starting from Weierstrass's, and plenty of functional equations connected with them.

9

Characterization of Groups, Loops, and Closure Conditions

Groupoids, quasigroups, loops, semigroups, and groups are considered in this chapter. Closure conditions, isotopy, and groups are characterized by various identities. Functional equations arising out of Bol, Moufang, and extra loops are considered. Mediality, the left inverse property, Steiner loops, and generalized bisymmetry are treated.

This chapter is devoted to algebraic identities (and their generalizations leading to the involvement of functional equations) connected to groups and well-known special loops such as Bol, Moufang, left inverse property (l.i.p.), and weak inverse property (w.i.p.).

An identity in a binary system (such as transitivity, bisymmetry, distributivity, Bol, Moufang, etc.) induces a generalized identity—a functional equation—in a class of quasigroups. We treat several of them in this chapter. First we give some notation and definitions.

9.1 Notation and Definitions

Groupoid A (nonempty) set G endowed with a binary operation $(\cdot)$ is called a *groupoid*; that is, for all $a, b \in G,\ a \cdot b \in G$.

Left (right) quasigroup A groupoid G in which $ax = b$ $(yc = d)$ has a unique solution for x (for y) for all a, b (c, d) in G is called a *left quasigroup* (*right quasigroup*).

Quasigroup A groupoid G in which the equations $ax = b$ and $yc = d$ are solvable uniquely for x and y, given a, b, c, d in G, is called a *quasigroup*; that is, for $ab = c$ in G, given any pair, the third element is uniquely determined. In other words, a groupoid that is both a left and a right quasigroup is a quasigroup.

Loop A quasigroup with an identity is called a *loop*.

Semigroup An associative groupoid is a *semigroup*.

Group An associative quasigroup (or loop) is a *group*.

Left cancellative (l.c.) A groupoid G is said to be *left cancellative* if $ax = ay$ in G implies that $x = y$. *Right cancellative* (r.c.) is defined similarly.

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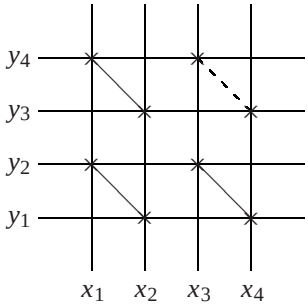
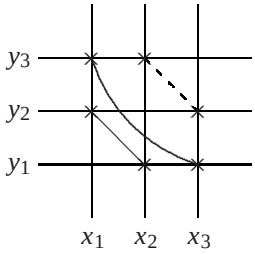
9.2 Closure Conditions

R-groupoid A groupoid G with the property that, for all $x_i, y_i \in G$ $(i = 1, 2, 3, 4)$, the equations $x_1y_2 = x_2y_1,\ x_1y_4 = x_2y_3,\ x_4y_1 = x_3y_2$ imply $x_3y_4 = x_4y_3$ is called an *R-groupoid*. This closure condition is called the *Reidemeister condition.* Geometrically, it means the following.

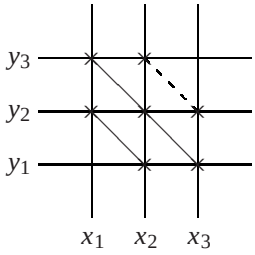


If there are third lines through (x_1y_2, x_2y_1), (x_1y_4, x_2y_3), and (x_3y_2, x_4y_1), there is a third line through (x_3y_4, x_4y_3); that is, the figure closes at (x_3y_4, x_4y_3).

T-groupoid A groupoid G is said to be a *T-groupoid* if, for all $x_i, y_i \in G$ $(i = 1, 2, 3)$, the equations $x_1y_2 = x_2y_1,\ x_1y_3 = x_3y_1$ imply $x_2y_3 = x_3y_2$. This closure condition is known as the *Thomsen condition.* The following figure is self-explanatory.



H-condition (Hexagonal condition) For $x_1, x_2, x_3, y_1, y_2, y_3 \in G$, if $x_1y_2 = x_2y_1,\ x_1y_3 = x_2y_2 = x_3y_1$ implies $x_2y_3 = x_3y_2$, G is said to satisfy the closure condition known as the *hexagonal condition*.



The H-condition implies power association, that is,

$$x \cdot y^{m+n} = xy^m \cdot y^n$$

for all $x, y \in G$ and all $m, n \in Z$.

Isotopy Let $G(\cdot)$ and $H(\circ)$ be two groupoids. If there exist three mappings $\alpha, \beta, \gamma : G \to H$ that are bijective (one-to-one and onto) such that $\gamma(x \cdot y) = \alpha x \circ \beta y$ for all $x, y \in G$, we say that $G(\cdot)$ is *isotopic* to $H(\circ)$ and the triplet (α, β, γ) is said to be an *isotope*. When $\alpha = \beta = \gamma$, we get the usual isomorphism (homomorphism). An isotopism (α, β, I_G) of $G(\cdot)$ onto $G(\circ)$ is called a *principal isotopism*, where I_G is the identity map of G onto G.

Fact 1. Every isotope of a quasigroup is a quasigroup [135, p. 56].

Isogroup Let $G(\cdot)$ be a groupoid. We say that $G(\cdot)$ is an isogroup (Abelian), provided there is a group (Abelian) $G(\circ)$ that is a (principal) isotope of $G(\cdot)$ such that $\cdot$ and $\circ$ are connected by one of the relations

$$x \circ y = \alpha(x) \cdot y \quad \text{or} \quad x \circ y = x \cdot \alpha(y) \quad \text{or} \quad x \circ y = \alpha(x \cdot \alpha(y)), \tag{9.1}$$

where α is the inverse operator of $G(\circ)$.

Left and right translation Mappings $L_a, R_a : G \to G$ defined by $L_a x = ax$, $R_a x = xa$, for all $x \in G$ (groupoid), are called *left* (*right*) *translations* or *multiplication* by a.

Fact 2. Suppose $\alpha, \beta, P : G \to G$ such that $P = \alpha\beta$ $(\alpha\beta)(x) = \alpha(\beta(x))$. If P is a permutation, then α is onto and β is 1–1.

Fact 3. (Aczél [11]). (R) A quasigroup $G(\cdot)$ is isotopic to a group if and only if the Reidemeister condition holds in $G(\cdot)$.

Fact 4. [11] (T) A quasigroup $G(\cdot)$ is isotopic to an Abelian group if and only if the Thomsen condition holds in $G(\cdot)$.

9.3 Groups

First we consider identities that characterize a group. A variety of groups is a class of groups that can be defined by means of a number of identical relations or laws, as they are sometimes called. Alternatively, it is a class of groups that is closed under the formation of direct products, homomorphic images, and subgroups.

One of the problems of the varieties is to define some class of varieties by single identity. This problem has been solved for groups, Abelian groups, loops with the inverse property, etc. It is known for groups that certain varieties of groups may be defined by a single law in terms of multiplication and inversion; of right division. It is known that each variety of groups that can be defined by a finite system of laws as a subvariety of the variety of groups can be defined by a single law as a subvariety of the variety of groups. In particular, the variety of all groups can be obtained from that of groupoids by a single law.

Here are some of the ways groups can be defined:

I In terms of multiplication $(\cdot)$ and inversion (-1).
$(\cdot)$ is associative, there is an e (identity) such that $xe = ex = x$, and there is x^{-1} (inverse to x) such that $xx^{-1} = x^{-1}x = e$.

II $(\cdot)$ is associative; equations $ax = b,\ yc = d$ are solvable for x, y.

III By four laws: $xx^{-1} = yy^{-1} = e;\ xe^{-1} = x = e(ex^{-1})^{-1};\ [x(ey^{-1})^{-1}] \cdot (ez^{-1})^{-1} = x[e \cdot \{y(ez^{-1})^{-1}\}^{-1}]^{-1};\ (xx^{-1} \cdot y)y^{-1} = e$.

IV $[\{y \cdot [z \cdot (xv)]\}^{-1} \cdot (yz)]^{-1} \cdot v^{-1} = x$ (defined as a variety of groups by a single law).

V (Wimmer and Ziebur [832]). Let G be a set with binary operation $(-)$ (called subtraction or difference) such that, for all x, y, z in G, (i) $x-y \in G$; (ii) there is an element $0 \in G$ such that $x-y = 0$ if and only if $x = y$; (iii) $(x-z)-(y-z) = x - y$ hold.
Define $+$ (called addition) in G such that

$$x + y = x - (0 - y).$$

Then $G(+)$ is a group.

9.4 Abelian Groups

The commutative law may be written as (iv) $x - (x - y) = y$ ((iv) is obviously equivalent to $x + y = y + x$ in any group).

A necessary and sufficient condition for G to be an Abelian group is that it satisfy (i), (iii), and (iv) or (i), (ii), and (v) $(x - z) - (x - y) = y - z$.

Characterization of Groups Satisfying a Single Identity

An identity in a binary system induces a generalized identity (functional equation) in a class of quasigroups. For example, the well-known identities of transitivity, distributivity, Bol, Moufang, l.c., and extra loop

(9.2)	$xy \cdot zy = xz,$	(transitivity)
(9.2a)	$xy \cdot xz = yz,$	
(9.3)	$xy \cdot zy = zx,$	
(9.3a)	$xy \cdot xz = zy,$	
(9.4)	$x \cdot yz = xy \cdot xz,$	(distributivity)
(9.5)	$y \cdot (z \cdot yx) = (y \cdot zy) \cdot x,$	(Bol (left))
(9.6)	$(xy \cdot x)z = x \cdot (y \cdot xz),$	(Moufang)
(9.7)	$(x \cdot xy) \cdot z = x \cdot (x \cdot yz),$	(l.c. identity)
(9.8)	$yx \cdot zx = (y \cdot xz) \cdot x,$	(extra identity)

give rise to the generalized associativity, bisymmetry, transitivity, distributivity, Bol, Moufang, l.c., and extra loop functional equations on quasigroups, and these equations on quasigroups will be investigated later. We study and solve the generalized functional equations on a more generalized algebraic structure than quasigroups, namely the GD-groupoids.

First we treat group identities.

Theorem 9.1. (See Aczél [12] and Kannappan [433, 435].) *Let $G(\cdot)$ be a groupoid satisfying the transitivity identity* (9.2) *with left cancellativity* (*l.c.*). *Then $G(\cdot)$ is an isogroup.*

Proof. Given $xz \cdot yz = xy +$ (l.c.), suppose $a \cdot a = e$ for some $a \in G$. Set $x = y = z = a$ in (9.2) to have $e \cdot e = e$.

Put $x = y = e$ in (9.2) and use l.c. to get $e \cdot ze = ez$; that is, $ze = z$ for all $z \in G$.

Set $z = y$ in (9.2) to obtain $xy \cdot yy = xy = xy \cdot e$ so that l.c. yields $yy = e$ for all $y \in G$.

Put $y = x,\ z = e$ in (9.2) to have $e \cdot ex = x$ for all x. Define $\circ$ on G by

$$x \circ y = x \cdot ey = x \cdot \beta(y), \quad \beta(y) = ey. \tag{9.1}$$

$\beta(ex) = x$ and l.c. implies β is 1–1. Hence $G(\cdot)$ is isotopic to $G(\circ)$ ((I, β, I) is the isotopism). We will show that $G(\circ)$ is a group; that is, $(\circ)$ is associative and $a \circ x = b,\ y \circ c = d$ are solvable.

$y = x$ in (9.2) gives

$$e \cdot zx = xz \quad \text{for all } x, z \in G. \tag{9.9}$$

Using transitivity, (9.9), and (9.1), we have

$$\begin{aligned}
(x \circ y) \circ z &= (x \cdot ey) \cdot ez \\
&= e[ez \cdot (x \cdot ey)] && \text{by (9.9)} \\
&= e \cdot [(ez \cdot y) \cdot \{(x \cdot ey) \cdot y\}] && \text{transitivity} \\
&= e \cdot [\{e \cdot (y \cdot ez)\} \cdot \{(x \cdot ey) \cdot (e \cdot ey)\}] \\
&= e \cdot [\{e \cdot (y \cdot ez)\} \cdot x] \\
&= x \cdot (e \cdot (y \cdot ez)) \\
&= x \circ (y \circ z);
\end{aligned}$$

that is, $\circ$ is associative.

Solvability

$$\begin{aligned}
a \circ (ea \cdot eb) &= a \cdot \{e(ea \cdot eb)\} \\
&= a \cdot (eb \cdot ea) && \text{by (9.9)} \\
&= (e \cdot ea) \cdot (eb \cdot ea) \\
&= e \cdot eb && \text{(transitivity)} \\
&= b
\end{aligned}$$

shows that $a \circ x = b$ is solvable. l.c. implies the uniqueness of the solution.

$ba \circ a = ba \cdot ea = be = b$ shows that $y \circ a = b$ is solvable. As before, l.c. implies the uniqueness of the solution. Thus $G(\circ)$ is a group and $G(\cdot)$ is an isogroup. Note that e is the identity for $(\circ)$.

This proves the theorem. □

Remark 9.2. Assume (9.2a) and r.c. By defining $a * b = b \cdot a$, (9.2a) goes over to (9.2) with l.c. Hence $G(\cdot)$ satisfying (9.2a) + r.c. is an isogroup.

We now show that $G(\cdot)$ satisfies the R-condition, thereby proving that $G(\cdot)$ is isotopic to a group.

Suppose $x_1y_2 = x_2y_1,\ x_1y_4 = x_2y_3,\ x_4y_1 = x_3y_2$. Then $y_2y_4 = x_1y_2 \cdot x_1y_4$ (by (9.2a)) $= x_2y_1 \cdot x_3y_3 = y_1y_3,\ y_2x_3 = x_3y_2 \cdot x_3x_3 = x_4y_1 \cdot x_4x_4 = y_1x_4$. Now $x_3y_4 = y_2x_3 \cdot y_2y_4 = y_1x_4 \cdot y_1y_3 = x_4y_3$. Thus the R-condition holds. By Fact 3, (note that $G(\cdot)$ is isotopic to a group, so a quasigroup $G(\cdot)$ is isotopic to a group. □

Now we consider identity for an Abelian group.

Theorem 9.3. [438, 433]. *Suppose* (9.3) *holds with l.c. in* $G(\cdot)$. *Then* $G(\cdot)$ *is a principal iso-Abelian group.*

Proof. We prove this by using the T-condition. It is easy to show that, as in Theorem 9.1,

$$xx = e, \quad ex = x, \quad xy = yx \cdot e, \quad \text{for all } x, y \in G. \tag{9.10}$$

Using (9.3) and (9.10), we have

$$xy \cdot uv = (yx \cdot e) \cdot (vu \cdot e) = vu \cdot yx. \tag{9.11}$$

To prove the T-condition, let

$$x_1y_2 = x_2y_1 \quad \text{and} \quad x_1y_3 = x_3y_1. \tag{9.12}$$

Now (9.3), (9.11), and (9.12) yield

$$y_2y_3 = y_3x_1 \cdot y_2x_1 \overset{(9.11)}{=} x_1y_2 \cdot x_1y_3 \overset{(9.12)}{=} x_2y_1 \cdot x_3y_1 = x_3x_2$$

and

$$x_3y_2 \overset{(9.3)}{=} y_2y_3 \cdot x_3y_3 = x_3x_2 \cdot x_3y_3 \overset{(9.11)}{=} y_3x_3 \cdot x_2x_3 \overset{(9.3)}{=} x_2y_3.$$

Hence the T-condition holds and $G(\cdot)$ is isotopic to an Abelian group. Isotopism is given by

$$\begin{aligned} x \circ y &= xe \cdot y, \\ y \circ x &= ye \cdot x \\ &= (x \cdot xe) \cdot (ye \cdot xe) \\ &= (x \cdot xe) \cdot (xy) \\ &\overset{(9.11)}{=} yx \cdot (xe \cdot x) \\ &\overset{(9.3)}{=} xe \cdot y = x \circ y. \end{aligned} \tag{9.1}$$

This proves the theorem. □

Theorem 9.4. (Hecke [362], Kannappan [435]). *The variety of groups is the variety of all groupoids $G(\cdot)$ defined by the single law*

$$x \cdot [\{(xx \cdot y) \cdot z\} \cdot \{(xx \cdot x) \cdot z\}] = y \quad \text{for all } x, y, z \in G. \tag{9.13}$$

$G(\cdot)$ satisfying (9.13) *is an isogroup.*

Proof. We will show that $G(\cdot)$ is a quasigroup satisfying the transitivity (9.2).

Using left and right multiplication, (9.13) can be written as

$$L_x R_{(xx\cdot x)z} \cdot R_z \cdot L_{xx} = Id, \tag{9.14}$$

for all $x, y, z \in G$, where Id is the identity map. By Fact 2, we see that L_{xx} is 1–1 and L_x is onto, so that L_{xx} is a permutation. By shifting L_{xx} to the right side, we conclude that R_z is 1–1.

Replace x by xx in (9.14) to get

$$L_{xx} R_{((xx\cdot xx)\cdot xx)z} \cdot R_z \cdot L_{xx\cdot xx} = Id. \tag{9.14a}$$

Since L_{xx} is a permutation, we see that $R_{((xx\cdot xx)\cdot xx)\cdot z}$ is onto. But R_z is 1–1. Hence $R_{((xx\cdot xx)\cdot xx)\cdot z}$ is a permutation. Then, from (9.14a), we obtain that R_z is a permutation. Now (9.14) shows that L_x is a permutation. Hence $G(\cdot)$ is a quasigroup.

Set $y = x$ in (9.13) to have

$$x \cdot [\{(xx \cdot x)z\} \cdot \{(xx \cdot x)z\}] = x.$$

Using z as a variable, we get $x \cdot uu = x = x \cdot vv$ for all $u, v \in G$, yielding

$$u \cdot u = \text{constant} = e \text{ (say)} \quad \text{for all } u \in G,$$

and $xe = x$. Then (9.13) becomes

$$\begin{aligned} x \cdot [(ey \cdot z) \cdot (ex \cdot z)] = y, \quad & \text{for all } x, y, z \in G, \\ = x \cdot (ey \cdot ex) & \end{aligned}$$

(take $z = e$) and also $e \cdot ex = x$ $(y = e)$. Thus, using l.c., we get

$$(ey \cdot z) \cdot (ex \cdot z) = ey \cdot ex,$$

which is $yz \cdot xz = yx$ (replace x, y by ex, ey). This is transitivity (9.2). By Theorem 9.1, $G(\cdot)$ is an isogroup. □

Theorem 9.4a. [362, 435]. *The variety of Abelian groups is the variety of all groupoids defined by the identity*

$$x \cdot (yz \cdot yx) = z \quad \text{for all } x, y, z \in G. \tag{9.13a}$$

A groupoid $G(\cdot)$ satisfying (9.13a) *is an iso-Abelian group.*

Proof. The proof is similar to that of Theorem 9.4. We will show that $G(\cdot)$ is a quasigroup satisfying (9.3a).

Using right and left multiplication, (9.13a) can be written as

$$L_x R_{yx} L_y = Id \quad \text{for } x, y \in G. \tag{9.14a}$$

By Fact 2, L_y is 1–1 and L_x is onto, so that L_x is a permutation and then R_{yx} is also a permutation. Using L_x as a permutation, we can get that yx is any $t \in G$ so that R_t is a permutation and $G(\cdot)$ is a quasigroup. Then (9.13a) yields (using y as variable)

$$yx \cdot yx = ux \cdot ux;$$

that is, $t \cdot t = e$ for any $t \in G$.

Putting $y = x = z$ in (9.13a), we have $xe = x$, and $y = x$ in (9.13a) gives $x \cdot xz = z$. Then (9.13a) becomes

$$yz \cdot yx = xz,$$

which is (9.3a). Thus $G(\cdot)$ is isotopic to an Abelian group. □

Result 9.5. (Legendre [604]). *A groupoid $G(\cdot)$ satisfying $xz \cdot yz = xy$ ((9.2) transitivity) and $x \cdot (yy \cdot y) = y$ is a group. Further, if it satisfies $x \cdot xy = y$, then it is an Abelian group* (*see* Theorems 9.1 *and* 9.3).

Result 9.6. (Stamate [765]). *A groupoid G in which to each element x there corresponds an element x' is a group if and only if $ab \cdot c = ad \cdot u$ implies $b = d \cdot uc'$ (a single axiom in terms of two operations, $\cdot$ and unary).*

Result 9.7. *Let $G(\cdot)$ be a groupoid in which to each element x there corresponds an element x' in G. Suppose the following postulate holds: If $aa' \cdot b' = rs' \cdot t'$ implies $b = tr' \cdot s$, then $G(\cdot)$ is an Abelian group (a single axiom in terms of $\cdot$ (multiplication) and unary operation (inversion)).*

Theorem 9.8. (Morgado [640]). *Let $G(\cdot)$ be a groupoid satisfying that the condition*

$$[a \cdot (bb \cdot b)] \cdot (cc \cdot c) = [a \cdot (dd \cdot d)] \cdot (uu \cdot u) \tag{9.15}$$

implies $b = d \cdot eu$. Then, if one defines

$$a \circ b = a \cdot (bb \cdot b) \quad \text{for } a, b \in G, \tag{9.1}$$

then $G(\circ)$ is a group isotopic to $G(\cdot)$.

Proof. Briefly outline the steps leading to the transitivity equation (9.2) using (9.15) a few times. Taking $d = b$, $u = c$ in (9.15), we get $b = b \cdot cc$ and in particular $bb = bb \cdot cc = bb \cdot bb$ and then $bb =$ constant $= e$ and $b = be$ and $e \cdot eb = b$. Now, for transitivity,

$$\begin{aligned} ab &= [e \cdot \{(ab \cdot ab) \cdot ab\}] \cdot (ee \cdot e) \\ &= [e \cdot (aa \cdot a)] \cdot [(eb \cdot eb) \cdot eb], \end{aligned}$$

which by (9.15) yields

$$\begin{aligned} a &= ab \cdot eb \\ &= [e \cdot \{(ab \cdot ab) \cdot ab\}] \cdot (bb \cdot b) \\ &= [e \cdot \{(ac \cdot ac) \cdot ac\}] \cdot (cc \cdot c). \end{aligned}$$

Again by (9.15), we get $ab = ac \cdot bc$, which is (9.2). Now use Theorem 9.1. This proves the result. □

A groupoid $G(\cdot)$ is said to satisfy a generalized associative law (g.a.l.) provided there exists a pair $a, b \in G$ such that

$$[(x \cdot by) \cdot a] \cdot z = x \cdot [b \cdot \{(ya \cdot z)\}] \tag{9.16}$$

for all $x, y, z \in G$.

Result 9.9. (Fotedar [286]). *If a groupoid $G(\cdot)$ satisfies* (9.16)*, then $G(\circ)$ is a semigroup, where $\circ$ is defined by*

$$x \circ y = xa \cdot by \quad \text{for } x, y \in G; \tag{9.17}$$

that is, $G(\cdot)$ is isotopic to a semigroup.

Further, if $G(\cdot)$ is a quasigroup satisfying (9.16)*, then $G(\cdot)$ is isotopic to a group $G(\circ)$, where $\circ$ is given by* (9.17)*.*

A groupoid $G(\cdot)$ satisfies the law (known in [264] as generalized associative) if there exist permutations P_i, Q_i $(i = 1, 2, 3, 4, 5)$ from G to G such that

$$P_5(P_4(P_1x \cdot (P_2y \cdot P_3z)) = Q_5(Q_3(Q_1x \cdot Q_2y)) \cdot Q_4z) \tag{9.18}$$

holds for all $x, y, z \in G$.

Result 9.10. (Evans [264]).

(a) *A finite groupoid with a unit and satisfying* (9.18)*, where P_i are permutations, is associative (that is, a semigroup).*
(b) *A groupoid with a unit, division on one side, and satisfying* (9.18)*, where P_i and Q_i are permutations, is a group.*

Abelian Groups

Suppose $G(\cdot)$ is a groupoid and

$$x \cdot [z \cdot \{y \cdot (x \cdot z)\}] = y \tag{9.19}$$

holds for $x, y, z \in G$. We will prove that $G(\cdot)$ is a quasigroup and (9.3a) holds.

Replace z by $z \cdot (y \cdot x)$ in (9.19) and use (9.19) to have

$$x \cdot [z \cdot (y \cdot x) \cdot \{y \cdot (x \cdot (z \cdot (yx)))\}] = y;$$

that is,

$$x \cdot [\{z \cdot (y \cdot x)\} \cdot z] = y \quad \text{for } x, y, z \in G, \tag{9.20}$$

which can be written as

$$L_x R_z L_z R_x = Id. \tag{9.20a}$$

So, R_x is 1–1, $(\cdot)$ is r.c., and L_x is onto.

Suppose $ya = yb$. Then (9.20) yields

$$a \cdot [\{z \cdot (ya)\} \cdot z] = y = b \cdot [\{z \cdot (yb)\} \cdot z].$$

r.c. implies $a = b$; that is, $(\cdot)$ is l.c. Hence L_x is a permutation and (9.20a) shows that R_x is a permutation.

Replace y by $x \cdot z$ in (9.19), and use l.c. to obtain

$$z \cdot [(x \cdot z) \cdot (x \cdot z)] = z;$$

that is,

$$z \cdot (t \cdot t) = z \quad \text{for } z, t \in G$$

(use solvability) and $tt = \text{constant} = e$ and $ze = z$. Now $z = x$ in (9.19) yields

$$x \cdot (x \cdot y) = y.$$

With y replaced by xy, (9.19) gives

$$z \cdot (xy \cdot xz) = y$$

or

$$z \cdot y = z \cdot [z \cdot (xy \cdot xz)] = xy \cdot xz,$$

which is (9.3a). Thus $G(\cdot)$ is isotopic to an Abelian group. Thus we have proved the following theorem.

Theorem 9.11. *A groupoid $G(\cdot)$ satisfying* (9.19) *is an iso-Abelian group.*

Theorem 9.11a. *A groupoid $G(\cdot)$ satisfying*

$$x \cdot (xz \cdot yz) = y, \quad \textit{for all } x, y, z \in G, \tag{9.19a}$$

is an iso-Abelian group.

Proof. We will show that (9.19) holds. Equation (9.19a) can be written as

$$L_x L_{xz} R_z = Id,$$

so that L_x is onto or $at = b$ is solvable for t. Set $y = x$ in (9.19a) and use $xz = t$ (any $t \in G$) to get

$$x \cdot (tt) = x.$$

Now $z = tt$ in (9.19a) yields

$$x \cdot (x \cdot y) = y.$$

Finally, replace z by xz in (9.19a) to have

$$x \cdot [z \cdot (y \cdot xz)] = y,$$

which is (9.19). This proves the theorem. □

Theorem 9.11b. *A groupoid in which*

(9.19b) $$[x \cdot (z \cdot y)] \cdot (xz) = y, \quad \textit{for } x, y, z \in G,$$

holds is an iso-Abelian group.

Proof. We will prove that (9.19) holds. Equation (9.19b) can be written as

$$R_{xz} L_x L_z = Id,$$

which implies that $(\cdot)$ is l.c. since L_z is 1–1. Suppose $a \cdot y = b \cdot y$. Then use (9.19b) to get

$$[x \cdot (ay)] \cdot (xa) = y = [x \cdot (by)] \cdot (xb),$$

which on using l.c. twice yields $a = b$; that is, $(\cdot)$ is r.c.

Replace y by $y \cdot xz$ in (9.19b) and use r.c. to have

$$[x \cdot \{z \cdot (y \cdot xz)\}] \cdot (x \cdot z) = y \cdot (xz),$$

which is (9.19); that is, $G(\cdot)$ is isotopic to an Abelian group. This completes the proof. □

Theorem 9.11c. (Hecke [362]). *In a groupoid $G(\cdot)$, suppose*

(9.19c) $$x \cdot (zy \cdot zx) = y, \quad \textit{for } x, y, z \in G,$$

holds. Then $G(\cdot)$ is an iso-Abelian group.

Proof. (See also Theorem 9.4a.) Suppose $zy_1 = zy_2$. Then (9.19c) shows that $y_1 = y_2$; that is, $(\cdot)$ is l.c. Replace y by xy in (9.19c) and use l.c. to obtain

$$(z \cdot xy) \cdot (zx) = y,$$

which is (9.19b). By Theorem 9.11b, the result follows. □

9.5 Functional Equation of Identities

In what follows, G will denote either the set of all real numbers $\mathbb{R}$ or the unit circle T (complex numbers z with $|z| = 1$).

Theorem 9.12. (Kannappan [438]). *Let $F : G \times G \to G$ be an onto, continuous map satisfying the functional equation*

$$F(F(x,y),F(x,z)) = F(y,z) \quad \textit{for all } x,y,z \in G. \tag{9.2a$'$}$$

Then the set of idempotents S of G is either G or consists of only one element and

$$F(x,y) = y \tag{9.21}$$

or

$$F(x,y) = f^{-1}(f(y) - f(x)), \quad \textit{when } G = \mathbb{R}, \tag{9.21a}$$

or

$$F(x,y) = f^{-1}(f(y)\,f(y)^{-1}), \quad \textit{when } G = T, \tag{9.21b}$$

accordingly, where $f : G \to G$ is some homeomorphism.

Proof. For $x, y \in G$, define $F(x,y) := xy$, and for subsets U, V of G, $UV := \{uv : u \in U,\ v \in V\}$. Then (9.2a$'$) can be rewritten as

$$xy \cdot xz = yz \quad \text{for } x,y,z \in G. \tag{9.2a}$$

Let $S = \{x \in G : x = xx = x^2\}$ be the set of idempotents of G. Evidently, from (9.2a) it follows that $x^2 \cdot x^2 = x^2$, so that $x^2 \in S$ for every $x \in G$. As every element in S is the square of itself, S is precisely the squares of elements in G.

The continuity of $(\cdot)$ implies that S is closed. Further, it is easy to see that S is connected. Hence S is a closed interval, which may be a single point. The cardinality of S is either > 1 or $= 1$.

First, let us suppose that $|S| > 1$. Let $e \in E$. Then we have

$$\begin{cases} ge = e & \text{for all } g \in G; \text{ that is,} \\ Ge = e. \end{cases} \tag{9.22}$$

Indeed, suppose $t \in Ge \cap S$. Then, by (9.2a) we have

$$t = tt = ge \cdot ge = ee = e. \tag{9.23}$$

First, let us suppose that e is an interior point of S. If $x\ (\neq e) \in Ge$, then since Ge is connected, Ge contains the interval from e to x. But then Ge contains more idempotents other than e. But this by (9.23) cannot be, so in this case (9.22) holds; that is, $Ge = e$.

Now, let us assume that e is an end point of S. Let $xe\ (\neq e) \in Ge$ for some $x \in G$. Then there are two open neighbourhoods U of xe and V of e such that $U \cap V = \phi$. By the continuity of $(\cdot)$, there are open neighbourhoods W of x and N of e such that $WN \subseteq U$. Since $e \in V \cap N$, there is a $t \in V \cap N$ such that t is an interior point of S. Since (9.22) holds for interior points of S, we have $xt = t$ and

that $xt \in WN \subseteq U$, implying $t \in U \cap V$, a contradiction. Thus (9.22) holds true in this case also.

Let $x, y \in G$. Then, by (9.2a) and (9.22), we have $yx \cdot yy = xy$ and $yx \cdot yy = yy$ so that, for every $x, y \in G$, $xy = yy \in S$. The hypothesis $GG = G$ will yield $G \subseteq S$; that is, $G = S$. Thus, when $|S| > 1$, $G = S$. Then, by (9.22), we have

$$xy = y \quad \text{for all } x, y \in G,$$

proving (9.21).

Let us suppose that $|S| = 1$. Let e be the unique idempotent. Then we have

$$xx = e \quad \text{for all } x \in G. \tag{9.24}$$

Now we will show that $Ga = G$ for any $a \in G$; that is, there is an $x \in G$ such that $xa = b$ holds for any $b \in G$.

By hypothesis, there are $y, z, u, v \in G$ such that $b = yz$ and $a = uv$. Now, using (9.2a) and (9.24), we have

$$\begin{aligned} b = yz &= zy \cdot zz = zy \cdot vv \\ \text{by (9.2a)} \quad &= (vu \cdot zy) \cdot (vu \cdot vv) \\ &= (vu \cdot zy) \cdot uv \\ &= (vu \cdot zy) \cdot a; \end{aligned}$$

that is, $xa = b$ is solvable for x.

Hence, by Kannappan [433], G is an isogroup; that is, $G(\circ)$ is isotopic to a group, where $\circ$ is given by

$$x \circ y = xe \cdot y \quad \text{or} \quad (x \cdot ye) \cdot e. \tag{9.1}$$

Further, $G(\circ)$ is a topological group. So, when $G = S$, there is an isomorphism

$$f : S(\circ) \to S(+)$$

such that

$$f(x \circ y) = f(x) + f(y)$$

and

$$f(ye) = -f(y).$$

Since, by (9.1), $xy = xe \cdot y$, it follows that

$$f(xy) = f(xe) + f(y) = f(y) - f(x),$$

so that (9.21a) holds.

Similarly, (9.21b) can be proved. This completes the proof of this theorem.

□

Result 9.13. [438]. *If $F : G \times G \to G$ is an onto, continuous function, satisfying the functional equation* (9.3′)

$$F(F(x, y), F(x, z)) = F(z, y) \quad \text{for } x, y, z \in G,$$

then

$$F(x, y) = f^{-1}(f(x) - f(y)) \quad \text{when } G = \mathbb{R}$$

or

$$F(x, y) = f^{-1}(f(x) \cdot f(y)^{-1}) \quad \text{when } G = T$$

where $f : G \to G$ is some homeomorphism.

Identity (9.3) *can be transformed into identity* (9.3a) *by defining a binary operation* $(*)$ *in G such that*

$$x * y = yx.$$

As a consequence, we have the following result.

Result 9.13a. [438]. *Let $F : G \times G \to G$ be an onto, continuous function satisfying*

$$F(F(x, y), F(z, y)) = F(z, x). \tag{9.3′}$$

Then F has the form

$$F(x, y) = f^{-1}(f(y) - f(x)) \quad \text{when } G \in \mathbb{R},$$
$$F(x, y) = f^{-1}(f(y) f(x)^{-1}) \quad \text{when } G = T.$$

For the functional equation

$$F(F(x, y), F(z, y)) = F(x, z) \tag{9.2′}$$

arising out of the transitivity equation (9.2), see [12, 148].

9.6 Functional Equations Arising Out of Bol, Moufang, and Extra Equations

9.6.1 Bol Equation

From the left Bol identity

$$x(y(xz)) = (x(yx))z \tag{9.5}$$

on a quasigroup, we obtain a generalized Bol identity on a class of quasigroups,

$$P_1(x, P_2(y, P_3(x, z))) = P_4(P_5(x, P_6(y, x)), z), \tag{9.5a}$$

where the P_i's are quasigroup operations on a set H. The general solution of this generalized Bol functional equation is obtained by reducing it to another functional equation,

$$P(x, y + S(x, z)) = P(x, y + \alpha(x)) + z, \tag{9.25}$$

where P and S are quasigroup operations on H and $\alpha(x) = S(x, 0)$. If the operations in the last functional equation are considered on real numbers (or groups), then the solution of this equation is obtained.

One of the most important identities considered in the theory of quasigroups is the Bol identity. A loop $H(\cdot)$ is called a left Bol loop [114] if the identity

$$x(y(xz)) = (x(yx))z \tag{9.5}$$

holds for every $x, y, z \in H$. This identity is called the left Bol identity. The right Bol identity is defined analogously:

$$((zx)y)x = z((xy)x). \tag{9.5$'$}$$

If a loop is both a right and a left Bol, then it is a Moufang loop; i.e., one of the following Moufang identities is satisfied:

$$x(y(xz)) = ((xy)x)z, \tag{9.6}$$

$$xy \cdot zx = (x \cdot yz)x, \tag{9.6a}$$

$$xy \cdot zx = x \cdot (yz \cdot x), \tag{9.6b}$$

$$((zx)y)x = z(x(yx)). \tag{9.6c}$$

It is easily seen that (9.6) is a particular case of (9.5$'$) if $H(\cdot)$ satisfies the elasticity law $(xy)x = x(yx)$, and then (9.5) implies (9.6). On the other hand, the left Moufang identity (9.6) does not imply (9.5), see, for example, [135].

Each identity in a universal algebra defines a generalized identity that is obtained from the given identity by replacing operations of the same arity (number of variables) by different operations of the same arity. Generalized associativity $P_1(P_2(x, y), z) = P_3(x, P_4(y, z))$, generalized bisymmetry $P_1(P_2(x, y), P_3(u, v)) = P_4(P_5(x, u), P_6(y, v))$, and generalized distributivity $P_1(x, P_2(y, z)) = P_3(P_4(x, y), P_5(x, z))$ are examples of such generalized identities. These identities will be considered as functional equations later (for references, see [12]).

To the left Bol identity there corresponds the generalized Bol (left identity) (9.5a). The corresponding identity for the right Bol identity is

$$Q_1(Q_2(Q_3(z, x), y), x) = Q_4(z, Q_5(Q_6(x, y), x)). \tag{9.5b}$$

Of course, all operations P_i, Q_i $(i = 1, 2, \ldots, 6)$ are defined on the same set H.

We shall consider equation (9.5a) on quasigroups; that is, we assume that all P_i and Q_i are quasigroups (quasigroup operations). In the following sections, we reduce equation (9.5a) to a simpler one containing two quasigroups and one loop, and we give a full solution of this equation under some suppositions.

We shall use the following notation. Let P be a binary operation defined on the set H. We denote the translations of P by

$$L_P(a)x = P(a,x), \quad R_P(a)(x) = P(x,a). \tag{9.26}$$

If P is one of the operations P_i $(i = 1,2,\ldots,6)$ from (9.5a), then we shall write $L_i(a)$ instead of $L_{P_i}(a)$ and, moreover, if a is a fixed element k of Q, then we shall write L_i instead of $L_i(k)$. Similar notation is used for right translations.

Let 0 be a fixed element of the set H. We denote $L_P(0) = L'$, $R_P(0) = R'$, and

$$x \underset{P}{+} y = P(R'^{-1}x, L'^{-1}y). \tag{9.26a}$$

Then $H(\underset{P}{+})$ is a loop (Bruck [135]) with the neutral element $P(0,0) = 0_P$.

Let all the operations in (9.5a) be quasigroup operations. Then L_i's and R_i's are permutations of H. If $x = k$ in (9.5a), then from (9.27) we have

$$L_1P_2(y, L_3z) = P_4(L_5R_6y, z);$$

that is,

$$P_4(y,z) = L_1P_2(R_6^{-1}L_5^{-1}y, L_3z).$$

Using this and (9.5a), we have

$$P_1(x, P_2(y, P_3(x,z))) = L_1P_2(R_6^{-1}L_5^{-1}P_5(x, P_6(y,x), L_3z)).$$

With $z = k$, it becomes

$$P_1(x, P_2(y, R_3x)) = L_1R_2'R_6^{-1}L_5^{-1}P_5(x, P_6(y,x)),$$

where $R_2' = R_2(L_3k)$. From these two equations, we obtain

$$L_1^{-1}P_1(x, P_2(y, P_3(x,z))) = P_2(R_2'L_1^{-1}P_1(x, P_2(y, R_3x)), L_3z). \tag{9.27}$$

Let

$$C_1(x,y) = L_1^{-1}P_1(x,y). \tag{9.28}$$

Now (9.27) and (9.28) yield

$$C_1(x, P_2(y, P_3(x,z))) = P_2(R_2'^{-1}C_1(x, P_2(y, R_3x)), L_3z);$$

that is,

$$\begin{aligned} &C_1(x, P_2(R_2'^{-1}y, R_3R_3^{-1}P_3(x,z))) \\ &\qquad = P_2(R_2'^{-1}C_1(x, P_2(R_2'^{-1}y, R_3x)), L_3z). \end{aligned} \tag{9.29}$$

Let

$$C_2(x,y) = P_2(R_2'^{-1}x, R_3y), \quad C_3(x,y) = R_3^{-1}P_3(x, L_3^{-1}R_3y). \tag{9.30}$$

With the help of (9.30), (9.29) can be rewritten as

$$C_1(x, C_2(y, C_3(x, R_3^{-1}L_3z))) = C_2(C_1(x, C_2(y, x)), R_3^{-1}L_3z);$$

that is,

$$C_1(x, C_2(y, C_3(x, z))) = C_2(C_1(x, C_2(y, x)), z). \tag{9.31}$$

From (9.28) and (9.30), it follows that C_1, C_2, and C_3 are quasigroup operations on H since L_1, L_3, R_3, and R_2' are permutations of H.

As every quasigroup is isotopic to a loop [135], we can assume that C_2 is isotopic to a loop; that is, C_2 satisfies

$$C_2(x, y) = R'x + L'y, \tag{9.32}$$

where R' and L' are as in (9.26a). Then $H(+)$ is a loop. By (9.32), (9.31) becomes

$$C_1(x, R'y + L'C_3(x, z)) = R'C_1(x, R'y + L'x) + L'z;$$

that is,

$$C_1(x, y + L'C_3(x, z)) = R'C_1(x, y + L'x) + L'z. \tag{9.33}$$

Now define

$$P(x, y) = C_1(x, y), \quad S(x, y) = L'C_3(x, L'^{-1}y). \tag{9.34}$$

Evidently, P and S are quasigroup operations on H. Using (9.34), we obtain from (9.33)

$$P(x, y + S(x, L'z)) = R'P(x, y + L'x) + L'z;$$

that is,

$$P(x, y + S(x, z)) + R'P(x, y + L'x) + z. \tag{9.35}$$

Putting $z = 0$ in (9.35), we have

$$P(x, y + S(x, 0)) = R'P(x, y + L'x),$$

and thus we get

$$P(x, y + S(x, z)) = P(x, y + \alpha(x)) + z, \tag{9.36}$$

where $\alpha(x) = S(x, 0)$.

Hence we obtain from (9.29), (9.30), and (9.34)

$$\begin{cases} P_1(x, y) = L_1C_1(x, y) = L_1P(x, y), \\ P_2(x, y) = C_2(R_2'x, R_3^{-1}y) = R'R_2'x + L'R_3^{-1}y, \\ P_3(x, y) = R_3C_3(x, R_3^{-1}L_3y) = R_3L'^{-1}S(x, L'R_3^{-1}L_3y), \\ P_4(x, y) = L_1P_2(R_6^{-1}L_5^{-1}x, L_3y) = L_1(R'R_2'L_5^{-1}x + L'R_3^{-1}L_3y), \end{cases}$$

where P and S satisfy (9.36). That is, with $L_1 = \phi,\ RR_2' = \lambda,\ LR_3^{-1} = \mu,\ L_3 = \psi$, and $R_2'R_6^{-1}L_5^{-1} = \theta$, we can rewrite them as

$$\begin{cases} P_1(x,y) = \phi P(x,y), \\ P_2(x,y) = \lambda x + \mu y, \\ P_3(x,y) = \mu^{-1} S(x, \mu\psi y), \\ P_4(x,y) = \phi(R'\theta x + \mu\psi y), \end{cases} \tag{9.37}$$

where P and S satisfy (9.36) and $\phi,\ \lambda,\ \mu,\ \psi,\ R'$, and θ are permutations on H.

From (9.37), we obtain

$$\begin{aligned} P_1(x, P_2(y, R_3x)) &= \phi P(x, \lambda y + \mu R_3 x) \\ &= \phi P(x, \lambda y + L'x) \\ &= \phi\theta P_5(x, P_6(y,x)), \end{aligned}$$

and thus

$$P_5(x, P_6(y,x)) = \theta^{-1} P(x, \lambda y + L'x). \tag{9.38}$$

Thus we have proved the following theorem.

Theorem 9.14. (Belousov and Kannappan [115]). *Let H be an arbitrary set. Let P_i $(i = 1, \ldots, 6)$ be quasigroup operations satisfying* (9.5a). *Then all the solutions of the functional equation* (9.5a) *are given by* (9.37) *and* (9.38), *where $\phi,\ \lambda,\ \mu,\ \psi,\ R',\ \alpha,\ \theta$, and L' are arbitrary permutations of H, and the loop operation $+$ and the quasigroup operations P and S satisfy* (9.36). *Conversely, the P_i's $(i = 1, 2, \ldots, 6)$ given by* (9.37) *and* (9.38), *where P and S satisfy* (9.36), *satisfy the generalized Bol equation* (9.5a).

By a straightforward computation, it is easy to verify the converse part.

Remark (i). The solutions of the right Bol functional equation can be obtained from (9.5a) by replacing all the Q_i's in (9.5b) by the P_i's, where

$$P_i(x,y) = Q_i(y,x).$$

Remark (ii). The generalized Moufang functional equation

$$P_1(x, P_2(y, P_3(x,z))) = P_4(P_5(P_6(x,y),x),z)$$

can also be reduced to (9.36). By the same computation, we obtain (9.27), from which we get (9.31) and finally (9.36). All the solutions are similar to (9.37). The only difference is (9.38), where instead of (9.33) we get

$$P_5(P_6(x,y),x) = \theta^{-1} P(x, \lambda y + L'x).$$

We have seen that the solution of the Bol functional equation (9.5a) is reduced to that of (9.36). Let us now consider equation (9.5a) on the set of real numbers $\mathbb{R}$,

and let us suppose that $H(+)$ is the additive group of real numbers. So, we have to consider (9.36) on $\mathbb{R}$.

Letting $S(x,z)=t$ in (9.36), we get, using S as a quasigroup operation,

$$P(x,y+t)=P(x,y+\alpha(x))+S^{-1}(x,t),\quad x,y,t\in\mathbb{R},$$

where $\alpha(x)=S(x,0)$. Thus, we obtain

$$\lambda_x(y+t)=\mu_x(y)+\nu_x(t)\quad\text{for all } y,t\in\mathbb{R}, \tag{9.39}$$

where

$$\lambda_x(u)=P(x,u),\quad \mu_x(u)=P(x,u+\alpha(x)),\quad \nu_x(u)=S^{-1}(x,u).$$

Equation (9.39) is the well-known Pexider equation. Hence there exists an additive function A_x on $\mathbb{R}$ satisfying

$$A_x(u+v)=A_x(u)+A_x(v)$$

for all $u,v\in\mathbb{R}$, such that

$$\begin{cases}\lambda_x(u)=c(x)+A_x(u),\\ \mu_x(u)=b(x)+A_x(u),\\ \nu_x(u)=d(x)+A_x(u),\end{cases} \tag{9.39a}$$

where $b(x)$, $c(x)$, and $d(x)$ are constants depending on x with $c(x)=b(x)+d(x)$. With the notation

$$F(x,u)=A_x(u),$$

we obtain

$$P(x,u)=c(x)+F(x,u), \tag{9.40}$$

$$S^{-1}(x,u)=d(x)+F(x,u), \tag{9.40a}$$

where F is additive in the second variable for each fixed x. From (9.40), we see that F is a right quasigroup; that is, $F(a,x)=b$ has a unique solution for all a,b. If in (9.40a) we put $S^{-1}(x,u)=w$, then we have

$$\begin{aligned} d(x)+F(x,u)&=w,\\ S(x,w)&=u.\end{aligned}$$

Thus,

$$S(x,d(x)+F(x,u))=u. \tag{9.41}$$

Since $S(x,0)=\alpha(x)$, we have $S^{-1}(x,\alpha(x))=0$. Thus, from (9.40a) with $u=\alpha(x)$, we get $d(x)=F(x,-\alpha(x))$ using F additive in the second variable. Hence (9.41) becomes

$$S(x,F(x,u-\alpha(x)))=u;$$

that is,

$$S(x, F(x, y)) = y + \alpha(x),$$

from which follows, using F as a right quasigroup,

$$S(x, y) = \alpha(x) + F^{-1}(x, y). \tag{9.42}$$

Therefore, we have proved the following theorem.

Theorem 9.14a. [115]. *Let $H(+)$ be the additive group of real numbers. Then the general solution of* (9.36) *is given by* (9.40) *and* (9.42)*, where F is an arbitrary right quasigroup that is additive in the second variable, and $c(x)$ and $\alpha(x)$ are arbitrary functions. Conversely, if P and S are given by* (9.40) *and* (9.42) *with F additive in the second variable, then* (9.36) *holds.*

The converse part can be obtained by easy computation.

Remark (iii). If we take P to be monotonic in the second variable, then we see that $A_x(u)$ is continuous and, for $A_x \not\equiv 0$, $A_x(u) = \beta(x)u$ for arbitrary $\beta(x)$. Hence $P(x, u) = C(x) + \beta(x)u$ and $S(x, u) = \alpha(x) + u\beta(x)$.

Remark (iv). Instead of the additive group of real numbers, we can take an arbitrary group and consider the Pexider type equation on this group. The general solution of (9.36) is given by (9.39a) and hence by (9.40) and (9.42). But the constant functions $c(x)$, $b(x)$, and $d(x)$ should be written in a proper way.

9.6.2 Moufang and Extra Loops

Let $G(\cdot)$ be a loop. Then it is known that the identities (9.6), (9.6a), (9.6b), and (9.6c) are equivalent.

A loop $G(\cdot)$ satisfying any one (and hence all) of these identities is called a Moufang loop (Bruck [135, p. 115]). Corresponding to the well-known identities above, the identical relations

$$(yx \cdot z) \cdot \lambda x = y \cdot (x \cdot (z \cdot \lambda x)), \tag{9.43}$$

$$xy \cdot (z \cdot \lambda x) = (x \cdot yz) \cdot \lambda x, \tag{9.43a}$$

$$xy \cdot (z \cdot \lambda x) = x \cdot (yz \cdot \lambda x), \tag{9.43b}$$

$$(xy \cdot \lambda x)z = x \cdot (y \cdot (\lambda x \cdot z)) \tag{9.43c}$$

are considered, where $\lambda : G \to G$ is any mapping.

We prove the following results regarding these equations. In what follows, we make use of the following fact about inverse property loops $G(\cdot)$: If (U, V, W) is an autotopism of G, then (W, JVJ, U) and (JUJ, W, V), where J is the inverse mapping, are also autotopisms of G [135, p. 112].

Result 9.15. (Kannappan and Taylor [522]). *If* (9.42) *holds in a loop $G(\cdot)$, then $G(\cdot)$ is Moufang and* (9.43a), (9.43b), *and* (9.43c) *also hold in $G(\cdot)$. Further, $\mu x \in N$ (nucleus), where $\mu : G \to G$ is a mapping such that $x \cdot \mu x = \lambda x$. Conversely, if $G(\cdot)$ is a Moufang loop with $\mu x \in N$ for all $x \in G$, then the identity* (9.43) *and* (9.43a), (9.43b), *and* (9.43c) *all hold in G.*

9.6.3 Extra Loop

In a loop $G(\cdot)$, the following identities are equivalent:

$$yx \cdot zx = (y \cdot xz) \cdot x, \tag{9.8}$$

$$(xy \cdot z) \cdot x = x \cdot (y \cdot zx), \tag{9.8a}$$

$$xy \cdot xz = x \cdot (yx \cdot z). \tag{9.8b}$$

A loop $G(\cdot)$ satisfying any one (and hence all) of these identities is called an extra loop (Fenyves [270]).

Let $\alpha : G \to G$ be any mapping. Let us consider the following identities in $G(\cdot)$ similar to the identities above:

$$yx \cdot (z \cdot \alpha x) = (y \cdot xz) \cdot \alpha x, \tag{9.44}$$

$$(xy \cdot z) \cdot \alpha x = x \cdot (y \cdot (z \cdot \alpha x)), \tag{9.44a}$$

$$xy \cdot (\alpha x \cdot z) = x \cdot ((y \cdot \alpha x) \cdot z). \tag{9.44b}$$

Result 9.16. [522]. *Let $G(\cdot)$ be a loop and let $\alpha : G \to G$ be any mapping such that the identity* (9.44) *holds in $G(\cdot)$. Then $G(\cdot)$ is Moufang and* (9.44a) *and* (9.44b) *also hold in $G(\cdot)$. Also $\mu x \in N$ (nucleus), where $\mu : G \to G$ is any mapping such that $\mu x = \alpha x \cdot x$. Conversely, if $G(\cdot)$ is Moufang and $\mu x \in N$, then* (9.44), *and so* (9.44a) *and* (9.44b) *also are satisfied in $G(\cdot)$.*

9.6.4 Characterizations

We give characterizations for Moufang loops, extra loops, and groups with the use of the results above.

Result 9.17. [522]. *Let $G(\cdot)$ be a loop and $\lambda, \mu, \alpha : G \to G$ satisfy $\lambda x = x \cdot \mu x = x \cdot (\alpha x \cdot x)$. Then the following conditions are equivalent:*

(a) *Identity* (9.43), (9.43a), *or* (9.43c) *is satisfied for all $x, y, z \in G$.*
(b) *$G(\cdot)$ is Moufang, and $\mu x \in N$ (nucleus) for all $x \in G$.*
(c) *Identity* (9.44) *or* (9.44a) *holds for all $x, y, z \in G$.*

9.6.5 Characterization of Moufang Loops

Let $G(\cdot)$ be a loop. Then

$$\begin{aligned} G(\cdot) \text{ is Moufang } &\Leftrightarrow yx \cdot zx^{-1} = (y \cdot xz)x^{-1} && \text{hold in } G(\cdot) \\ &\Leftrightarrow (xy \cdot z)x^{-1} = x \cdot (y \cdot (z \cdot x^{-1})) && \text{hold in } G(\cdot) \\ &\Leftrightarrow xy \cdot (x^{-1}z) = x \cdot ((y \cdot x^{-1})z) && \text{hold in } G(\cdot). \end{aligned}$$

9.6.6 Extra Loops

Let $G(\cdot)$ be a loop. Then $G(\cdot)$ is an extra loop $\Leftrightarrow$ $G(\cdot)$ is Moufang and $x^2 \in N \Leftrightarrow$. Identity (9.43), (9.43a), or (9.43b) holds with $\lambda x = x^3$.

9.6.7 Groups

Let $G(\cdot)$ be a loop. Then $G(\cdot)$ is a group $\Leftrightarrow$. Identity (9.43), (9.43a), or (9.43b) holds with $\lambda x = x^2$.

The Moufang identities (9.6), (9.6a), (9.6b), and (9.6c) induce generalized Moufang identities. Here we consider the Moufang functional equation

$$C_1(x, C_2(y, C_3(x, z))) = C_4(C_5(C_6(x, y), y), z) \tag{9.45}$$

on GD-groupoids $(S_1, S_5, S; C_1)$, $(S_2, S_4, S_5; C_2)$, $(S_1, S_3, S_4; C_3)$, $(S_1, S_3, S; C_4)$, $(S_6, S_1, S_7; C_5)$, $(S_1, S_2, S_6; C_6)$. The solution is obtained by reducing it to a simpler functional equation

$$C(x, v \cdot P(x, v')) + C(x, v \cdot \psi x) \cdot v', \tag{9.45a}$$

$(\cdot)$ being a loop operation and C and P right G-quasigroup operations (see Theorem 9.14).

9.7 GD-groupoid

$(S_1, S_2, S_3; \alpha)$ is said to be a GD-groupoid, where S_i $(i = 1, 2, 3)$ are distinct nonempty sets and $\alpha : S_1 \times S_2 \to S_3$ is onto (surjective) such that the translations $L_\alpha(a)y = \alpha(a, y)$ and $R_\alpha(b)x = \alpha(x, b)$ are onto for every fixed $a \in S_1,\ b \in S_2$.

If the mappings $L_\alpha(a)$ and $R_\alpha(a)$ are also 1–1 for every $a \in S_1$ and $b \in S_2$, we obtain a right and left G-quasigroup, respectively. If a GD-groupoid is both a left and a right G-groupoid, then it is called a G-quasigroup.

Result 9.18. (Pavlović [665]). *All solutions of the functional equation* (9.45) *with the assumption that C_1 and C_2 are right G-quasigroups and C_4 is a left G-quasigroup are given by*

$$C_1(x, v) = \gamma\, C(x, v), \quad C_2(y, u) = \alpha y \cdot \beta u, \quad C_3(x, z) = \beta^{-1} P(x, \phi z),$$
$$C_4(r, z) = \gamma\, (\delta^{-1}\gamma \cdot \phi z), \quad C_5(C_6(x, y), x) = \delta C(x, \alpha y \cdot \psi x),$$

where the mappings $\beta : S_4 \to S_5,\ \gamma : S_5 \to S,\ \delta : S_5 \to S_7$ are 1–1 *and onto the mappings $\alpha : S_2 \to S_5,\ \phi : S_3 \to S_5,\ \psi : S_1 \to S_5$ are onto, and the loop $S_5(\cdot)$ and the right G-quasigroups $(S_1, S_5, S_5; C)$ and $(S_1, S_5, S_5; P)$ satisfy equation* (9.45a).

Similarly, functional equations induced by extra l.c. identities can be considered.

9.8 More Identities

9.8.1 Entropic, Bisymmetric, or Mediality Identity

A groupoid G is said to be *entropic* provided

$$ab \cdot cd = ac \cdot bd \quad \text{holds for all } a, b, c, d \in G. \tag{9.46}$$

Obviously, every Abelian group is an entropic groupoid. We will consider the converse.

Theorem 9.19. (Morgado [639]). *Let G be an entropic groupoid. Then $G(\cdot)$ is an iso-Abelian group if and only if*

$$a = b \cdot (ba \cdot cc), \quad \text{for } a, b, c \in G, \tag{9.47}$$

holds.

Proof. We present a simple, direct, shorter proof different from that in [639].

Suppose (9.47) holds. $(\cdot)$ is l.c. Let $ba = bd$. Replacing a by d in (9.47), we get $a = d$. Applying l.c. to (9.47) results in $cc =$ constant $= e$, say. Now $a = b$ in (9.47) gives $b = be$. Then

$$xz = xz \cdot e = xz \cdot yy = xy \cdot zy$$

(the only place where an entropic groupoid is used), which is transitivity (9.3). We will now show that the T-condition holds.

Suppose $x_1y_2 = x_2y_1$ and $x_1y_3 = x_3y_1$, $x_1, x_2, x_3, y_1, y_2, y_3 \in G$. Then, using (9.46), $x_2y_1 \cdot x_3y_1 = x_1y_2 \cdot x_1y_3$ implies $x_2x_3 = e \cdot y_2y_3 = y_3y_3 \cdot y_2y_3 = y_3y_2$. Finally,

$$x_2y_3 \overset{(9.46)}{=} x_2x_3 \cdot y_3x_3 = x_3y_2 \cdot y_3x_3 = e \cdot y_2x_3 = x_3x_3 \cdot y_2x_3 = x_3y_2.$$

Thus the T-condition holds. Hence $G(\cdot)$ is isotopic to an Abelian group $G(\circ)$. The isotopy may be given by

$$a \circ b = a \cdot eb.$$

This completes the proof. □

Remark 9.19a. Suppose $a = (cc \cdot ab) \cdot b$ holds instead of (9.47). Then define $a * b = b \cdot a$. Then

$$a = b * (cc \cdot ab) = b * (ab * cc) = b * ((b * a) * (c * c)),$$

which is (9.47).

Theorem 9.19b. *Instead of* (9.47) *in Theorem* 9.19, *suppose*

$$a = b \cdot (cc \cdot ab), \quad \text{for } a, b, c \in G, \tag{9.47b}$$

holds.

Proof. Suppose $ab = db$. Equation (9.47b) imples r.c. Let $ba = bd$. Equation (9.47b) yields

$$\begin{aligned} a &= b \cdot (ca \cdot cb) \quad \text{(by (9.46))} \\ &= b \cdot (ba \cdot bb), \end{aligned}$$

but

$$d = b \cdot (bd \cdot db),$$

which shows that $a = d$; that is, l.c. Use l.c. and r.c. to have $cc =$ constant $= e$, and $b = a$ in (9.47b) shows that $a = ae$.

$yz = yz \cdot xx = yx \cdot zx$, which is transitivity. As in Theorem 9.19, it can be shown that the T-condition holds. Thus $G(\cdot)$ is an iso-Abelian group. □

Result 9.20. (Bednarek and Wallace [109]). *Suppose G is a Hausdorff groupoid in which $(x, y) \to xy$ is continuous and $xy \cdot yz = xz$ holds. Then*

(i) *mediality* (9.47) *holds;*
(ii) $S^2 = S \neq \phi$, *where $S =$ the set of all idempotents in G;*
(iii) $e \in S \Rightarrow xe = xe \cdot e$ *and* $ey = e \cdot ey$;
(iv) $GS = SG = S^2$;
(v) $e \in S \Rightarrow Ge$ *is a subsemigroup of S. Moreover, $Ge = Ge \cdot Ge$ and $x \to xe$ is a retractioning mapping* (*continuous homomorphism f such that $f^2 = f$*);
(vi) *G is a semigroup;*
(vii) *the minimal ideal of G is $G^2 = (S \cap Ge) \cdot (eSe) \cdot (eG \cap S)$;*
(viii) *suppose $xy \cdot yz = xz$ is implied by $y \cdot (xy \cdot zx) = yz$. Moreover, if G is a compact connected Hausdorff space, then the mapping $(x, y) \to xy$ is constant.*

9.9 Left Inverse Property (l.i.p.), Crossed-Inverse (c.i.), and Weak Inverse Property (w.i.p.) Loops

We now give characterizations for (i) groupoids (loops) with identities satisfying l.i.p., (ii) the c.i. property, and (iii) w.i.p. as subvarieties of groupoids satisfying certain conditions (single identities).

9.9.1 l.i.p. Loops

A groupoid $G(\cdot)$ is said to have the *left inverse property* if, for every $x \in G$, there is at least one $a \in G$ such that $a \cdot xy = y$ for all $y \in G$. Consequently, the equation $cx = d$ is solvable for x. Similarly, we can define *right inverse property* in a groupoid that implies the solvability of $yc = d$. If $G(\cdot)$ satisfies both the l.i.p. and r.i.p., then G is said to have the *inverse property* and in this case $G(\cdot)$ is a quasigroup (Bruck [135, p. 111]).

Let $G(\cdot)$ be a groupoid. $G(\cdot)$ is an iso-l.i.p. (r.i.p.) groupoid (or loop) provided there is an l.i.p. (r.i.p.) groupoid (or loop) $G(*)$ with unity that is a principal isotope of $G(\cdot)$ such that $*$ and $\cdot$ are connected by the relation $x*y = x \cdot \lambda(y)$ $(x*y = \rho(x) \cdot y)$ for all $x, y \in G$, where $\lambda(\rho)$ is the left (right) inverse operator of $G(*)$.

Result 9.21. (Kannappan [434]). *A necessary and sufficient condition that a groupoid $G(\cdot)$ is an iso-l.i.p. groupoid is that the identity*

$$vv \cdot [\{(tt)x \cdot (uu)(xy)\} \cdot (ww)] = y$$

or

$$[vv \cdot \{(tt)x \cdot (uu)(xy)\}] \cdot (ww) = y$$

holds for all $x, y, u, v, w, t \in G$.

9.9.2 c.i. Loops

Loops $G(\cdot)$ in which the equivalent identities $xy \cdot \rho(x) = y$ or $\lambda(x) \cdot yx = y$ hold for all $x, y \in G$ are called *crossed-inverse* (c.i.) loops, where $\rho(\lambda)$ is the right (left) inverse operator [72, 135].

Let $G(\cdot)$ be a groupoid. Then $G(\cdot)$ is called an iso-c.i. loop if there is a c.i. loop $G(*)$ that is a principal isotope of $G(\cdot)$ such that $\cdot$ and $*$ are related either by $x * y = \rho(x) \cdot y$ or by $x * y = x \cdot \lambda(y)$.

Result 9.22. [434]. *Let $G(\cdot)$ be a right quasigroup. Then $G(\cdot)$ is an iso-c.i. loop provided*

$$x \cdot yx = y \cdot (tt)$$

holds for all $x, y, t \in G$.

9.9.3 w.i.p. Loops

Let $G(\cdot)$ be a loop with identity 1. Then G is said to be a loop with *weak inverse properties* if whenever three elements $x, y, z \in G$ satisfy the relation $xy \cdot z = 1$, they also satisfy the relation $x \cdot yz = 1$. The properties $y \cdot \rho(xy) = \rho(x)$ or $\lambda(xy) \cdot x = \lambda(y)$ are equivalent to the definition of w.i.p. loop above (Basarab [104]).

Let $G(\cdot)$ be a groupoid. We say that $G(\cdot)$ is an iso-w.i.p. loop provided there is a w.i.p. loop $G(*)$ that is a principal isotope of $G(\cdot)$ such that $*$ and $\cdot$ are connected either by $x \cdot y = \lambda(x) * y$ or by $x \cdot y = x * \rho(y)$.

Result 9.23. [589]. *A necessary and sufficient condition that a left quasigroup $G(\cdot)$ is an iso-w.i.p. loop is that*

$$[(tt) \cdot x] \cdot (zx) = (uu) \cdot z$$

holds for all $x, z, t, u \in G$.

9.10 Steiner Loops

Mendelsohn [634] has defined the concept of a generalized triple system as follows. Let S be a set of v elements. Let T be a collection of b subsets of S, each of which contains three elements arranged cyclically and such that any ordered pair of elements of S appears in exactly one cyclic triplet (note the cyclic triplet $\{a, b, c\}$

contains the ordered pairs $ab,\ bc,\ ca$ but not $ba,\ cb,\ ac$). When such a configuration exists, we will refer to it as a generalized triple system. If we ignore the cyclic order of the triples, the generalized triple system is a B.I.B.D.

There is a one-to-one correspondence between generalized triple systems of order v and quasigroups of order v satisfying the identities $x^2 = x,\ (xy)x = x(yx) = y$. The term generalized Steiner quasigroup means a quasigroup that satisfies the identities above.

Let G be a generalized Steiner quasigroup of order v. From G, a loop G^* with operator $*$ is constructed as follows. The elements of G^* are the same as those of G together with an extra element e. Multiplication in G^* is defined as follows: $a * e = e * a = a;\ a * a = e$; and for $a, b \in G$ with $a \neq b$, define $a * b = a \cdot b$. It follows easily that G^* is a loop with identities $x * e = e * x = x,\ x * x = e,\ x * (y * x) = (x * y) * x = y$. Also, the correspondence between generalized Steiner quasigroups and generalized Steiner loops is a bijection.

A loop that satisfies the identities

$$xx = e, \quad xe = x = ex, \quad x \cdot yx = y = xy \cdot x, \quad \text{for } x, y \in G, \tag{9.48}$$

is called a *generalized Steiner loop* (g.s.l.). In [640] is given the identity (9.49) characterizing g.s.l. Five equivalent identities were found immediately afterwards in 1970 to characterize g.s.l.s. Now the time has come to put them in the following theorem.

Theorem 9.24. *A groupoid $G(\cdot)$ is a generalized Steiner loop if and only if G satisfies any one of the identities*

$$a \cdot [((bb) \cdot c) \cdot a] = c, \tag{9.49}$$

$$[a \cdot c(bb)] \cdot a = c, \tag{9.49i}$$

$$a \cdot (ca \cdot bb) = c, \tag{9.49ii}$$

$$(a \cdot ca) \cdot bb = c, \tag{9.49iii}$$

$$bb \cdot (a \cdot ca) = c, \tag{9.49iv}$$

$$(bb \cdot a) \cdot (ca \cdot dd) = c, \tag{9.49v}$$

for $a, b, c, d \in G$.

Proof. First we consider (9.49), treated in [640]. Here we present a different, simpler proof.

In (9.49), replace c by $(dd \cdot k) \cdot bb$ and use (9.49) to get

$$a \cdot ka = (dd \cdot k) \cdot bb,$$

and

$$bb \cdot (a \cdot ka) = k, \quad \text{for } a, b, k \in G. \tag{9.50}$$

Suppose $ka = ua$. Then (9.50) shows that $k = u$; that is, $(\cdot)$ is r.c. Apply r.c. in (9.50) to obtain $bb = \text{constant} = e$ (say). Then (9.49) becomes

$$a \cdot (ec \cdot a) = c.$$

Put $c = e$ to obtain $a \cdot ea = e = ea \cdot ea$, implying $ea = a$. So, $a \cdot ca = c$.

First, $a = e$ in (9.49) yields $ce = c$, and replacing a by ac gives $ac \cdot (c \cdot ac) = c$; that is, $ac \cdot a = c$. This proves it is a g.s.l.

Now we take up (9.49i):

(9.49i) $$[a \cdot c(bb)] \cdot a = c \quad \text{for } a, b, c \in G.$$

Put $c = dd \cdot bb \cdot (c \cdot dd)$ in (9.49i) to have

$$ac \cdot a = bb \cdot (c \cdot dd),$$

(9.50i) $$(ac \cdot a) \cdot bb = c, \quad \text{for } a, b, c \in G.$$

Now $ac = ad$ in (9.50i) gives $c = d$ and l.c. and

$$(ac \cdot a) \cdot bb = (ac \cdot a) \cdot dd.$$

l.c. implies $bb =$ constant $= e$ and $(ac \cdot a) \cdot e = c$. Now $c = e$ gives $(ae \cdot a) \cdot e = e = (ae \cdot a) \cdot (ae \cdot a)$; that is, $ae \cdot a = e = ae \cdot ae$, implying $ae = a$. Equation (9.49i) yields $ac \cdot a = c$.

First $a = e$ gives $ec = c$, and replacing a by ca shows that $(ca \cdot c) \cdot ca = c$ or $a \cdot ca = c$. This proves it is a g.s.l.

Next we consider

(9.49ii) $$a \cdot (ca \cdot bb) = c \quad \text{for } a, b, c \in G.$$

$ca = da$ in (9.49ii) yields $c = d$ and thereby r.c. Suppose $ca = cd$. Equation (9.49ii) is $d \cdot (cd \cdot bb) = c$, which by using r.c. gives $a = d$; that is, l.c. Apply l.c. to $a \cdot (ca \cdot bb) = c = a \cdot (ca \cdot dd)$ to get $bb = e$. Now (9.49ii) becomes $a \cdot (ca \cdot e) = c$. As before, $c = a$ shows that $ae = a$ and $a \cdot ca = c$. As before, $a = e$ gives $ec = c$ and changing a to ac gives $ac \cdot (c \cdot ac) = c$ or $ac \cdot a = c$. This proves it is a g.s.l.

Now we take up

(9.49iii) $$(ac \cdot a) \cdot bb = c.$$

Now $ac = ad$ in (9.49iii) gives $c = d$ and l.c. Then (9.49iii) yields $bb =$ constant $= e$ and $(ac \cdot a) \cdot e = c$. First, as before, $c = e$ gives $(ae \cdot a) \cdot e = e = (ae \cdot a) \cdot (ae \cdot a)$; that is, $ae \cdot a = e$. As in the proof of (9.49i), we obtain $ae = a = ea$ and $ac \cdot a = c = a \cdot ca$.

Next in line is

(9.49iv) $$bb \cdot (a \cdot ca) = c \quad \text{for } a, b, c \in G.$$

First $ca = da$ gives $c = d$ and r.c. and then $bb = e$ and $e \cdot (a \cdot ca) = c$. Now $c = e$ yields $a \cdot ea = e = ea \cdot ea$, implying $ea = a$ and $a \cdot ca = c$. Use previous arguments to get $ae = a$, $ac \cdot a = c$.

Finally, we consider

(9.49v) $$(bb \cdot a)(ca \cdot dd) = c \quad \text{for } a, b, c, d \in G.$$

First $ca = ua$ shows $c = u$ and r.c. and then $bb = e$. Now $ea \cdot (ca \cdot e) = c$. Set $a = c$ to have $ec \cdot e = c$. Put $c = e$ to get $ea \cdot (ea \cdot e) = e$ or $ea \cdot a = e$. Hence $ea = a$. So, $ce = c$ and $a \cdot ca = a$. Finally, a to ac yields $ac \cdot (c \cdot ac) = a$ or $ac \cdot a = a$. This proves it is a g.s.l. □

9.11 Bol Loop and Power Associativity

It is known that a right (left) Bol loop is power associative. Here we prove that the right Bol loop satisfying (9.5′) is power associative by using the hexagonal closure condition.

Suppose (9.5′) holds. Put $y = x^{-1}$ in (9.5′) to obtain

$$((zx) \cdot x^{-1}) \cdot x = z \cdot (xx^{-1} \cdot x) = zx,$$

implying $zx \cdot x^{-1} = z$; that is, the right inverse property holds in G. Suppose

$$x_1y_2 = x_2y_1, \quad x_1y_3 = x_2y_2 = x_3y_1, \quad \text{for } x_i, y_i \in G \ (i = 1, 2, 3). \tag{9.51}$$

Use (9.51) and the right inverse property to have

$$\begin{aligned} x_1 &= x_2y_2 \cdot y_3^{-1}, \\ x_1y_2 &= (x_2y_2 \cdot y_3^{-1}) \cdot y_2 \overset{(9.5')}{=} x_2 \cdot (y_2y_3^{-1} \cdot y_2) \\ &= x_2y_1; \end{aligned}$$

that is, $y_2y_3^{-1} \cdot y_2 = y_1$. Now

$$\begin{aligned} x_3y_1 &= x_3 \cdot (y_2y_3^{-1} \cdot y_2) \\ &= (x_3y_2 \cdot y_3^{-1}) \cdot y_2 \quad \text{by (9.5′)} \\ &= x_2y_2, \end{aligned}$$

so that $x_2 = x_3y_2 \cdot y_3^{-1}$; that is, $x_2y_3 = x_3y_2$. Hence the H-condition holds. Thus a right Bol loop is power associative.

9.12 More Functional Equations

We consider the generalized equations of associativity, transitivity, and bisymmetry

$$(x1y)2z = x3(y4z), \tag{9.52}$$

$$(x1z)2(y3z) = x4y, \tag{9.53}$$

$$(x1y)2(z3u) = (x4z)5(y6u), \tag{9.54}$$

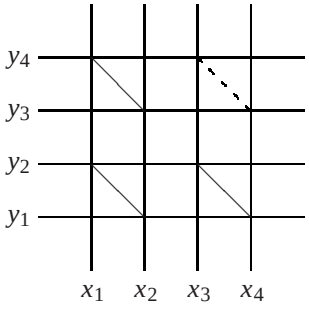
where $1, 2, 3, 4, 5, 6$ are binary operations and x, y, z, u belong to certain sets.

First we consider the generalized associativity (9.52) on the quasigroups $G(i)$, $i = 1, 2, 3, 4$.

Theorem 9.25. *If* (9.52) *holds for all* $x, y, z \in G$ *and the* $G(i)$ $(i = 1, 2, 3, 4)$ *are quasigroups, then each* $G(i)$ $(i = 1$ *to* $4)$ *is isotopic to the same group.*

Proof. (See also Taylor [802].) We will prove that the R-condition holds. Suppose

(9.52a) $$x_1 4 y_2 = x_2 4 y_1, \quad x_1 4 y_4 = x_2 4 y_3, \quad \text{and } x_3 4 y_2 = x_4 4 y_1.$$



Set $y = x_2,\ z = y_1$ and $y = x_1,\ z = y_2$; $y = x_2, z = y_3$ and $y = x_1,\ z = y_4$; and $y = x_3,\ z = y_2$ and $y = x_4,\ z = y_1$ separately in (9.52) and use (9.52a) to get

(9.55) $$(x 1 x_1) 2 y_2 = (x 1 x_2) 2 y_1,$$

(9.56) $$(x 1 x_2) 2 y_3 = (x 1 x_1) 2 y_4,$$

(9.57) $$(x 1 x_3) 2 y_2 = (x 1 x_4) 2 y_1.$$

Choose $z_1, z_2 \in G$ such that $z_1 1 x_3 = z_2 1 x_1$ (note that $G(1)$ is a quasigroup).

Put $x = z_2$ in (9.56) to get

(9.56a) $$(z_2 1 x_2) 2 y_3 = (z_2 1 x_1) 2 y_4.$$

Let $x = z_1$ in (9.57) and $x = z_2$ in (9.55) to have

$$(z_1 1 x_3) 2 y_2 = (z_1 1 x_4) 2 y_1;$$

that is,

$$(z_2 1 x_1) 2 y_2 = (z_1 x_4) 2 y_1$$

and

$$(z_2 1 x_1) 2 y_2 = (z_2 1 x_2) 2 y_1.$$

Thus

$$(z_1 1 x_4) 2 y_1 = (z_2 1 x_2) 2 y_1$$

or

$$z_1 1 x_4 = z_2 1 x_2.$$

Now (9.56a) becomes

$$(z_1 1 x_4) 2 y_3 = (z_2 1 x_1) 2 y_4 = (z_1 1 x_3) 2 y_4;$$

that is, by (9.52),

$$z_1 3(x_4 4 y_3) = z_1 3(x_3 4 y_4).$$

Since $G(1)$ is a quasigroup, $x_4 4 y_3 = x_3 4 y_4$; that is, the R-condition holds in $G(4)$. So $G(4)$ is isotopic to a group. Similarly, that $G(i)$ $(i = 1, 2, 3)$ are isotopic to a group can be proved. This completes the proof.

Result 9.25a. [802]. *Generalized equation of transitivity. If* (9.53) *holds for* x, y, z *in* G *and* $G(i)$ $(i = 1$ *to* $4)$ *are quasigroups, then each* $G(i)$ *is isotopic to the same group.*

Result 9.25b. [802]. *If* (9.54) *holds for all* $x, y, z, u \in G$ *and each* $G(i)$ $(i = 1, 2, \ldots, 6)$ *is a quasigroup, then every* $G(i)$ *is isotopic to the same Abelian group.*

9.13 Generalized Groupoids

Now we consider the generalized equations of associativity and bisymmetry on GD-groupoids; that is, on different sets. A generalized groupoid (X, Y, V, i) will be denoted by "i".

Let $(X, Y, V, 1)$, $(V, W, S, 2)$, $(X, U, S, 3)$, and $(Y, W, U, 4)$ be GD-groupoids.

An element $b \in Y$ is a *right generator* of the generalized groupoid (X, Y, V, i) if $Xib = \{xib \mid x \in X\} = V$.

An element $c \in V$ is a *right solution element* of i if $xiy = c$ has a solution $y \in Y$ for all $x \in X$.

9.13.1 Generalized Associativity

Result 9.25c. (See Taylor [803].) *Let* 1, 2, 3, *and* 4 *be generalized groupoids that satisfy* (9.52)*. If* 1 *has a left generator,* 2 *is right cancellative,* 3 *is left cancellative and has a left solution element, and* 4 *has a right solution element, then the solution of the generalized associativity equation is given by*

$$x1y = \phi^{-1}(f(x)g(y)), \quad v2z = \phi(v)h(z),$$
$$x3u = f(x)\psi(u), \qquad y4z = \psi^{-1}(g(y)h(z)),$$

where $(S, \cdot)$ *is a group,* $f : X \to S$, $g : Y \to S$, $h : Z \to S$ *are surjections, and* $\phi : V \to S$, $\psi : U \to S$ *are bijections* (*see also* [113]).

9.13.2 Generalized Bisymmetry

Let $(X, Y, H, 1)$, $(H, S, G, 2)$, $(W, U, S, 3)$, $(X, W, T, 4)$, $(T, V, G, 5)$, and $(Y, U, V, 6)$ be generalized groupoids that satisfy (9.54).

Result 9.25d. [803]. *Let* 1, 2, 3, 4, 5, *and* 6 *be generalized groupoids that satisfy* (9.54)*. If* 1, 4, *and* 6 *each have both a right solution element and a right generator and if* 2 *and* 5 *are cancellative, then there exists an Abelian group* $G(+)$, *surjections*

$g : X \to G$, $f : Y \to G$, $h : W \to G$, $k : U \to G$, and bijections $\alpha : H \to G$, $\beta : S \to G$, $\gamma : T \to G$, $\delta : V \to G$ that satisfy

$$x1y = \alpha^{-1}(f(x) + g(y)), \quad r2s = \alpha(r) + \beta(s),$$
$$z3u = \beta^{-1}(h(z) + k(u)), \quad x4z = \gamma^{-1}(f(x) + h(z)),$$
$$t5v = \gamma(t) + \delta(v), \quad y6u = \delta^{-1}(g(y) + k(u)).$$

10

Functional Equations from Information Theory

In this chapter, information theory using desirable properties and representation, Shannon's entropy, directed divergence, generalized directed divergence, entropy of order α and degree β, and weighted entropy are treated. The Fundamental equations of entropy and their generalizations and sum form equations and their generalizations are treated. Distance measures and inset measures are performed, and applications are treated.

10.1 Introduction

In this chapter, we briefly consider various measures of information and their salient properties and bring out their natural connections to many functional equations and their characterizations. It is not an exaggeration to say that the concept of information has played an important role from its beginning since it has so many aspects and applications in various fields. A fundamental question in information theory is how to measure the amount of information, in other words, how to define information. Several approaches are possible. One is a pragmatic approach that starts from certain particular problems of information theory and accepts as measures of the amount of information the quantities which present themselves in the solution. Another approach may be described as the axiomatic or postulational approach in which one starts with certain properties that a reasonable measure of information has to possess, and after this it is a purely mathematical question of determining all the expressions that possess these postulates. This is the point of view adopted by C.E. Shannon in 1948 in his now famous fundamental paper "A mathematical theory of communication". This led to the study and application of functional equations in information theory. These two points of view are according to the opinion of Rényi [699]—they are not as opposed to each other as they seem to be but rather are compatible and even complement each other. As our interest is focussed on functional equations, this chapter follows the point of view of the postulational approach.

Information is what is provided by a descriptive statement that affects our knowledge; that is, it is the information content of the statement. "Information" is also

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known as uncertainty. When one faces an alternative, one finds oneself in a state of uncertainty. Any process of knowledge or decision entails the transition from one state of uncertainty to another. This cannot take place without losing or gaining something. The term involved in this process is what is called information. The end result is information gain. Note that we can interpret a decrease or loss of information as an increase or gain of information. This was the reason for saying that a measure of the amount of uncertainty is the same as a measure of the amount of information.

There is another word that is associated with information, which is entropy, suggested by von Neumann and (originating with Boltzmann, Gibbs, and Shannon) introduced in physics. The entropy of a probability distribution can be interpreted not only as a measure of uncertainty but also as a measure of information. As a matter of fact, the amount of information that we get when we observe the result of an experiment can be taken to be numerically equal to the amount of uncertainty concerning the outcome of the experiment before carrying it out. We shall use the word entropy to indicate a measure of uncertainty.

Since the advent of the notion of information, due originally to Hartley, it has been much developed by Shannon, and it is with Shannon's name that it is associated, as *Shannon entropy*. Since then, quite a large number of papers have appeared on the subject. It was developed by many authors, including Feinstein, Kullback, Rényi, Wiener, and others, and was also advanced into the ergodic theory by Gelfand, Khinchin, Kolmogoroff, Yaglon, and others and applied to investigate the theory of transformation with invariant measures by Kolmogoroff and others. The study of information theory, broadly speaking, includes entropy theory, coding theory, modulation theory, noise reduction theory, theory of channels, and applications in ecology, psychology, detective work in tracking down criminals, forecasting theory, statistical mechanics and thermodynamics, statistics, marketing, economics, image processing, etc. Mathematicians, physicists, electrical engineers, physiologists, phoneticists, psychologists, statisticians, and others have all contributed to this area.

The first definition of the notion of entropy was given in its generality by Shannon, well known as *Shannon's entropy* (SE) [732], as

$$\text{(SE)} \qquad H_n(P) = -\sum_{i=1}^{n} p_i \log p_i,$$

where $P = (p_1, p_2, \ldots, p_n)$, $0 \le p_i \le 1$, $\sum_{i=1}^{n} p_i = 1$, $n \ge 2$, $n \in Z^+$, has been deduced from various statements of axioms. Hartley's entropy is a measure of uncertainty with a minimum amount of advance information. More precisely, it is a measure of the uncertainty one has about the outcome of the experiment when the only relevant information is the total number of possible outcomes n. Hartley introduced the measure [344]

$$H_n^*(P) = \log\{\text{number of positive } p_i\text{'s}\} = \log N(P),$$

where $P = (p_1, \ldots, p_n)$, $0 \le p_i \le 1$, $\sum p_i = 1$; that is, the measure depends only on the number of nonzeros in P, and $N(P)$ is the number of nonzeros among $p_1, p_2, \ldots, p_n$ (that is, the number of events in an experiment with probability greater than 0).

A key feature of Shannon's information theory is the discovery that the colloquial term information can often be given a mathematical meaning as a numerically measurable quantity, on the basis of a probabilistic model, in such a way that the solution of many important problems of information storage and transmission can be formulated in terms of this measure of the amount of information. This information measure has a very concrete operational interpretation: roughly, it equals the minimum number of binary digits needed, on average, to encode the message in question. The coding theorems of information theory provide such overwhelming evidence for the adequacy of Shannon's information measure that to look for essentially different measures of information might appear to make no sense at all. Moreover, it has been shown by several authors, starting with Shannon, that the measure of the amount of information is uniquely determined by some rather natural postulates.

The aim of this chapter is to characterize entropy and other measures of information uniquely by means of physically plausible properties; that is, axiomatic characterization of measures of information from the functional equation point of view, following Shannon's lead. The most common approach is based on some probability theory that includes dependence on the outcomes of an experiment and the corresponding probabilities. The idea is to define an information measure by a set of intuitively natural properties leading to a system of functional equations and functional inequalities, the solution of which gives rise to characterization of entropies. This axiomatic procedure advanced by Shannon is probably the most effective and is a most useful one for practical applications. The literature in this field is voluminous. A partial list is [43, 699, 700, 732, 698, 97, 283, 48, 804, 569, 570, 344, 238, 239, 242, 571, 245, 57, 164].

10.2 Notation, Basic Notions, and Preliminaries

Let $\Gamma_n = \{P = (p_1, p_2, \ldots, p_n);\ p_i \ge 0 \text{ with } \sum_{i=1}^{n} p_i = 1\}$ be the set of all complete, discrete, nonnegative probability distributions of length n ($n \ge 2$); $\Gamma_n^0 = \{P = (p_1, p_2, \ldots, p_n);\ p_i > 0 \text{ and } \sum_{i=1}^{n} p_i = 1\}$ be the interior of Γ_n ($n \ge 2$); $\Delta_n = \Gamma_n$ or Γ_n^0; $I = [0, 1]$, $I_0 =]0, 1[$, $I_1 = I$ or I_0; $0 \cdot \log 0 = 0$; $0^\beta = 0$; $0 \cdot \log \frac{0}{0} = 0$; logarithms are to base 2;

$$\text{(SF)} \quad s(x) = -x \log x - (1-x)\log(1-x) \qquad \text{for } x \in I_1, \text{ Shannon function,}$$

$$\text{(10.1)} \quad s(x, y) = -x \log y - (1-x)\log(1-y) \qquad \text{for } x, y \in I_1,$$

$$\text{(10.1a)} \quad s_\beta(x) = \frac{1}{2^{1-\beta} - 1}(x^\beta + (1-x)^\beta - 1) \qquad \text{for } x \in I,\ \beta \ne 0,$$

(10.1b) $$s_L(x) = xL(x) + (1-x)L(1-x),$$

(10.1c) $$s(x,y,z) = x(a_1 \log x + b_1 \log y + c_1 \log z) + y(a_2 \log y + b_2 \log z) + c_2 z \log z$$

for $P \in \Delta_n$, $Q \in \Delta_m$ define the product $P * Q = (p_i q_j) \in \Delta_{mn}$ $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$; and a sequence of map $\mu_n : \Delta_n^k \to \mathbb{R}$ $(n \geq 2,\ k \in Z_+^*,\ k \geq 1)$ is called an *information measure*.

10.2.1 Properties, Postulates, and Axioms

Shannon emphasized that "the hard core of information theory is essentially a branch of mathematics, a strictly deductive system; a thorough understanding of the mathematical foundations and its communication application is surely a prerequisite to other applications". Many of the inventions deal with the storage, transmission, transformation, and retrieval of information. Information occurs in various forms: oral, written, visual, electronic, mechanical, etc. From a mathematical point of view, the essence of information is its quantity, and the basic problem is how to measure it. As mentioned earlier, we adopt the axiomatic (mathematical) approach of Shannon. Axiomatic characterization of measures of information in general and Shannon's entropy in particular is done by judiciously choosing the properties (algebraic and analytic) satisfied by them. These lead to the study of many functional equations. Apart from Cauchy functional equations, which occur in various characterizations, and numerous other functional equations, there are two categories of functional equations that play a prominent role. One is known as the *fundamental equation of information theory*,

(FEI) $$f(x) + (1-x) f\left(\frac{y}{1-x}\right) = f(y) + (1-y) f\left(\frac{x}{1-y}\right),$$

for $x, y \in [0, 1[$ with $x + y \in I$, and its generalizations, and the other is referred to as the *sum form equation*,

(SFE) $$\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i y_j) = \sum_{i=1}^{n} f(x_i) + \sum_{j=1}^{m} f(y_j),$$

for $(x_i) \in \Delta_n$, $(y_j) \in \Delta_m$, and its variations. These equations will be studied in the subsequent sections, pointing out the close connection between these two types of functional equations.

10.2.2 Desirable Properties—Postulates

Algebraic Properties

***n*-symmetry**

$\mu_n(P_1, P_2, \ldots, P_k) = \mu_n(P_{\alpha(1)}, P_{\alpha(2)}, \ldots, P_{\alpha(k)})$, where $P_j \in \Delta_n$ $(j = 1, 2, \ldots, k)$ and α is a permutation on $\{1, 2, \ldots, k\}$.

Expansibility $(k = 1)$

$\mu_{n+1}(p_1, p_2, \ldots, p_n, 0) = \mu_n(p_1, p_2, \ldots, p_n)$ for all $(p_i) \in \Gamma_n$; that is, adding an outcome with zero probability does not change the amount of information expected.

Additivities

(m, n)**-Additive, Subadditive, and Strong Additive**

Consider an experiment P with the possible outcomes c_i $(i = 1, 2, \ldots, n)$ of probabilities $p_i = p(c_i)$ and another experiment Q with possible outcomes c'_j $(j = 1, 2, \ldots, m)$ of probabilities $q_j = p(c'_j)$; denote by $P * Q$ the combination of the two experiments with $c_i \cap c'_j$, $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$ as possible outcomes. It is natural to expect when the two experiments are independent (that is, every outcome c_i of the first experiment is independent of every outcome c'_j of the second experiment) that the information expected from the combination of the two experiments is equal to the sum of the information expected from the individual experiments. That is,

$$\begin{aligned}\mu_{mn}(P^{(1)} * Q^{(1)} \| P^{(2)} * Q^{(2)} \| \cdots P^{(k)} * Q^{(k)}) \\ = \mu_m(Q^{(1)} \| Q^{(2)} \cdots \| Q^{(k)}) + \mu_n(P^{(1)} \| P^{(2)} \| \cdots P^{(k)})\end{aligned}$$

for $P^{(1)}, P^{(2)}, \ldots, P^{(k)} \in \Delta_n$, $Q^{(1)}, Q^{(2)}, \ldots, Q^{(k)} \in \Delta_m$, $P^{(i)} * Q^{(i)} \in \Delta_{nm}$. This is called (m, n)-*additive*.

In general, we cannot expect more information from a combination of two experiments than the sum of the information that can be obtained from the individual experiments. For $k = 1$, this is called *subadditive* and is expressed by

$$\mu_{mn}(P * Q) \leq \mu_n(P) + \mu_m(Q)$$

for $P \in \Delta_n$, $Q \in \Delta_m$.

(m, n)**-Strong Additivity**

$$\begin{aligned}\mu_{mn}(p_1q_{11}, p_1q_{12}, \ldots, p_1q_{1n}, p_2q_{21}, \ldots, p_2q_{2n}, \ldots, p_mq_{m1}, \ldots, p_mq_{mn}) \\ = \mu_m(p_1, p_2, \ldots, p_m) + \sum_{j=1}^{m} p_j \mu_n(q_{j1}, q_{j2}, \ldots, q_{jn})\end{aligned}$$

for $(p_i) \in \Delta_m$, $(q_{jk})_{k=1}^n \in \Delta_n$, $j = 1, 2, \ldots, m$.

n**-Recursivity or Branching**

$$\mu_n \begin{pmatrix} p_{11}, p_{12}, \ldots, p_{1n} \\ p_{21}, p_{22}, \ldots, p_{2n} \\ \vdots \\ p_{k1}, p_{k2}, \ldots, p_{kn} \end{pmatrix} = \mu_{n-1} \begin{pmatrix} p_{11} + p_{12}, p_{13}, \ldots, p_{1n} \\ p_{21} + p_{22}, p_{23}, \ldots, p_{2n} \\ \vdots \\ p_{k1} + p_{k2}, p_{k3}, \ldots, p_{kn} \end{pmatrix} + \phi_{n-1} \begin{pmatrix} p_{11}, p_{12} \\ p_{21}, p_{22} \\ \vdots \\ p_{k1}, p_{k2} \end{pmatrix},$$

$P_i = (p_{i1}, p_{i2}, \ldots, p_{in}) \in \Delta_n$ $(i = 1, 2, \ldots, k)$, and some function $\phi_{n-1} : I_1^2 \to \mathbb{R}$.

In a special case $(k = 1)$, the sequence $\{\mu_n\}$ is said to be *recursive* provided

$$\mu_n(p_1, p_2, \ldots, p_n) = \mu_{n-1}(p_1 + p_2, p_3, \ldots, p_n) + \phi_{n-1}(p_1, p_2)$$

for all $P = (p_1, p_2, \ldots, p_n) \in \Delta_n$ and for some function $\phi_{n-1} : I_1^2 \to \mathbb{R}$.

A special form of ϕ_{n-1} used in many characterizations is $\phi_{n-1}(p_1, p_2) = (p_1 + p_2)\mu_2\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right)$ with $p_1 + p_2 > 0$.

Suppose we want to know the outcome of a given experiment P with outcomes $c_1, c_2, \ldots, c_n$ of probabilities $p_1, p_2, \ldots, p_n$. Alternatively, we could perform the experiment Q with outcomes $c_1 \cup c_2, c_3, \ldots, c_n$ and corresponding probabilities $p_1 + p_2, p_3, \ldots, p_n$. This may result in some loss of information. The basic assumption underlying the recursivity is that the amount of information that is lost by observing the outcome of Q instead of P depends only on the probabilities p_1 and p_2 of the respective events c_1 and c_2.

Normality

In the case $k = 1$, $\mu_2\left(\frac{1}{2}, \frac{1}{2}\right) = 1$.

Representation

Sum Property

A sequence of measures $\mu_n : \Delta_n^k \to \mathbb{R}$ $(k \geq 1)$ is said to have the *sum property* provided there exists a function $f : I_1^n \to \mathbb{R}$ such that

$$\mu_n(P_1 \| P_2 \| \cdots \| P_k) = \sum_{i=1}^{n} f(p_{1i}, p_{2i}, \ldots, p_{ki})$$

for all $P_j = (p_{j1}, p_{j2}, \ldots, p_{jn}) \in \Delta_n$ $(j = 1, 2, \ldots, k)$ and for all $n \geq 2$, and the function f is called a *generating function* of the sequence $\{\mu_n\}$.

When $k = 1$, the representation is

$$\mu_n(P) = \sum_{i=1}^{n} f(p_i) \quad \text{for } P \in \Delta_n,$$

and when $k = 2$, the sum property is

$$\mu_n(P \| Q) = \sum_{i=1}^{n} f(p_i, q_i) \quad \text{for } P, Q \in \Delta_n.$$

Quasilinearity

For the case $k = 1$, there exists a continuous, strictly monotonic function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\mu_n(P) = \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi(-\log p_j)\right) \quad \text{for } p_j > 0$$

for $P \in \Gamma_n^0$.

Regularity

We illustrate regularity for the case $k = 1$. $\mu_n(p_1, p_2, \ldots, p_n)$ is continuous on Δ_n. Define

$$f(p) = \mu_2(p, 1-p) \quad \text{for } p \in I_1. \tag{10.2}$$

f is called the *Shannon information function.* Some of the regularity properties used are:

- f is Lebesgue integrable on I_1,
- f is measurable,
- f is continuous at 0,
- f is continuous on I_1,
- $\lim_{p \to 0^+} \mu_2(p, 1-p) = 0$ (smallness for small probabilities), which gives the intuitive statement that we obtain very little information out of an experiment with two possible outcomes, one of which is almost certain and the other almost impossible,
- f is increasing or decreasing on $]0, \frac{1}{2}[$,
- $0 \le f(p) \le k$ for $p \in I_1$, k some constant,
- f is bounded on an interval or bounded on a set of positive measure,
- f is differentiable, etc.

10.2.3 Characterization of Information Measures

Some of the well-known measures of information are

(SE) $H_n(P) = -\sum_{i=1}^{n} p_i \log p_i$ (Shannon's entropy),

(KI) $I_n(P\|Q) = -\sum p_i \log q_i$ (Kerridge's inaccuracy),

(dd) $D_n(P\|Q) = \sum p_i \log \frac{p_i}{q_i}$ (directed divergence),

(gdd) $GD_n(P\|Q\|R) = \sum p_i \log \frac{q_i}{r_i}$ (generalized directed divergence),

(10.3) $H_n^{\alpha}(P) = \frac{1}{1-\alpha} \log\left(\sum_{i=1}^{n} p_i^{\alpha}\right)$ (entropy of order α),

$\alpha \neq 1,\ \alpha > 0$

$$H_n^{\beta}(P) = \frac{1}{2^{1-\beta}-1}\left(\sum_{i=1}^{n} p_i^{\beta} - 1\right) \quad \text{(entropy of degree or type } \beta\text{)}, \tag{10.4}$$
$$\beta \neq 1$$

$$H_n(P, W) = -\sum w_i p_i \log p_i \qquad \text{(weighted entropy)}, \tag{10.5}$$

etc., for $P, Q, R \in \Delta_n$, $W = (w_1, \ldots, w_n)$, $w_i \geq 0$.

There are many algebraic properties that are shared by these measures. It is easy to check that all these measures are symmetric; whereas the first four measures are additive and recursive, the last three are nonadditive; all the measures above have representation or the sum property except Rényi's entropy of order α; the generating functions are $f(p) = -p \log p$, $g(p, q) = -p \log q$, or $p \log \frac{p}{q}$, $h(p, q, r) = p \log \frac{q}{r}$, $k(p, w) = -wp \log p$, $f^{\beta}(p) = c(p^{\beta} - p)$, etc. There are properties that are special to individual measures.

Now we deal with the axiomatic characterization of these measures through some of the properties they possess. We start off with the Shannon entropy.

10.2.4 Shannon Entropy and Some of Its Generalizations

Shannon's entropy satisfies the inequality known as *Shannon's inequality*,

$$-\sum_{i=1}^{n} p_i \log p_i \leq -\sum_{i=1}^{n} p_i \log q_i, \tag{SI}$$

$P = (p_i)$, $Q = (q_i) \in \Delta_n$, which will be dealt with in Chapter 16, on inequalities.

We will show that whereas symmetry and recursivity lead to the fundamental equation of information (FEI), the additivity and the sum representation lead to the sum form functional equation (SFE). First we take up (FEI).

10.2a Fundamental Equation of Information—Axiomatic Characterizations

Several authors have given different sets of postulates to characterize Shannon's entropy (SE). Let $\mu_n : \Delta_n \to \mathbb{R}$, $n \geq 2$, be a sequence of information measures.

Shannon [732] used the postulates μ_n is n-symmetric, μ_n is continuous in each variable, $\mu_2(\frac{1}{2}, \frac{1}{2}) = 1$, $\mu_n(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) \leq \mu_{n+1}(\frac{1}{n+1}, \ldots, \frac{1}{n+1})$, and

$$\begin{aligned} &\mu_n(p_1, p_2, \ldots, p_{m-1}, p_m q_1, \ldots, p_m q_{n-m+1}) \\ &\quad = \mu_m(p_1, p_2, \ldots, p_m) + p_m \mu_{n-m+1}(q_1, q_2, \ldots, q_{n-m+1}) \end{aligned} \tag{10.6}$$

to characterize (SE), whereas Kinchin [534] showed that n-symmetry, continuity in each variable of μ_n, strong additivity, expansibility, normality, nonnegativity, $\mu_n(P) \geq 0$, and $\mu_n(P) \leq \mu_n(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ characterize (SE). But Faddeev ([267, 266]; see also Rathie and Kannappan [688] and Aczél [13]) gave the simplest of postulates and proved the following theorem that is of basic importance in information theory.

Theorem 10.1. *Suppose μ_n on Γ_n is n-symmetric, f given by* (10.2) *is continuous for $p \in [0,1]$, $\mu_2(\frac{1}{2},\frac{1}{2}) = 1$, and μ_n is n-recursive,*

$$\begin{aligned}\mu_n(p_1, p_2, \ldots, p_n) &= \mu_{n-1}(p_1 + p_2, p_3, \ldots, p_n) \\ &\quad + (p_1 + p_2)\mu_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right),\end{aligned} \tag{10.7}$$

for $P = (p_i) \in \Gamma_n$ with $p_1 + p_2 > 0$. Then $\mu_n(P) = H_n(P)$.

Proof (1). The proof given by Faddeev consists of the following steps.

The function $L : Z_+^* \to \mathbb{R}$ defined by

$$L(n) = \mu_n\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right), \quad n \geq 1,$$

(i) satisfies (L) (called a completely additive number-theoretical function), and
(ii) $\lim_{n\to\infty} (L(n+1) - L(n)) = 0$.

(i) and (ii) together with $L(2) = 1$ imply $L(n) = \log n$. Now claim μ_n is expansible. Using symmetry and recursivity (10.7), for $p \in I$,

$$\begin{aligned}\mu_3(p, 1-p, 0) &= \mu_2(1, 0) + \mu_2(p, 1-p) \\ &= \mu_2(p, 0, 1-p) \\ &= \mu_2(p, 1-p) + p\mu_2(1, 0).\end{aligned}$$

Thus $\mu_2(1,0) = 0$ and $\mu_3(p, 1-p, 0) = \mu_2(p, 1-p)$. Further,

$$\begin{aligned}\mu_{n+1}(p_1, \ldots, p_n, 0) &= \mu_{n+1}(p_1, 0, p_2, \ldots, p_n) \\ &= \mu_n(p_1, p_2, \ldots, p_n) + p_1\mu_2(1, 0) \\ &= \mu_n(p_1, p_2, \ldots, p_n);\end{aligned}$$

that is, μ_n is expansible. μ_n is also strongly additive (see [43, p. 62]). Let $r = \frac{m}{n}$ be a rational, $m \leq n$. Now applying $(2, n)$-strong additivity using symmetry and expansibility,

$$\mu_{2n}\left(\underbrace{\left(1 - \frac{m}{n}\right)\frac{1}{n-m}, \ldots, \left(1 - \frac{m}{n}\right)\cdot\frac{1}{n-m}}_{(n-m)\text{ times}}, \underbrace{0, \ldots, 0}_{m\text{ times}}, \underbrace{\frac{m}{n}\cdot\frac{1}{m}, \ldots, \frac{m}{n}\cdot\frac{1}{m}}_{m\text{ times}}, \underbrace{0, \ldots, 0}_{(n-m)\text{ times}}\right)$$

$$= \mu_2\left(1-\frac{m}{n},\frac{m}{n}\right)+\left(1-\frac{m}{n}\right)\mu_n\left(\underbrace{\frac{1}{n-m},\ldots,\frac{1}{n-m}}_{(n-m)\text{ times}},\underbrace{0,\ldots,0}_{m\text{ times}}\right)$$
$$+\frac{m}{n}\mu_n\left(\underbrace{\frac{1}{m},\ldots,\frac{1}{m}}_{m\text{ times}},\underbrace{0,\ldots,0}_{(n-m)\text{ times}}\right)$$
$$= \mu_2\left(1-\frac{m}{n},\frac{m}{n}\right)+\left(1-\frac{m}{n}\right)\mu_{n-m}\left(\frac{1}{n-m},\ldots,\frac{1}{n-m}\right)$$
$$+\frac{m}{n}\mu_m\left(\frac{1}{m},\ldots,\frac{1}{m}\right).$$

Hence

$$\begin{aligned} f(r) &= \mu_2\left(\frac{m}{n},1-\frac{m}{n}\right)\\ &= \mu_n\left(\frac{1}{n},\ldots,\frac{1}{n}\right)-\frac{m}{n}\mu_m\left(\frac{1}{m},\ldots,\frac{1}{m}\right)\\ &\quad -\left(1-\frac{m}{n}\right)\mu_{n-m}\left(\frac{1}{n-m},\ldots,\frac{1}{n-m}\right)\\ &= \log n-\frac{m}{n}\log m-\left(1-\frac{m}{n}\right)\log(n-m)\\ &= -r\log r-(1-r)\log(1-r)\quad \text{for } r \text{ rational in } I_1. \end{aligned}$$

Since f is continuous, $f(x)=s(x)$, the Shannon function (SF).

Now apply recursivity (10.7) to obtain $\mu_n(P)=H_n(P)$.

Proof (2). Now we present a simple proof of the theorem involving (FEI), but first we give the motivation of (FEI) [196].

10.2b The Fundamental Equation of Information Theory

Let α be a random event, and denote the probability of α by $x=p(\alpha)$. If we perform an experiment concerning α, then we denote the measure of information obtained by the experiment by $I(\alpha)$. We suppose that the unknown quantity $I(\alpha)$ depends only on the probability x of α; that is,

$$I(\alpha)=f(x),$$

where $f(x)$ is a real-valued function that is defined in the closed interval $[0,1]$. Let β be another event such that $\alpha\beta=0$ and let $y=p(\beta)$ be its probability. After the experiment concerning α, we perform a second experiment concerning β. The second experiment yields the relative information $I(\beta|\overline{\alpha})$, and we define it as the information belonging to the probability

$$\frac{y}{1-x} = \frac{p(\beta)}{p(\overline{\alpha})}$$

taken with the weight $1 - x = p(\overline{\alpha})$; that is,

$$I(\beta|\overline{\alpha}) = (1-x)\, f\left(\frac{y}{1-x}\right).$$

Having performed both experiments, we obtain the gain of information

$$I(\alpha, \beta) = I(\alpha) + I(\beta|\overline{\alpha}) \quad (\alpha\beta = 0),$$

and we suppose that this is independent of the order in which the experiments concerning α and β were performed. This assumption means that the unknown function $f(x)$ satisfies the functional equation

$$\text{(FEI)} \qquad f(x) + (1-x)\, f\left(\frac{y}{1-x}\right) = f(y) + (1-y)\, f\left(\frac{x}{1-y}\right)$$

for all pairs (x, y) such that

$$(x, y) \in D = \{(x, y) : x, y \in [0, 1[, (x + y) \in I\}.$$

Now we will derive (FEI) from the axioms. Using 3-symmetry, 3-recursivity, (10.7), and (10.2), we have for $P = (p_1, p_2, p_3) \in \Gamma_3$,

$$\begin{aligned} H_3(p_1, p_2, p_3) &= H_2(p_1 + p_2, p_3) + (p_1 + p_2) H_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right) \\ &= f(p_3) + (1 - p_3)\, f\left(\frac{p_1}{1 - p_3}\right) \\ &= H_3(p_2, p_3, p_1) \\ &= H_2(p_2 + p_3, p_1) + (p_2 + p_3) H_2\left(\frac{p_2}{p_2 + p_3}, \frac{p_3}{p_2 + p_3}\right) \\ &= f(p_1) + (1 - p_1)\, f\left(\frac{p_3}{1 - p_1}\right). \end{aligned}$$

Thus we obtain the fundamental equation of information theory (FEI) for $x, y \in [0, 1[$ with $x + y \in [0, 1]$, where $f : [0, 1] \to \mathbb{R}$.

Now we determine the regular solution of (FEI).

Solution of (FEI). Suppose $f : I \to \mathbb{R}$ satisfies (FEI) for $x, y \in [0, 1[$ with $(x+y) \in I$ and is integrable or measurable. Then f is given by

$$\text{(10.8)} \qquad f(x) = cs(x) + dx \quad \text{for } x \in I,$$

where $s(x)$ is the Shannon function (SF).

Note that the integrability or measurability of f implies the continuity of f, which in turn implies the differentiability of f (Aczél and Daróczy [43]), and then f has derivatives of all orders. Here we need the derivative of order two only.

Now differentiate (FEI) with respect to x and then the resulting equation with respect to y to get

$$\frac{y}{(1-x)^2} f''\left(\frac{y}{1-x}\right) = \frac{x}{(1-y)^2} f''\left(\frac{x}{1-y}\right).$$

Now put $\frac{y}{1-x} = u$, $\frac{x}{1-y} = v$, so that $u, v \in]0,1[$, $y = \frac{u(1-v)}{1-uv}$, $x = \frac{v(1-u)}{1-uv)}$,

$$u \cdot \frac{1-uv}{1-v} f''(u) = v \cdot \frac{1-uv}{1-u} f''(v)$$

or

$$u(1-u) f''(u) = \text{constant} = c,$$

and

$$f''(u) = \frac{c}{u} + \frac{c}{1-u}.$$

Hence

$$\begin{aligned} f(u) &= c[-u \log u - (1-u)\log(1-u)] + bu + d \\ &= cs(u) + bu + d, \end{aligned}$$

where b, c, d are constants. This f satisfies (FEI) provided $d = 0$. Thus the solution of (FEI) is (10.8).

Continuation of Proof (2) of the Theorem. If $f = \mu_2$ and since μ_2 is symmetric, f is symmetric; that is, $f(x) = f(1-x)$ and then

$$f(x) = cs(x).$$

If we use normality, then $c = 1$ and $f(x) = s(x)$. Then apply recursivity to obtain $\mu_n(P)$.

This completes the proof of the theorem. □

Remark 10.2. The (FEI) has been studied by several authors under various regularity conditions (see [43, 13, 193, 202, 129, 530, 477, 603, 806]) such as being integrable in I (Tverberg [806]), measurable in $]0,1[$ (Lee [603]), monotonically increasing in $]0, \frac{1}{2}]$ (Kendall [530]), under continuous at 0 (Daróczy [193]), continuous in I, nonnegative, bounded in I (Daróczy [199]), under continuity of $f(p) = \mu_2(p, 1-p)$, $\lim_{p\to 0} f(p) = 0$, and $f(p)$ is monotone in $]0, \frac{1}{2}]$ (Daróczy [193], Borges [129]), and boundedness on an interval or on a set of positive measure (Diderrich [223]). Boundedness from one side, like nonnegativity, is not enough to characterize the Shannon entropy for $f(x) = s(x) + \frac{1}{x(x-1)} D(x)^2$, where D is a derivative satisfying (D) (Diderrich [224], Aczél [35]).

Problem. Determine all nonnegative solutions of (FEI).

Now we find the general solution of (FEI).

Theorem 10.3. (Aczél and Daróczy [43]). *The most general solution of* (FEI) *is given by*

$$f(x) = \begin{cases} xL(x) + (1-x)L(1-x) + bx & \text{for } x \in]0,1[, \\ 0 & \text{for } x = 0 \text{ and } x = 1, \end{cases} \tag{10.8a}$$

where L satisfies the logarithmic equation (L) *and b is a constant.*

Proof. Without loss of generality, we can take

$$f(x) = f(1-x) \quad \text{for } x \in I. \tag{10.9}$$

Then f is said to be *symmetric*.

Indeed, $y = 1 - x$ in (FEI) gives

$$f(1-x) = f(x) + (1-2x)f(1) \quad \text{for } x \in I.$$

Define

$$\begin{aligned} g(x) &= f(x) + f(1-x) - f(1) \quad \text{for } x \in I, \\ &= 2f(x) - 2xf(1). \end{aligned}$$

Then

$$\begin{aligned} g(x) + (1-x)g\left(\frac{y}{1-x}\right) &= 2f(x) - 2xf(1) \\ &\quad + (1-x)\left[2f\left(\frac{y}{1-x}\right) - \frac{2y}{1-x}f(1)\right] \\ \text{by (FEI)} \quad &= 2f(y) + 2(1-y)f\left(\frac{x}{1-y}\right) - (2x+2y)f(1) \\ &= g(y) + (1-y)g\left(\frac{x}{1-y}\right); \end{aligned}$$

that is, g satisfies (FEI) and is symmetric.

We can take $f(1) = 0$. Define

$$k(x) = f(x) - f(1)x \quad \text{for } x \in I.$$

Then k is a solution of (FEI) with $k(1) = 0$. With $x = 0$, (FEI) shows that $f(0) = 0$. So, when f is symmetric, $f(0) = 0 = f(1)$. Now, for symmetric $f : I \to \mathbb{R}$ satisfying (FEI), define $F : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}$ by

$$F(x,y) = (x+y)f\left(\frac{x}{x+y}\right) \quad \text{for } x, y > 0. \tag{10.10}$$

Then

(10.11) $F(x,y) = F(y,x)$ for $x, y > 0$ (symmetric; use f symmetric),

(10.11a) $F(tx,ty) = tF(x,y)$ for $x, y, t > 0$ (homogeneity).

Further,

(10.11b) $F(x+y,z) + F(x,y) = F(x,y+z) + F(y,z)$ for $x, y, z > 0$

(see also [395]). Using (10.10) and (FEI),

$$\begin{aligned}
&F(x+y,z) + F(x,y)\\
&\quad = (x+y+z)\, f\left(\frac{x+y}{x+y+z}\right) + (x+y)\, f\left(\frac{x}{x+y}\right)\\
&\quad = (x+y+z)\left[f\left(1 - \frac{x+y}{x+y+z}\right) + \frac{x+y}{x+y+z}\, f\left(\frac{\frac{x}{x+y+z}}{\frac{(x+y)}{x+y+z}}\right)\right]\\
&\quad = (x+y+z)\left[f\left(\frac{x}{x+y+z}\right) + \frac{y+z}{x+y+z}\, f\left(\frac{1-\frac{x+y}{x+y+z}}{1-\frac{x}{x+y+z}}\right)\right]\\
&\quad = (x+y+z)\, f\left(\frac{x}{x+y+z}\right) + (y+z)\, f\left(\frac{z}{y+z}\right)\\
&\quad = F(x,y+z) + F(z,y)\\
&\quad = F(x,y+z) + F(y,z) \quad \text{(use (10.11))}.
\end{aligned}$$

F on $\mathbb{R}_+^* \times \mathbb{R}_+^*$ has an extension $F_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying (10.11), (10.11a), and (10.11b) with $F_1(x,y) = F(x,y)$ for $x, y > 0$ (see [43]). For this F_1, there exists a function $\beta : \mathbb{R} \to \mathbb{R}$ such that

(10.12) $F_1(u,v) = \beta(u+v) - \beta(u) - \beta(v)$ for $u, v \in \mathbb{R}$,

(10.13) $\beta(u,v) = u\beta(v) + v\beta(u)$ $u, v \in \mathbb{R}$.

Since F_1 satisfies (10.11), (10.11a), and (10.11b), put $t = 0$ and $y = 0$ in (10.11b) to get

$$F_1(0,0) = 0,\ F_1(x,0) = F_1(0,z) = F_1(0,0) = 0.$$

Now define operations $\oplus$ and $\odot$ on $\mathbb{R}^2$ by

$$\begin{aligned}
(u,x) \oplus (v,y) &= (u+v, x+y+F_1(u,v)),\\
(u,x) \odot (v,y) &= (uv, uy+vx), \quad \text{for } u, v, x, y \in \mathbb{R}.
\end{aligned}$$

Then $\mathbb{R}^2$ with these operations is a commutative ring, and the mapping $\alpha : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\alpha(u,x) = u, \quad u, x \in \mathbb{R},$$

is a homomorphism onto $\mathbb{R}$.

Then there exists a subring $S(\oplus, \odot)$ of $\mathbb{R}^2(\oplus, \odot)$ that is isomorphic to $\mathbb{R}(+, \cdot)$ under the mapping α and a function $\beta : \mathbb{R} \to \mathbb{R}$ such that every element of S is of the form $(t, \beta(t))$ for $t \in \mathbb{R}$ (see [43]). Since $S(\oplus, \odot)$ is a subring, we have

$$(u, \beta(u)) \oplus (v, \beta(v)) = (u + v, \beta(u + v)),$$
$$(u, \beta(u)) \odot (v, \beta(v)) = (uv, \beta(uv)), \quad u, v \in \mathbb{R}.$$

But using the definition of $(\oplus, \odot)$, we obtain (10.12) and (10.13).

Since $F_1|\mathbb{R}_+^* \times \mathbb{R}_+^* = F$, there is a $\delta : \mathbb{R}_+^* \to \mathbb{R}$ such that

$$F(u, v) = \delta(u + v) - \delta(u) - \delta(v), \tag{10.12a}$$
$$\delta(uv) = u\delta(v) + v\delta(u), \quad \text{for } u, v > 0. \tag{10.13a}$$

Define $L : \mathbb{R}_+^* \to \mathbb{R}$ by $L(u) = -\frac{\delta(u)}{u}$ for $u > 0$. Then (10.13a) shows that L is logarithmic and (10.12a) yields

$$F(u, v) = -(u + v)L(u + v) + uL(u) + vL(v), \quad u, v > 0.$$

Hence, from (10.10),

$$(x + y) f\left(\frac{x}{x + y}\right) = -(x + y)L(x + y) + xL(x) + yL(y).$$

With $\frac{x}{x+y} = t$, $1 - t = \frac{y}{x+y}$, and $t \in]0, 1[$, we see that

$$\begin{aligned} f(t) &= -L(x + y) + \frac{x}{x + y}L(x) + \frac{y}{x + y}L(y) \\ &= -L(x + y) + tL(x) + (1 - t)L((1 - t)(x + y)) \\ &= tL(x) + (1 - t)L(1 - t) - tL(x + y) \\ &= tL(t) + (1 - t)L(1 - t), \quad \text{for } t \in]0, 1[, \end{aligned}$$

which is to be proved. This proves the theorem. □

Remark 10.4. [43]. The converse also holds. That is, for $F : \mathbb{R}^* \times \mathbb{R}_+^* \to \mathbb{R}$ satisfying (10.11), (10.11a), and (10.11b), $F(\frac{1}{2}, \frac{1}{2}) = 1$, $f : I \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} F(1 - x, x) & \text{for } x \in]0, 1[, \\ 0 & \text{for } x = 0,\ x = 1, \end{cases}$$

satisfies (FEI) and (10.10).

For the general solution of (FEI) on open domain $]0, 1[$, see Maksa and Ng [619].

Remark. If L is measurable, so is f, but not the converse. If (10.8a) holds for L, if holds for $xL(x) + D(x)$ also, where D is a derivation.

10.2c Some Generalizations of (FEI)

Result 10.5. (Rathie and Kannappan [688]). *Suppose* $f, g : I \to \mathbb{R}$ *satisfy*

$$f(x) + g(x)\, f\left(\frac{y}{1-x}\right) = f(y) + g(y)\, f\left(\frac{x}{1-y}\right) \tag{10.14}$$

for $x, y \in [0, 1[$ *with* $(x + y) \in I$ *and*

$$g(x + y - xy) = g(x)g(y) \quad \text{for } x, y \in I. \tag{10.15}$$

The most general solution f *of the functional equation* (10.14) *satisfying* $f(0) = f(1)$ *and* $f(\frac{1}{2}) = 1$ *depending on* g*, where* g *satisfies* (10.15)*, is given as follows.*

If g *is constant on* $]0, 1[$*, then there arise the following four cases:*

(i) *When* g *is zero in* $[0, 1]$, $f(x) = 1$ *for* $x \in I$.
(ii) *When* $g(x) = 0$ *for* $x \in]0, 1]$, $g(0) = 1$, *then* $f(x) = 1$ *in* $]0, 1[$ *and* $f(0) = f(1) = 0$.
(iii) *When* g *is unity in* $[0, 1]$, $f(x) = 1$ *on* $]0, 1[$ *and* $f(0) = f(1)$.
(iv) *When* $g(x) = 1$ *for* $x \in [0, 1[$, $g(1) = 0$, f *is as in* (iii).

If g *is not constant on* $]0, 1[$:

(v) *When* $g(\frac{1}{2}) = \frac{1}{2}$, f *is a solution of* (FEI).
(vi) *When* $g(\frac{1}{2}) \neq \frac{1}{2}$, f *is given by*

$$f(p) = \frac{g(p) + g(1-p) - 1}{2g\left(\frac{1}{2}\right) - 1} \quad \text{for } p \in I. \tag{10.16}$$

Proof. We treat case (vi). In this case, $g(x) > 0$, $g(0) = 1$, $f(0) = f(1) = 0$, $f(x) = f(1-x)$, and (10.9) and $p = 1 - x$ and $q = \frac{y}{1-x}$ in (10.14) yield

$$f(p) + g(1-p)\, f(q) = f(pq) + g(pq)\, f\left(\frac{1-p}{1-pq}\right) \quad \text{for } p \in (0, 1),\ q \in I. \tag{10.17}$$

Let

$$G(p, q) = f(p) + [g(p) + g(1-p)]\, f(q) \quad \text{for all } p, q \in]0, 1[. \tag{10.18}$$

Then, from (10.18), with the help of (10.15), (10.9), (10.17), and $g(x) > 0$, we obtain

$$\begin{aligned} G(p, q) &= f(pq) + g(pq)\left[f\left(\frac{1-p}{1-pq}\right) + \frac{g(p)}{g(pq)}\, f(q)\right] \\ &= f(pq) + g(pq)\left[f\left(\frac{1-p}{1-pq}\right) + g\left\{\frac{p(1-q)}{1-pq}\right\} f(q)\right] \\ &= f(pq) + g(pq)\left[f\left\{\frac{q(1-p)}{1-pq}\right\} + g\left\{\frac{q(1-p)}{1-pq}\right\} f(p)\right] \end{aligned}$$

$$= f(pq) + g(pq)\left[f\left(\frac{1-q}{1-pq}\right) + g\left\{\frac{q(1-p)}{1-pq}\right\} f(p)\right]$$
$$= G(q, p).$$

Thus, we obtain (10.16). We will connect to H_n^β later.

Second Generalization

Result 10.6. (Kannappan [442]). *Let* $f, h : [0, 1[\to \mathbb{R},\ g, k : I \to \mathbb{R}$ *satisfy the functional equation*

$$f(x) + (1-x)^\beta g\left(\frac{y}{1-x}\right) = h(y) + (1-y)^\beta k\left(\frac{x}{1-y}\right) \tag{10.19}$$

for $x, y \in [0, 1[$ *with* $(x + y) \in [0, 1],\ \beta\ (> 0) \neq 2$. *Then* f, g, h, k *are given by*

$$\begin{aligned}
g(x) &= u(x) - d_1 x^\beta + c_3, && x \in I,\\
h(y) &= g(y) + b_1(1-y)^\beta + b_2 && \text{for } y \in [0, 1[,\\
f(x) &= g(1-x) + c_1 x^\beta + c_2(1-x)^\beta + b_2 && \text{for } x \in]0, 1[,\\
k(x) &= g(1-x) + c_1 x^\beta + d_1(1-x)^\beta + d_2 && \text{for } x \in]0, 1[,
\end{aligned}$$

where $b_1, b_2, c_1, c_2, d_1, d_2$ *are constants and* u *is a solution of*

$$u(1-x) + (1-x)^\beta u\left(\frac{y}{1-x}\right) = u(y) + (1-y)^\beta u\left(\frac{1-x}{1-y}\right) \tag{10.20}$$

for $x \in]0, 1[,\ y \in [0, 1[$, *with* $(x + y) \in]0, 1[$.

Now we consider the most general generalization of (FEI),

$$f(x) + M(1-x)g\left(\frac{y}{1-x}\right) = h(y) + M(1-y)k\left(\frac{x}{1-y}\right), \tag{10.21}$$

for all $x, y \in [0, 1[$ with $x + y \in [0, 1]$, where M is multiplicative. It turns out that nontrivial embeddings of the reals in the complex generate some interesting solutions (Kannappan and Ng [495]).

In many applications, various special cases of (10.21) have occurred. The special case where $f = g = h = k$ and $M =$ the identity map is known as the fundamental equation of information and has been extensively investigated by many authors [43]. The case where $f = g = h = k$ and M is multiplicative was treated in [688]. The general solution of (10.21) when $M(1-x) = (1-x)^\beta$ was obtained in [442]. With these in mind, it is desirable to know the general solution of (10.21). Due to some technical difficulties, the case where M is multiplicative but not a power remained open. This is resolved by using embeddings of the reals in the complex, giving rise to new solutions.

10.2.5 General Solution of Equation (10.21)

Result 10.7. (Kannappan and Ng [495]). *Let* $f, h : [0, 1[\to \mathbb{R},\ g, k : I \to \mathbb{R}$, *and* $M : [0, 1] \to \mathbb{R}$ *be functions satisfying* (10.21)*, where* M *satisfies*

$$\begin{cases} M(xy) = M(x)M(y) \quad \textit{for all } x, y \in]0, 1[, \\ M(0) = 0 \quad \textit{and} \quad M(1) = 1. \end{cases} \tag{10.22}$$

Then they are given by

$$\begin{aligned} f(x) &= \phi(1-x) + a_1 M(x) + b_1 M(1-x) + c \quad \textit{on } [0, 1[, \\ g(x) &= \phi(x) + a_2 M(x) + b_2 M(1-x) + b_1 \quad \textit{on } [0, 1], \\ h(x) &= \phi(x) + a_2 M(x) + b_3 M(1-x) + c \quad \textit{on } [0, 1], \\ k(x) &= \phi(1-x) + a_1 M(x) + b_2 M(1-x) - b_3 \quad \textit{on } [0, 1], \end{aligned} \tag{10.23}$$

where

$$\begin{aligned} &\textit{either } M(x) = x^2 \textit{ and } \phi(x) = D(x), \\ &\quad \textit{or} \quad M(x) = |\phi(x)|^2 \textit{ and } \phi(x) = a \operatorname{Im} \phi(x), \\ &\quad \textit{or} \quad M(x) = x \textit{ and } \phi(x) = xL(x) + (1-x)L(1-x), \\ &\quad \textit{or} \quad M \textit{ is an arbitrary map satisfying } (10.22) \textit{ and } \phi = 0, \end{aligned}$$

with constants a, a_i, b_i, c *and function* $D : \mathbb{R} \to \mathbb{R}$ *is a derivation,* $L : \mathbb{R}_+ \to \mathbb{R}$ *is logarithmic, and* $\phi : \mathbb{R} \to \mathbb{C}$ *is an injective embedding satisfying*

$$\phi(x + y) = \phi(x) + \phi(y), \quad \phi(xy) = \phi(x)\phi(y).$$

The converse is also true.

Corollary 10.7a. (Kannappan and Ng [489], Maksa [618], Aczél and Ng [58]). *The most general measurable solution of*

$$f(x) + (1-x)g\left(\frac{y}{1-x}\right) = h(y) + (1-y)k\left(\frac{x}{1-y}\right), \tag{10.21a}$$

for $x, y \in [0, 1[$ *with* $(x + y) \in I$, *where* $f, h : [0, 1[\to \mathbb{R}$, *and* $g, k : I \to \mathbb{R}$ *has the form*

$$\begin{aligned} f(x) &= as(x) + b_1 x + d, \\ g(y) &= as(y) + b_2 y + b_1 - b_4, \\ h(x) &= as(x) + b_3 x + b_1 + b_2 - b_3 - b_4 + d, \\ k(y) &= as(y) + b_4 y + b_3 - b_2, \end{aligned}$$

for $x \in [0, 1[$ *and* $y \in I$, *where* s *is the Shannon function and* a, b_1, b_2, b_3, b_4, *and* d *are arbitrary constants.*

Corollary 10.7b. *When all* f, g, h, k *are the same in* (10.21a) *(that is, when* f *satisfies* (10.21a) *and is measurable) it is easy to see that*

$$f(x) = as(x) + bx \tag{10.8}$$

for some constants a *and* b.

Corollary 10.7c. (Ebanks, Kannappan, and Ng [238]). *Let* $I_0^n =]0,1[^n$ *be the n-dimensional open unit interval and* $C = \{(x,y) \mid x, y, x+y \in I_0\}$. *Now we give the general solution of the functional equation*

$$f(x) + M(1-x)g\left(\frac{y}{1-x}\right) = h(y) + M(1-y)k\left(\frac{x}{1-y}\right) \tag{10.21b}$$

for all $(x,y) \in C$, *where* $f, g, h, k, M : I_0 \to \mathbb{R}$ *are unknown functions with M multiplicative, i.e.,* $M(xy) = M(x)M(y)$ *for all* $x, y \in I_0$. *Here, all vector operations are to be done componentwise and* 1 *stands for* $(1, 1, \ldots, 1) \in \mathbb{R}^n$. *The special case* $f = g = h = k$ *is known as the n-dimensional fundamental equation of information of multiplicative type.*

The general solution of equation (10.21b) *with nonzero multiplicative M is given by*

$$\begin{aligned}
f(x) &= M(x)L(x) + M(1-x)L(1-x) + \eta_3 M(x) - \eta_2 M(1-x) + \eta_5,\\
g(x) &= M(x)L(x) + M(1-x)L(1-x) + \eta_1 M(x) + \eta_2,\\
h(x) &= M(x)L(x) + M(1-x)L(1-x) + \eta_1 M(x) - \eta_4 M(1-x) + \eta_5,\\
k(x) &= M(x)L(x) + M(1-x)L(1-x) + \eta_3 M(x) + \eta_4,
\end{aligned}$$

for additive M; or

$$\begin{aligned}
f(x) &= L_1(1-x) + L_2(1-x) + L_3(x) + \eta_2 + \eta_3,\\
g(x) &= L_2(1-x) + L_1(x) + \eta_1,\\
h(x) &= L_2(1-x) + L_3(1-x) + L_1(x) + \eta_1 + \eta_3,\\
k(x) &= L_2(1-x) + L_3(x) + \eta_2,
\end{aligned}$$

for $M \equiv 1$; *or*

$$\begin{aligned}
f(x) &= -a(x) + \eta_4 M(x) - \eta_3 M(1-x) + \eta_6,\\
g(x) &= a(x) + \eta_1 M(x) + \eta_2 M(1-x) + \eta_3,\\
h(x) &= a(x) + \eta_1 M(x) - \eta_5 M(1-x) + \eta_6,\\
k(x) &= -a(x) + \eta_4 M(x) + \eta_2 M(1-x) + \eta_5,
\end{aligned}$$

for $M \not\equiv 1$ *and nonadditive, where M and a have one of the following forms:*

(i) $a(\zeta_1, \zeta_2, \ldots, \zeta_n) = d(\zeta_k)$, $M(\zeta_1, \zeta_2, \ldots, \zeta_n) = \zeta_k^2$ *for some fixed k;*
(ii) $a(\zeta_1, \zeta_2, \ldots, \zeta_n) = \eta \operatorname{Im}\phi(\zeta_k)$, $M(\zeta_1, \zeta_2, \ldots, \zeta_n) = |\phi(\zeta_k)|^2$ *for some fixed k;*
(iii) $a(\zeta_1, \zeta_2, \ldots, \zeta_n) = \eta(\zeta_j - \zeta_k)$, $M(\zeta_1, \zeta_2, \ldots, \zeta_n) = \zeta_j \zeta_k$ *for some fixed* $j \neq k$;
(iv) $a = 0$.

In the above, the L's are logarithmic and the η*'s are constants. The map d is a derivation, and* $\operatorname{Im}\phi$ *stands for the imaginary part of a nontrivial embedding* $\phi : \mathbb{R} \to \mathbb{C}$.

Remark. [442, 495]. The proof involves solving the equations

$$m(x)+m(y)=M(1-x)m\left(\frac{y}{1-x}\right)+M(1-y)m\left(\frac{x}{1-y}\right),$$
$$m(x)+m(1-x)=0,$$
$$M(uv)=M(u)M(v),$$

for all $x, y, u, v \in]0,1[$ with $x+y \in]0,1[$, where $m\ (\neq 0),\ M :]0,1[\to \mathbb{R}$. The solutions are

$$\begin{cases} M(x)=x^2 \text{ and } m(x)=D(x), \\ \text{or } M(x)=|\phi(x)|^2 \text{ and } m(x)=a\,\mathrm{Im}\,\phi(x), \\ \text{or } M \text{ is arbitrary and } m=0, \end{cases}$$

where D is a derivation and ϕ is an embedding of $\mathbb{R}$ into $\mathbb{C}$.

Before treating the sum form equation (SFE), we give two results connected to recursivity.

(i) Kaminski and Mikusinski [418], from 3-symmetry, 3-recursiveness, and expansibility and

$$\mu_3(p_1,p_2,p_3)=\mu_2(p_1+p_2,p_3)+(p_1+p_2)\mu_2\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\right)$$
$$=\mu_3(p_1+p_2,0,p_3)+(p_1+p_2)\mu_3\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2},0\right),$$

derived the functional equation

$$H(x,y,z)=H(x+y,0,z)+H(x,y,0), \tag{10.24}$$

called the entropy equation, where $H : C=\{(x,y,z)\in\mathbb{R}^3 : x,y,z\geq 0,\ xy+yz+zx>0\} \to \mathbb{R}$ is defined by

$$H(x,y,z)=(x+y+z)\mu_3\left(\frac{x}{x+y+z},\frac{y}{x+y+z},\frac{z}{x+y+z}\right).$$

Evidently, $H(\lambda x,\lambda y,\lambda z)=\lambda H(x,y,z),\ \lambda>0$ (H is positively homogeneous of degree 1).
It is proved [418] that if H is continuous, symmetric, and positively homogeneous of degree 1 on C and satisfies on the interior C^0 of C the functional equation (10.24), then

$$H(x,y,z)=(x+y+z)\log(x+y+z)-x\log x-y\log y-z\log z.$$

A simpler proof connected to (FEI) is given in [17]. Also, the general solution of (10.24) is given by

$$H(x,y,z)=(x+y+z)L(x+y+z)-xL(x)-yL(y)-zL(z),$$

where H defined on C is positively homogeneous of degree 1, satisfies (10.24), and is symmetric, and L is logarithmic.

(ii) Instead of the recursivity (10.7), assume the general version

(10.7a) $\mu_n(p_1, p_2, \ldots, p_n) = \mu_{n-1}(p_1 + p_2, p_3, \ldots, p_n) + \phi_{n-1}(p_1, p_2)$,

where ϕ_{n-1} is a function of (p_1, p_2) defined on $S = \{(x, y) : x, y \geq 0,\ 0 < x + y \leq 1\}$. Since this axiom is weaker than (10.7), it is necessary to supplement it with additional axioms. As a matter of fact, we prove the following theorem.

Theorem 10.8. *Suppose μ_4 is symmetric, μ_n is expansible,* $(2, n)$ *is strong additive,* (10.7a) *is recursive with ϕ_2 continuous on S, and $\mu_2(\frac{1}{2}, \frac{1}{2}) = 1$. Then $\mu_n(P) = H_n(P)$ for $P \in \Gamma_n$ with $p_1 + p_2 > 0$.*

Proof. It is enough to show that

$$\phi_{n-1}(x, y) = (x + y)\mu_2\left(\frac{x}{x + y}, \frac{y}{x + y}\right) \quad \text{for } (x, y) \in S.$$

Then the result will follow from Theorem 10.1.

As the first step, we will show that ϕ_{n-1} is independent of n and $\phi_{n-1}(p_1, p_2) = \phi_2(p_1, p_2)$. For $(p_1, p_2) \in S$, choose p_3 such that $\sum_{i=1}^{3} p_i = 1$. Then, using expansibility, we get

$$\begin{aligned}\phi_{n-1}(p_1, p_2) &= \mu_n(p_1, p_2, p_3, 0, \ldots, 0) - \mu_{n-1}(p_1 + p_2, p_3, 0, \ldots, 0)\\ &= \mu_3(p_1, p_2, p_3) - \mu_2(p_1 + p_2, p_3)\\ &= \phi_2(p_1, p_2).\end{aligned}$$

Next we prove that μ_n is n-symmetric by induction. Since μ_4 is symmetric, so are μ_2 and μ_3 (use expansibility). Thus ϕ_2 is symmetric. Suppose μ_{n-1} is symmetric for $n \geq 3$. Evidently, μ_n is symmetric in p_1, p_2 by (10.7a). Further, from

$$\mu_n(p_1, p_2, \ldots, p_n) = \mu_{n-1}(p_1 + p_2, p_3, \ldots, p_n) + \phi_2(p_1, p_2),$$

by the induction hypothesis, μ_n is symmetric in p_i $(i = 3, 4, \ldots, n)$. Now μ_n will be symmetric in p_2, p_3 provided

$$\mu_{n-1}(p_1 + p_2, p_3, \ldots, p_n) + \phi_2(p_1, p_2) = \mu_{n-1}(p_1 + p_3, p_2, \ldots, p_n) + \phi_2(p_1, p_3)$$

holds; that is, if

$$\phi_2(p_1 + p_2, p_3) + \phi_2(p_1, p_2) = \phi_2(p_1 + p_3, p_2) + \phi_2(p_1, p_3)$$

holds, which indeed is true from μ_4. Hence μ_n is symmetric. Now we will show that ϕ_2 is additive. From n-symmetry, (10.7a), and $(2, 4)$, $(2, 3)$ additive, we have from $\mu_8(p_1 p, p_2(1 - p), \ldots, p_4(1 - p))$ that

$$\sum_{i=1}^{4} \phi_2(p_i p, p_i(1 - p)) = \mu_2(p, 1 - p) \quad \text{for } p \in I, \tag{10.25}$$

and from $\mu_6((p_1+p_2)p, (p_1+p_2)(1-p), p_3p, \ldots, p_4(1-p))$

$$\phi_2((p_1+p_2)p, (p_1+p_2)(1-p)) + \sum_{i=3}^{4} \phi_2(p_i p, p_i(1-p)) = \mu_2(p, 1-p).$$

Thus

$$\phi_2(p_1+p_2)p, (p_1+p_2)(1-p) = \phi_2(p_1p, p_1(1-p)) + \phi_2(p_2p, p_2(1-p)),$$

which is additive in the sense that if, for fixed p, we define $A_p(x) = \phi_2(xp, x(1-p))$, for $x \in I$,

$$A_p(p_1+p_2) = A_p(p_1) + A_p(p_2) \quad \text{for } p_1, p_2, p_1+p_2 \in I.$$

Since ϕ_2 is continuous, so is A_p, and from Theorems 1.1 and 1.2 it follows that

$$\phi_2(xp, x(1-p)) = xA_p(1) = x\phi_2(p, 1-p).$$

But then (10.25) gives $\phi_2(p, 1-p) = \mu_2(p, 1-p)$.

Set $xp = p_1$ and $x(1-p) = p_2$ to get

$$\phi_2(p_1, p_2) = (p_1+p_2)\mu_2\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right), \quad (p_1, p_2) \in S.$$

This proves the result (see also [196, 204, 13]). □

Now we turn our attention to the other equation, the sum form equation (SFE), and connect it to (FEI).

10.2d Sum Form Functional Equation (SFE) and Its Generalizations

Ever since Chaundy and McLeod [149] considered (SFE), which arose in statistical thermodynamics, it has been extensively studied (see [41, 204, 432, 448]). First we will obtain the continuous solution of (SFE) and then its measurable solution holding for the fixed pair of integers m, n (≥ 3).

10.2d.1 Representation and Characterization

A sequence of measures $\mu : \Delta_n \to \mathbb{R}$ is said to have the *sum property* if there exists a function $f : I \to \mathbb{R}$ such that

$$\mu_n(P) = \sum_{i=1}^{n} f(p_i) \tag{10.26}$$

for $P \in \Delta_n$ $(n \geq 2)$. In this case, f is said to be a *generating function* of $\{\mu_n\}$.

Theorem 10.9. (Kannappan [432]). *Suppose* $\mu_n : \Gamma_n \to \mathbb{R}$ *is* (m, n)*-additive and has representation* (10.26) *holding for all* $m, n\ (\geq 2)$ *with continuous generating function* f*. Then* f *has the form*

$$f(x) = cx \log x \quad \text{for } x \in I, \tag{10.27}$$

and $\mu_n(P) = cH_n(P)$ *for* $P \in \Gamma_n$.

Proof. (See also Aczél and Daróczy [43] and Chaundy and McLeod [149].) From (m, n)-additive and the sum representation (10.26), we obtain the (SFE) for $(x_i) \in \Gamma_n,\ (y_j) \in \Gamma_m$. Now we describe all the continuous solutions of (SFE) by giving a simple proof.

Define a function

$$h(x) = xf\left(\frac{1}{x}\right) \quad \text{for all real } x \geq 1. \tag{10.28}$$

Evidently, h is continuous. Let m, n be any integers ≥ 1. Then putting $x_i = 1/n$ $(i = 1, 2, \ldots, n)$ and $y_j = 1/m$ $(j = 1, 2, \ldots, m)$ in (SFE) and using (10.28), we get

$$h(mn) = h(m) + h(n) \quad \text{for positive integers } m, n \geq 1. \tag{10.29}$$

It is known that h in (10.29) has a unique extension to the multiplicative group of positive rational numbers. The question is whether this extension is the same as h in (10.28). That it is indeed the case is shown as follows.

Taking any rational $r \in]0, 1[$ as m/n $(m \leq n)$ and letting $x_1 = m/n,\ x_2 = \cdots = x_{n-m+1} = 1/n$, and $y_1 = \cdots = y_m = 1/m$ in (SFE), we obtain (by taking n as $n - m + 1$ and m as n)

$$mf\left(\frac{1}{n}\right) + (n-m)mf\left(\frac{1}{mn}\right) = f\left(\frac{m}{n}\right) + (n-m)f\left(\frac{1}{n}\right) + mf\left(\frac{1}{m}\right). \tag{10.30}$$

Using (10.28), (10.29), and (10.30), we find that

$$h\left(\frac{n}{m}\right) = h(n) - h(m) \quad \text{for } m < n. \tag{10.31}$$

From (10.29) and (10.31) there results that

$$h(xy) = h(x) + h(y) \quad \text{for all rationals } x, y \geq 1. \tag{10.32}$$

The continuity of h implies that (10.32) holds for all real $x, y \geq 1$. It follows that h is logarithmic and $h(x) = -c \log x$ for some real constant c. This together with (10.28) gives the required solution f of (SFE). From the representation (10.26) follows $\mu_n(P) = cH_n(P)$. This proves the theorem.

Remark. The solution (10.27) of (SFE) is obtained in [41] holding for $m = n\ (> 2)$, in Daróczy and Maksa [204] for $m = 2,\ n \geq 3$, and in [43] for $m = 2,\ n = 3$. Now we obtain the solution of (SFE) holding for fixed m, n, showing that the solution depends on m and n also.

Now we consider the measurable solution of (SFE) holding for a fixed pair $m, n\ (\geq 3)$.

By letting

$$F(x,y) = f(xy) - yf(x) - xf(y) \tag{10.33}$$

for $x, y \in [0,1]$, we see that F is measurable in each variable and that (SFE) takes the form

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F(x_i, y_j) = 0$$

for $(x_i) \in \Gamma_n,\ (y_j) \in \Gamma_m$, holding for some fixed pair $m, n\ (\geq 3)$. Then F is given by [448]

$$F(x,y) = F(0,y)(1-nx) + F(x,0)(1-my) - F(0,0)(1-nx)(1-my)$$

and here

$$= f(0)(1-y)(1-nx) + f(0)(1-x)(1-my) - f(0)(1-nx)(1-my).$$

Thus we obtain

$$\begin{aligned} f(xy) = {} & yf(x) + xf(y) + f(0)(1-y)(1-nx) + f(0)(1-x)(1-my) \\ & - f(0)(1-nx)(1-my) \end{aligned}$$

for $x, y \in [0,1]$, which can be rewritten as

$$f(xy) = yf(x) + xf(y) + d[1 - x - y + (m+n-mn)xy],$$

where $d = f(0)$; that is,

$$f(xy) - d - d(m+n-mn)xy = x(f(y)-d) + y(f(x)-d)$$

or

$$h(xy) = xk(y) + yk(x) \quad \text{for } x, y \in [0,1], \tag{10.34}$$

where

$$h(x) = f(x) - d(1+m+n-mn)x, \quad k(x) = f(x) - d,$$

are measurable functions.

Now (10.34) can be rewritten as

$$\frac{h(xy)}{xy} = \frac{k(y)}{y} + \frac{k(x)}{x}, \quad x, y \in]0,1],$$

a Pexider equation, so that the measurability of the functions implies

$$k(x) = bx\log x + cx, \quad x \in]0,1],$$

where b, c are constants; that is,

$$f(x) = bx \log x + cx + d, \quad x \in]0, 1]. \tag{10.35}$$

But obviously (10.35) holds for $x = 0$ in view of $f(0) = d$ and the fact that $0 \log 0 = 0$.

Now, f given by (10.35) is a solution of (SFE) provided $c = (mn - m - n)d$. Thus we have proved the following theorem.

Theorem 10.10. (Kannappan [432]). *Let $f : [0, 1] \to \mathbb{R}$ be measurable and satisfy* (SFE) *for a fixed pair m, n (≥ 3). Then f is a solution of* (SFE) *if and only if*

$$f(x) = bx \log x + d(mn - m - n)x + d, \tag{10.27a}$$

where b, d are constants.

Remark 10.11. We will show the close connection between (SFE) and (FEI). As a first step, we prove the following lemma.

Lemma 10.11a. *Let $f_i : I \to \mathbb{R}$ ($i = 1, 2, 3, 4$) satisfy*

$$f_1(xy) - f_2((1 - x)y) - f_3(y) = y f_4(x) \quad \text{for } x, y \in I. \tag{10.36}$$

Then $k(x) = f_4(x) - (f_4(0) - f_4(1)x - f_4(0))$ is a solution of (FEI).

Proof. Set $x = 0$ and $x = 1$ separately to obtain

$$f_1(0) - f_2(y) - f_3(y) = y f_4(0),$$
$$f_1(y) - f_2(0) - f_3(y) = y f_4(1).$$

Substitution of $f_3(y)$ and $f_2(y)$ from these equations in terms of $f_1(y)$ in (10.36) yields

$$f(xy) + f((1 - x)y) - f(y) = yk(x) \quad \text{for } x, y \in I, \tag{10.37}$$

where $f(x) = f_1(x) - f(0)$ and $k(x)$ is as given above.

With

$$k(x) = 2\left[f\left(\frac{x}{2}\right) + f\left(\frac{1-x}{2}\right) - f\left(\frac{1}{2}\right)\right]$$

$$k\left(\frac{y}{1-x}\right) = \frac{2}{1-x}\left[f\left(\frac{y}{1-x} \cdot \frac{1-x}{2}\right) + f\left(\frac{1-x-y}{1-x} \cdot \frac{1-x}{2}\right) - f\left(\frac{1-x}{2}\right)\right],$$

from (10.37) (using $y = \frac{1}{2}$ and $y = \frac{1-x}{2}$) we get

$$k(x) + (1 - x)k\left(\frac{y}{1-x}\right) = 2\left[f\left(\frac{x}{2}\right) + f\left(\frac{1-x}{2}\right) - f\left(\frac{1}{2}\right)\right] + 2\left[f\left(\frac{y}{2}\right) + f\left(\frac{1-x-y}{2}\right) - f\left(\frac{1-x}{2}\right)\right].$$

Since the right side is symmetric in x and y, k satisfies (FEI). This proves the lemma. □

Now we obtain (FEI) from (SFE). Suppose $f : [0, 1] \to \mathbb{R}$ is measurable and is a solution of (SFE) holding for $m = 2,\ n = 3$ with $f(0) = 0 = f(1)$.

For $p \in I$, take $x_1 = p,\ x_2 = 1 - p$, so that (SFE) can be written as

$$\sum_{j=1}^{3}[f(py_j) + f(1-p)y_j] = f(p) + f(1-p) + f(y_1) + f(y_2) + f(y_3). \tag{10.38}$$

For fixed p, define $A(y) = f(py)+f((1-p)y)-f(y)-y(f(p)+f(1-p)),\ y \in I$. Note that $A(0) = 0$. Then (10.38) becomes $A(y_1) + A(y_2) + A(y_3) = 0$, which is an "additive" equation on I. Since f and thus A is measurable, $A(x) = c(p)x$ for $x \in I$. Since $A(1) = 0 = c(p)$, we have

$$0 = A(y) = f(py) + f((1-p)y) - f(y) = y(f(p) + f(1-p)),$$

which is (10.37) with $k(x) = f(x) + f(1-x)$. Hence $f(x) + f(1-x)$ is a solution of (FEI).

With some manipulation from $f(x) + f(1-x) = cs(x)$ and (10.37), we can obtain $f(x) = cx \log x$.

In Daróczy [194], the measurable solution of (SFE) holding for $m = n = 2$ is obtained.

Result 10.12. *Let $f :]0, 1[\to \mathbb{R}$ be a measurable solution of*

$$\begin{aligned} f(pq) + f((1-p)q) + f(p(1-q)) + f[(1-p)(1-q)] \\ = f(p) + f(1-p) + f(q) + f(1-q) \end{aligned} \tag{10.39}$$

for $p, q \in]0, 1[$. Then f is bounded and integrable in each closed interval and infinitely differentiable (*it need only be six times differentiable*) *and has the form*

$$f(x) = b(4x^3 - 9x^2 + 5x) + cx \log x + d, \quad x \in]0, 1[, \tag{10.40}$$

where b, c, d are constants.

The measures μ_n having the sum property with a measurable generating function f are $(2, 2)$-additive if and only if

$$\mu_n(P) = 4bH_n^3(P) - 9bH_n^2(P) + cH_n(P) + dn \quad \text{for } P \in \Gamma_n^0.$$

Some Generalizations

Result 10.13. (Kannappan and Ng [496]). *The measurable solutions $f_i :]0, 1[\to \mathbb{R}$ ($i = 1$ to 6) of the functional equation*

$$f_1(pq)+f_2(p(1-q))+f_3((1-p)q)+f_4((1-p)(1-q)) = f_5(p)+f_6(q) \tag{10.39a}$$

for $p, q \in]0,1[$ *are given by*

$$f_i(p) = 4ap^3 + (b-9a)p^2 + c_i p + cp\log p + e_i \log p + d_i, \quad i = 1,4,$$
$$f_i(p) = 4ap^3 - (b+9a)p^2 + c_i p + cp\log p + e_i \log p + d_i, \quad i = 2,3,$$
$$f_5(p) = -6ap^2 + (6a - 2b + c_2 - c_4)p + c[p\log p + (1-p)\log(1-p)] + (e_1+e_2)\log p + (e_3+e_4)\log(1-p) + d_5,$$
$$f_6(p) = -6ap^2 + (6a - 2b + c_3 - c_4)p + c[p\log p + (1-p)\log(1-p)] + (e_1+e_3)\log p + (e_2+e_4)\log(1-p) + d_6,$$

where a, b, c, c_i, d_i, e_i *are constants with*

$$4b + c_1 - c_2 - c_3 + c_4 = 0 \quad \textit{and} \quad b - 5a + c_4 + d_1 + d_2 + d_3 + d_4 = d_5 + d_6.$$

This solution is obtained by determining the measurable solutions of

$$f(pq) + f(p(1-q)) + f((1-p)q) + f((1-p)(1-q)) = 0 \tag{10.39b}$$

and

$$f_1(pq) + f_2(p(1-q)) + f_3((1-p)q) + f_4((1-p)(1-q)) = 0 \tag{10.39c}$$

as $f(p) = 4bp - b$ *and*

$$\begin{cases} f_i(p) = bp^2 + (a-b)p - c\log p + d_i, & i = 1,4, \\ f_i(p) = -bp^2 + (a+b)p + c\log p + d_i, & i = 2,3, \end{cases}$$

where a, b, c, d_i*'s are constants with* $a + d_1 + d_2 + d_3 + d_4 = 0$.

Problem. Determine the general solutions of (SFE), (10.39), and (10.39a).

Result 10.14. (Kannappan [448, 436], Kannappan and Sahoo [501]). *Suppose* $g : I \to \mathbb{R}$ *is measurable. Then* g *is a solution of the equation*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} g(x_i y_j) = \sum_{i=1}^{m} g(x_i) + \sum_{j=1}^{n} g(y_j) + c \cdot \sum g(x_i) \cdot \sum g(y_j) \tag{10.41}$$

holding for fixed $m, n \geq 3$ *(for* $(x_i) \in \Gamma_m$, $(y_j) \in \Gamma_n$*),* $c \neq 0$*, if and only if* g *has either the form*

$$g(p) = \frac{(a-1)}{c}p + \frac{b}{c} \tag{10.42}$$

or

$$g(p) = \frac{p^\beta - p}{c}, \quad \textit{for } p \in I, \tag{10.42a}$$

where β *is an arbitrary constant, while the constants* a, b *satisfy the equation*

$$(a + mnb) = (a + nb)(a + mb).$$

Remark. If $b = 0 = g(0)$ (then $a = 0$), and $g(1) \neq 0$, then (10.42) yields $g(x) = -\frac{x}{c}$ (a solution independent of m and n). Note that $c = 0$ gives (SFE).

Now we consider the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_i p_i^\alpha \sum_j f(q_j) + \sum_j q_j^\beta \cdot \sum_i f(p_i), \tag{10.43}$$

where $P \in \Gamma_n$, $Q \in \Gamma_m$, α, β $(\neq 0)$, $f :]0,1[\to \mathbb{R}$, and obtain the following result [436, 502, 443].

Result 10.15. *Let $f :]0,1[\to \mathbb{R}$ be measurable. Then f satisfies the functional equation* (10.43) *for all $P \in \Gamma_n$ and $Q \in \Gamma_m$ with arbitrary but fixed m, n (≥ 3) if, and only if,*

$$f(p) = \begin{cases} a(p^\alpha - p^\beta), & \alpha \neq \beta, \\ ap\log p + b(mn - m - n)p + b, & \alpha = \beta = 1, \\ ap^\alpha \log p, & \alpha = \beta\ (\neq 1), \end{cases}$$

where a, b are arbitrary constants.

Result 10.15a. (Kannappan [456]). *The most general solution of the functional equation* (10.43) *holding for all $P \in \Gamma_n$, $Q \in \Gamma_m$ with an arbitrary but fixed pair of integers m, n greater than* 2 *is*

$$f(p) = \begin{cases} A(p) + d(p^\alpha - p^\beta), & \alpha \neq \beta, \\ A(p) + L(p)p^\alpha, & \alpha = \beta \neq 1, \end{cases}$$

where A is additive with $A(1) = 0$ and $L :]0,1[\to \mathbb{R}$ is logarithmic.

More Generalizations

Result 10.16. (Kannappan [452], Ng [649]). *Let $f_{ij}, g_i, h_j : I \to \mathbb{R}$ $(i = 1, 2;\ j = 1, 2, \ldots, n)$ be measurable and satisfy the equation*

$$\sum_{i=1}^{2}\sum_{j=1}^{n} f_{ij}(p_i q_j) = \sum_{i=1}^{2} g_i(p_i) + \sum_{j=1}^{n} h_j(q_j) \tag{10.44}$$

for fixed $n \geq 3$. Then f_{ij}, h_i, g_j are given by

(10.44a)

$$\begin{cases} f_{1j}(x) = ax\log x + (b + d_j)x + d_{1j}, & \\ f_{2j}(x) = ax\log x + (d_j + c)x + d_{2j} - d_{1j}, & \\ h_j(x) = ax\log x + d_j x + d_{2j} - c_j, & j = 1, 2, \ldots, n, \\ g_1(x) = g(x), & \text{arbitrary}, \\ g_2(x) = -g(1-x) + a[x\log x & \\ \qquad +(1-x)\log(1-x)] + (b-c)x + c + d, & \end{cases}$$

where $a, b, c, d, C_j, d_j, d_{1j}$, and d_{2j} are arbitrary constants.

Next we treat the functional equation

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(p_i q_j) = \sum_{j=1}^{n} g_j(q_j) + \sum_{j=1}^{n} q_j^{\beta} \sum_{i=1}^{m} h_i(p_i) \tag{10.45}$$

for $P \in \Gamma_m,\ Q \in \Gamma_n,\ f_{ij}, h_i, g_j : I \to \mathbb{R}$ measurable and obtain the following result.

Result 10.17. [443]. *Suppose* (10.45) *holds for* $m = 2,\ n = 3$. *Then they are given by*

$$\begin{cases} f_{1j}(x) = ax\log x + (b + d_j)x + d_{1j}, & j = 1,2,3, \\ f_{2j}(x) = ax\log x + (d_j + C)x + d_{2j}, & j = 1,2,3, \\ g_3(x) = ax\log x + (d_j - d)x + d_{1j} + d_{2j} - b_j, & \text{for } j = 1,2,3, \\ h_1(x) = h(x), & \\ h_2(x) = -h(1-x) + a[x\log x & \\ \qquad +(1-x)\log(1-x)] - (b-C)x + b, & \text{for } \beta = 1, \\ f_{1j}(x) = (a+b)x^{\beta} + d_j x + d_{1j}, & j = 1,2,3, \\ f_{2j}(x) = (a+C)x^{\beta} + d_j x + d_{2j}, & j = 1,2,3, \\ g_j(x) = ax^{\beta} + (d_j - d)x + d_{1j} + d_{2j} - b_j, & j = 1,2,3, \\ h_1(x) + h_2(x) = a[x^{\beta} + (1-x)^{\beta} - 1] & \\ \qquad +bx^{\beta} + C(1-x)^{\beta}, & \end{cases} \tag{10.45a}$$

where $a, b, c, d, d_j, d_{ij}, b_j$ $(i = 1,2,\ j = 1,2,3)$ *are constants with* $b_3 + b_1 + b_2 + d = 0$.

Remark. Here again we connect the "sum form" equation and "fundamental" equation. The proof uses the (measurable) solution of the equation

$$f(xy) - g((1-x)y) - h(y) = y^{\beta}k(x), \quad x, y \in I,\ \beta \neq 0, \tag{10.36a}$$

where $f, g, h, k : I \to \mathbb{R}$ (see (10.36)). If we define $k_1(x) = k(x) - c(1-x)^{\beta},\ c = g(0)$, then k_1 satisfies [456]

$$k_1(x) + (1-x)^{\beta}k_1\left(\frac{y}{1-x}\right) = k_1(y) + (1-y)^{\beta}k_1\left(\frac{x}{1-y}\right),$$

and its solutions are

$$k_1(x) = \begin{cases} a[x\log x + (1-x)\log(1-x)] + bx & \text{for } \beta = 1, \\ a[x^{\beta} + (1-x)^{\beta} - 1] + bx^{\beta} & \text{for } \beta \neq 1. \end{cases}$$

Another generalization of (SFE) is the equation [443, 436, 516]

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij}(p_i q_j) = \sum_{i=1}^{n} p_i^{\alpha} \cdot \sum_{j=1}^{m} g_j(q_j) + \sum_{j=1}^{m} q_j^{\beta} \cdot \sum_{i=1}^{n} h_i(p_i), \tag{10.46}$$

where $P \in \Gamma_n,\ Q \in \Gamma_m$, and $\alpha, \beta\ (\neq 0) \in \mathbb{R}$.

Result 10.18. (Kannappan [456]). *Let* $f_{ij}, g_j, h_i : I \to \mathbb{R}$ *be measurable* ($i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$). *Then these functions satisfy the functional equation* (10.46) *for fixed* m, n (≥ 3) *if and only if they are of the form given by*

$$\text{(10.46a)} \qquad \begin{cases} f_{ij}(x) = a_j x^\alpha + b_i x^\beta + d_{ij} x + c_{ij}, \\ g_j(x) = a_i x^\alpha - C x^\beta + d_j x + e_j, \qquad \text{for } \alpha \neq \beta, \\ h_i(x) = C x^\alpha + b_i x^\beta + \alpha_i x + \gamma_i, \end{cases}$$

$i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$, *where* $a_j, b_i, C, d_{ij}, c_{ij}, d_j, e_j, \alpha_i, \gamma_i$ *are constants satisfying some conditions, or are of the form given by*

$$\text{(10.46b)} \qquad \begin{cases} f_{ij}(x) = a x^\alpha \log x + (b_j + C_i) x^\alpha + d_{ij} x + c_{ij}, \\ g_j(x) = a x^\alpha \log x + b_j x^\alpha + d_j x + e_j, \qquad \text{for } \alpha = \beta, \\ h_i(x) = a x^\alpha \log x + C_i x^\alpha + \alpha_i x + \beta_i, \end{cases}$$

for $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n$, *where the constants* $a, b_j, C_i, d_{ij}, c_{ij}, d_j, e_j, \alpha_i, \gamma_i$ *satisfy some conditions.*

10.2e Other Measures of Information—Entropy of Type β, H_n^β

In Havrda and Charvat [361], the entropy of type β (10.3) $H_n^\beta(P)$, $\beta \neq 1$ was introduced (see also Daróczy and Kiesewetter [202] and Kannappan [448]).

(i) Suppose $\mu_n : \Gamma_n \to \mathbb{R}$ is 3-symmetric and n-recursive of type β; that is,

$$\mu_n(p_1, p_2, \ldots, p_n) = \mu_{n-1}(p_1 + p_2, p_3, \ldots, p_n) + (p_1 + p_2)^\beta \mu_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right),$$

with $\beta \neq 1$, $p_1 + p_2 > 0$, $\mu_2(\frac{1}{2}, \frac{1}{2}) = 1$. Then $\mu_n(P) = H_n^\beta(P)$. From 3-symmetry and 3-recursivity of type β there results the functional equation

$$f(x) + (1 - x)^\beta f\left(\frac{y}{1 - x}\right) = f(y) + (1 - y)^\beta f\left(\frac{x}{1 - y}\right),$$

where $f : I \to \mathbb{R}$ is defined by $f(x) = \mu_2(x, 1 - x)$, which is a special case of (10.14), and the solutions are given by (10.16), $f(x) = \frac{1}{(2^{1-\beta} - 1)}(x^\beta + (1 - x)^\beta - 1) = s_\beta(x)$. Then recursivity yields $\mu_n(P) = H_n^\beta(P)$. Note that no regularity condition on f is needed.

(ii) H_n^β is not additive but nonadditive, satisfies

$$\text{(10.47)} \qquad H_{mn}^\beta(P * Q) = H_n^\beta(P) + H_m^\beta(Q) + c H_n^\beta(P) H_m^\beta(Q),$$

for $P \in \Gamma_n$, $Q \in \Gamma_m$, and has representation with $f(p) = c(p^\beta - p)$, where $c = 2^{1-\beta} - 1$.

Suppose $\mu_n : \Gamma_n \to \mathbb{R}$ has representation $\mu_n(P) = \sum_{i=1}^{n} f(p_i)$ and satisfies (10.47). Then f satisfies (10.41) and is given by (10.42a), $f(x) = \frac{1}{c}(x^\beta - x),\ x \in I$. Thus $\mu_n^\beta(P) = H_n^\beta(P)$.

10.2f Directed Divergence (dd) and Inaccuracy (KI)

The measures of information known as directed divergence (dd) (Kannappan and Sahoo [510], Kannappan and Ng [491]) and inaccuracy (Kerridge [532]) are given by D_n and I_n, respectively. There are many algebraic properties satisfied by D_n and I_n, such as symmetry, recursivity, and additivity, among others.

Postulates a

Let $\mu_n : \Gamma_n^2 \to \mathbb{R}$ $(n \geq 2)$ be a sequence of functions satisfying the following postulates.

(i) **Symmetry.** $\mu_n(P\|Q)$ is symmetric.
(ii) **Branching.**

$$\mu_n\begin{pmatrix} p_1, \ldots, p_n \\ q_1, \ldots, q_n \end{pmatrix} = \mu_{n-1}\begin{pmatrix} p_1 + p_2, p_3, \ldots, p_n \\ q_1 + q_2, q_3, \ldots, q_n \end{pmatrix} + (p_1 + p_2)\mu_2\begin{pmatrix} \frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2} \\ \frac{q_1}{q_1+q_2}, \frac{q_2}{q_1+q_2} \end{pmatrix}, \quad p_1 + p_2, q_1 + q_2 > 0,$$

with

(iii) $F(p,q) = K_2\begin{pmatrix} p, 1-p \\ q, 1-q \end{pmatrix}$ measurable for $(p,q) \in J$, where

$$J =]0,1[\times]0,1[\cup \{(0,y)\} \cup \{(1,z)\} \quad \text{for } y \in]0,1[,\ z \in]0,1].$$

Then

$$F(x,y) = as(x) + d[x \log y + (1-x)\log(1-y)], \tag{10.48}$$

where a and d are arbitrary constants.

From (i), (ii), and (iii) can be derived the functional equation

$$F(x,y) + (1-x)F\left(\frac{u}{1-x}, \frac{v}{1-y}\right) = F(u,v) + (1-u)F\left(\frac{x}{1-u}, \frac{y}{1-v}\right) \quad \text{for } (x,y) \in J, \tag{10.49}$$

a two-variable version of (FEI).

Now we obtain the most general solution of (10.49) [442, 450, 491]. Let $F : J \to \mathbb{R}$ satisfy the functional equation (10.49) for $x, u \in [0,1[$ with $x + u \in I$, for $y, v, y + v \in I_0$, where $J = I \times I_0 \cup \{(0,0),(1,1)\}$.

Without loss of generality, we can assume F is symmetric; that is,

$$F(x, y) = F(1 - x, 1 - y) \quad \text{for } (x, y) \in J.$$

The function $H(x, y) = F(x, y) - F(1, 1)x$ also satisfies (10.49) and that H is symmetric.

So, let F satisfy (10.49) and be symmetric. For fixed $\upsilon, y \in]0, 1[$, (10.49) can be rewritten as

$$g(x) + (1 - x)h\left(\frac{u}{1 - x}\right) = k(u) + (1 - u)N\left(\frac{x}{1 - u}\right) \tag{10.21a}$$

(which is a special case of (10.21)) for $x, u \in [0, 1[$ with $(x + u) \in I$, where

$$g(x) = F(x, y),\ h(x) = F\left(x, \frac{\upsilon}{1 - y}\right),\ k(x) = F(x, \upsilon),\ N(x) = F\left(x, \frac{y}{1 - \upsilon}\right).$$

As in (10.22), for $(x, y) \in D$, we have

$$\begin{aligned}
g(x) &= s_L(x) + a_1 x + a_2 && \text{for } x \in I,\\
h(x) &= s_L(x) + b_1 x + b_2 && \text{for } x \in I_0,\\
k(x) &= s_L(x) + c_1 x + b_2 + c_1 - a_1 - a_2 && \text{for } x \in [0, 1[,\\
N(x) &= s_L(x) + (b_1 + a_2)x - c_1 + a_1 + a_2 && \text{for } x \in I_0,
\end{aligned}$$

where $s_L(x)$ is given by (10.16) and a_i and b_i are functions of υ and y. Thus F is of the form

$$F(x, y) = xL(x) + (1 - x)L(1 - x) + a(y)x + b(y), \tag{10.50}$$

for $x, y \in I_0$, where a and b are functions of y only (use the forms of g and k so that a and b are functions of y only).

First, using the symmetry of F, we have

$$a(y) + a(1 - y) = 0$$

and

$$a(y) + b(y) = b(1 - y), \quad \text{for } y \in I_0. \tag{10.51}$$

Substituting this value of F into (10.49) and equating the coefficient of x and the constant terms, we obtain

$$a(y) - b\left(\frac{\upsilon}{1 - y}\right) = a\left(\frac{y}{1 - \upsilon}\right) \tag{10.52}$$

and

$$b(y) + b\left(\frac{\upsilon}{1 - y}\right) = b(\upsilon) + b\left(\frac{y}{1 - \upsilon}\right) \tag{10.53}$$

for $y, \upsilon, y + \upsilon \in I_0$.

Use (10.51), (10.52), and (10.53) to get

$$b(1-y) = b(v) + b\left(1 - \frac{y}{1-v}\right);$$

that is, b satisfies the Cauchy functional equation (logarithmic)

$$b(1-pq) = b(1-p) + b(1-q) \quad \text{for } p, q \in I_0.$$

Taking $\ell(x) = b(1-x)$ for $x \in I_0$, it follows that ℓ is logarithmic and from (10.51) that

$$a(y) = \ell(y) - \ell(1-y).$$

Thus, F is of the form

(10.50a) $$F(x,y) = xL(x) + (1-x)L(1-x) + x\ell(y) + (1-x)\ell(1-y)$$

for $x, y = I_0$, where L, ℓ satisfy (L).

Substitutions $x = 0 = y;\ u = 1-x,\ v = 1-y;\ x = 0;\ u = 1-x$ in (10.49) yield

$$F(0,0) = 0 = F(1,1);\ F(0,y) = \ell(1-y);\ F(1,y) = \ell(y).$$

Thus (10.50a) holds for $(x,y) \in J$.

Remark. If $F : J \to \mathbb{R}$ satisfies the functional equation (10.49) only (no symmetry), then the most general form of F is given by

$$F(x,y) = xL(x) + (1-x)L(1-x) + x\ell(y) + (1-x)\ell(1-y) + cx,$$

where c is an arbitrary constant and L, ℓ are solutions of (L). This proves the conjecture in Kannappan [478]. □

The measurable solutions of (10.49) are given by (10.48) (which follows from (10.50a)).

To obtain directed divergence D_n, use

$$\textbf{Unit.} \quad \mu_2 \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix} = \tfrac{1}{3},$$
$$\textbf{Nullity.} \ \mu_2 \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = 0,$$

so that $a = 1 = -d$ and

$$F(x,y) = x \log \frac{x}{y} + (1-x) \log \frac{(1-x)}{(1-y)},$$

$\mu_n(P \| Q) = D_n(P \| Q)$.

To obtain inaccuracy I_n, use

$$\textbf{Unit. } \mu_2 \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix} = 1,$$
$$\mu_2 \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = 1,$$

so that $a = 0,\ d = -1$,

$$F(x, y) = -x \log y - (1 - x) \log(1 - y),$$

$\mu_n(P \| Q) = I_n(P \| Q)$.

Postulates b

Let $\mu_n : \Gamma_n^2 \to \mathbb{R}$ be a sequence of functions satisfying the following properties:

- $\mu_n(P \| Q)$ is symmetric.
- (m, n)-**additive**. $\mu_{mn}(P * Q \| R * S) = \mu_m(P \| R) + \mu_n(Q \| S),\ P, R \in \Gamma_m,\ Q, S \in \Gamma_n$, for all $m, n \geq 2$.
- **Representation-sum property**. There exists a continuous (or measurable in each variable) function $g : I^2 \to \mathbb{R}$ such that

$$\mu_n(P \| Q) = \sum_{i=1}^{n} g(p_i, q_i), \quad P, Q \in \Gamma_n \tag{10.54}$$

(f is called a generating function of the sequence $\{\mu_n\}$). Then g has the form

$$g(x, y) = ax \log x + bx \log y + cy \log y + a_1 x + a_2 y + d \tag{10.54a}$$

for $(x, y) \in J$ with $a_1 + a_2 = (mn - m - n)d$.

From the (m, n)-additivity and the sum representation there results the "sum form" equation

$$\sum_{i=1}^{m} \sum_{j=1}^{n} g(p_i q_j, r_i s_j) = \sum_{i=1}^{m} g(p_i, r_i) + \sum_{j=1}^{n} g(q_j, s_j), \tag{10.55}$$

whose measurable solution is given by (10.54) (see Kannappan [452, 443]).

To illustrate the link between the "sum form" (10.55) and the "fundamental equation" (10.49), consider the equation (10.55) holding for $n = 2,\ m = 3$,

$$\begin{aligned} &\sum_{j=1}^{3} g(p \cdot r_j, q \cdot s_j) + \sum_{j=1}^{3} g((1 - p)r_j, (1 - q)s_j) \\ &\qquad = g(p, q) + g(1 - p, 1 - q) + \sum_{j=1}^{3} g(r_j, s_j), \end{aligned} \tag{10.55a}$$

for $p, q \in I,\ R, S \in \Gamma_3$. Then g has the form (10.54a) with $a_1 + b_1 = d$ [452, 494].

Proof. Fix $p, q \in I$, and define

$$G(r,s) := g(pr, qs) + g((1-p)r, (1-q)s) - r[g(p,q) + g(1-p, 1-q)] - g(r,s) \tag{10.56}$$

for $(r,s) \in J = I \times I_0 \cup \{(0,0),(1,1)\}$. Then G satisfies $\sum_{i=1}^{n} G(r_i, s_i) = c$, and by [452], $G(r,s) = ar + bs + d$ with $a + b + 3d = 0$, where a, b, and d are functions of p and q. First, let $r = 0 = s$ to get $g(0,0) = d$ and then $r = 0$ to obtain

$$g(0, qs) + g(0, (1-q)s) - g(0,s) = bs + d = b(q) + d,$$

which is of the form (10.36) [452]. Hence, from [452] we have $b(q) = -a_1 s(q)$. Then (10.56) becomes

(10.57)

$$K(pr, qs) + K((1-p)r, (1-q)s) - K(r,s) = r[K(p,q) + K(1-p, 1-q) - d],$$

where

$$K(r,s) = g(r,s) - a_1 s \log s - d \quad \text{for } (r,s) \in J.$$

Define

$$F(x,y) = K(x,y) + K(1-x, 1-y) - d, \quad (x,y) \in J. \tag{10.58}$$

Then

$$\begin{aligned} F(x,y) &+ (1-x)F\left(\frac{u}{1-x}, \frac{v}{1-y}\right) \\ &= K(x,y) + K(1-x, 1-y) - d \\ &\quad + (1-x)\left[K\left(\frac{u}{1-x}, \frac{v}{1-x}\right) + K\left(1 - \frac{u}{1-x}, 1 - \frac{v}{1-y}\right) - d\right] \\ &= K(x,y) + K(1-x, 1-y) - d + K(u,v) \\ &\quad + K(1-u-x, 1-v-y) - K(1-x, 1-y). \end{aligned}$$

Take $r = 1-x,\ s = 1-y$ in (10.57), which is symmetric in x and u and y and v. Hence F satisfies (10.49).

This shows the connection between these two types of functional equations. Note that F is measurable in each variable and is symmetric, and F is given by (10.48). From this F and (10.57) follows $K(x,y) = -a_2 x \log x - a_3 x \log y - a_4 x - a_5 y - a_6$, and g is given by (10.54). Then, from the representation, we get $\mu_n(P \| Q)$. For various characterizations of directed divergence, see Kannappan and Rathie [497] and Kannappan [478, 444].

10.2f.1 Generalized Directed Divergence

Now we consider generalized directed divergence, GD_n, briefly (Ng [649], Aczél and Nath [56]).

Let $\mu_n : \Gamma_n^3 \to \mathbb{R}$ $(n \geq 2)$ be a sequence of measures.

(i) As before, using the properties of 3-symmetry, n-recursivity, or branching with measurable

$$H(p,q,r) = \mu_2 \begin{pmatrix} p, 1-p \\ q, 1-q \\ r, 1-r \end{pmatrix}, \quad \text{for } (p,q,r) \in K,$$

where

$$K =]0,1[\times]0,1[\times]0,1[\cup \{(0,y,z)\} \cup \{(1,y',z')\},$$

for $y, z \in [0,1[,\ y', z' \in]0,1[$, we obtain the "fundamental" equation

$$\begin{aligned} H(x,y,z) + (1-x)H\left(\frac{u}{1-x}, \frac{v}{1-y}, \frac{w}{1-z}\right) \\ = H(u,v,w) + (1-u)H\left(\frac{x}{1-u}, \frac{y}{1-v}, \frac{z}{1-w}\right) \end{aligned} \tag{10.59}$$

for $(x,y,z) \in K$. The general solution of (10.59) is given by [450]

$$\begin{aligned} H(x,y,z) = s_L(x) + x\ell_1(y) + (1-x)\ell_1(1-y) + x\ell_2(z) \\ + (1-x)\ell_2(1-z), \quad \text{for } (x,y,z) \in K, \end{aligned} \tag{10.59a}$$

where L, ℓ_1, ℓ_2 are logarithmic, $s_L(x)$ is given by (10.16), and the measurable solution is given by [490]

$$\begin{aligned} H(x,y,z) = a(x\log x + (1-x)\log(1-x)) \\ + b(x\log y + (1-x)\log(1-y)) \\ + c(x\log z + (1-x)\log(1-z)). \end{aligned} \tag{10.59b}$$

To get the form GD_n, use nullity $H(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 0 = H(\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$, so that

$$H(x,y,z) = b\left[x\log\frac{y}{z} + (1-x)\log\frac{(1-y)}{(1-z)}\right]$$

and $\mu_n(P\|Q\|R) = GD_n(P\|Q\|R)$ (from recursivity).

(ii) There is a real-valued continuous (measurable or measurable in each variable) $F : K \to \mathbb{R}$ where $K = I_0^3 \cup \{(0,y,z)\} \cup \{(1,y',z')\}$ with $y, z \in [0,1[,\ y', z' \in]0,1]$ such that the following hold:

- **Representability.** $\mu_n(P\|Q\|R) = \sum_{i=1}^{n} F(p_i, q_i, r_i)$, for all $n \geq 2$.
- (m,n)-**Additivity.** $\mu_{mn}(P*U\|Q*V\|R*W) = \mu_n(P\|Q\|R) + \mu_m(U\|V\|W)$ for $P, Q, R \in \Gamma_n,\ U, V, W \in \Gamma_m$.

These two properties lead to the "sum form" equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F(p_i u_j, q_i v_j, r_i w_j) = \sum_{i=1}^{n} F(p_i, q_i, r_i) + \sum_{j=1}^{m} F(u_j, v_j, w_j), \tag{10.60}$$

$P, Q, R \in \Gamma_n$, $U, V, W \in \Gamma_m$, $(p_i, q_i, r_i) \in K$, whose measurable solution is given by

$$F(x, y, z) = s(x, y, z) + d_1 x + d_2 y + d_3 z + d_4$$

with $d_1 + d_2 + d_3 = (mn - m - n)d_4$, where $s(x, y, z)$ is given by (10.1c).

Remark. As before, we can link the "sum form" (10.60) to the fundamental equation (10.59) (see Kannappan [455]).

For various characterizations of generalized directed divergence, see Aczél and Nath [56], and Kannappan [437, 444, 455].

Remark. The measures H_n, I_n, D_n, GD_n, H_n^{β}, $H_n(P; W)$ possess in particular the properties of symmetry, being expansible, and branching. It is shown in Ng [650] that these three properties are equivalent to the representability of H_n, I_n, D_n, GD_n, etc. All these measures H_n, I_n, D_n, H_n^{β}, $D_n(P\|Q) + D_n(Q\|P)$ (called symmetric difference) possess the sum property with regular generating functions. With the exception of entropy of degree β, all are (m, n)-additive for all $m, n \geq 2$.

We now intend to find all measures μ_n generated by Lebesgue measurable functions that are $(2, 2)$-additive. This amounts to solving the functional equation

$$\begin{aligned} f(pr, qs) + f(p(1-r), q(1-s)) + f((1-p)r, (1-q)s) \\ + f((1-p)(1-r), (1-q)(1-s)) \\ = f(p, q) + f(1-p, 1-q) + f(r, s) + f(1-r, 1-s) \end{aligned}$$

for $f :]0, 1] \times]0, 1[\to \mathbb{R}$, measurable in each variable, which in turn leads to the following (representation) result (see [494, 507]).

Result 10.19. *The measures μ_n having the sum property with a Lebesgue measurablegenerating function f are $(2, 2)$-additive if, and only if, they are given by*

$$\begin{aligned} \mu_n(P\|Q) = 4aH_n^3(P) + 4a'H_n^3(Q) - 9aH_n^2(P) - 9a'H_n^2(Q) \\ + bH_n(P) + b'H_n(Q) + cI_n(P\|Q) + c'I_n(Q\|P) + dn, \end{aligned}$$

where a, a', b, b', c, c', and d are arbitrary constants.

Furthermore, it is easy to see from the result above that if the measures μ_n are (m, n)-additive for all $m, n \geq 2$, then $a = a' = d = 0$, and that μ_n has the form

$$\mu_n(P\|Q) = bH_n(P) + b'H_n(Q) + cI_n(P\|Q) + c'I_n(Q\|P). \tag{10.61}$$

In what follows, we intend to examine additional distinct properties that distinguish various measures of information that are of the form (10.61) and isolate them as special cases. It is possible to obtain characterizations of various known measures by giving suitable postulates and using methods adapted in [494].

- **Shannon's entropy.** In the event that μ_n depends only on P or only on Q, then μ_n is of the form $bH_n(P)$ or $b'H_n(Q)$ accordingly.
- **Inaccuracy.** We utilize two of the properties of inaccuracy I_n cited in [532] to isolate inaccuracy out of the measures (10.61).
  "If two sets of alternatives are asserted to have probabilities which are independent, the inaccuracy of the joint assertion is the sum of the separate inaccuracies." This motivates the following stronger version of the additivity.
- **Strong additivity.** A measure μ_n is said to be strongly additive provided

$$\mu_{mn}((P_1, P_2)\|Q_1 * Q_2) = \mu_n(P_1\|Q_1) + \mu_n(P_2\|Q_2),$$

where the joint distribution (not necessarily independent)

$$(P_1, P_2) = (p_{11}, \ldots, p_{1n}, p_{21}, \ldots, p_{2n}, \ldots, p_{m1}, \ldots, p_{mn}) \in \Gamma^0_{mn}$$
$$P_1 = \left(\sum_j p_{ij}\right)_{i=1}^m \in \Gamma^0_m,\ P_2 = \left(\sum_i p_{ij}\right)_{j=1}^n \in \Gamma^0_n, \quad \text{for all } m, n \geq 2.$$

It is observed that $(P_1, P_2) = P_1 * P_2$ when P_1 and P_2 are independent, and hence the strong additivity implies the (m, n)-additivity for all $m, n \geq 2$. The measures (10.61) are strongly additive in Q if, and only if,

$$\mu_n(P\|Q) = b'H_n(Q) + cI_n(P\|Q).$$

Indeed, by considering $(P_1, P_2) = (\frac{1}{4}, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}),\ P_1 = (\frac{3}{8}, \frac{5}{8}),\ P_2 = (\frac{1}{2}, \frac{1}{2}), Q_1 = Q_2 = (q, 1-q),\ (q \in]0, 1[)$, it is easy to see that the measures (10.61) are strongly additive only when $b = 0 = c'$.
"The value of I_n is a minimum, for fixed P, when $Q = P$ so that the error term $(\sum p_i \log p_i/q_i)$ is zero. It then reduces to Shannon entropy." This motivates the *minimum property* (in the variable Q)

$$\mu_n(P\|Q) \geq \mu_n(P\|P) \quad \text{for all } P, Q \in \Gamma^0_n.$$

Among the measures (10.61), those that possess the minimum property (in Q) are given by

$$\mu_n(P\|Q) = bH_n(P) - b'D_n(Q\|P) + cI_n(P\|Q), \quad \text{with } b' \leq 0,\ c \geq 0.$$

Indeed, by considering the measures (10.61) with $n = 2$, we see that as a consequence of the minimum property, the map $q \to \mu_2((p, 1-p)\|(q, 1-q))$ has derivative equal to zero at p, from which we conclude that $b' + c' = 0$ and so $\mu_n(P\|Q) = bH_n(P) - b'D_n(Q\|P) + cI_n(P\|Q)$. Then the minimum property $\mu_n(P\|Q) \geq \mu_n(P\|P)$ is equivalent to $b'D_n(Q\|P) + cD_n(P\|Q) \geq 0$. Since $D_n \geq 0,\ \sup D_n(P\|Q) = +\infty$, and $\sup_p D_n(P\|Q) < +\infty$, the inequality $-b'D_n(Q\|P) + cD_n(P\|Q) \geq 0$ holds if, and only if, $-b', c \geq 0$. Thus the measures (10.61) possess the strong additivity and the minimum property if, and only if,

$$\mu_n(P\|Q) = cI_n(P\|Q), \quad (c \geq 0).$$

- **Symmetric divergence or J-divergence.** [498, 411, 468, 492]. The measures (10.61) are symmetric in P and Q (i.e., $\mu_n(P\|Q) = \mu_n(Q\|P)$) if, and only if, $b = b'$, $c = c'$; and $\mu_n(P\|P) = 0$ if, and only if, $b + c + b' + c' = 0$. Thus the measures (10.61) are symmetric and have the property "the divergence is zero when $P = Q$" if, and only if, it is given by

$$\mu_n(P\|Q) = c[D_n(P\|Q) + D_n(Q\|P)].$$

- **Directed divergence.** A stronger version of the sum property is the f-divergence introduced in Csiszar [185]. The measure μ_n is an *f-divergence* if, and only if, it has the representation

$$\mu_n(P\|Q) = \sum q_i\, f(p_i/q_i)$$

for some $f :]0, \infty[\to \mathbb{R}$ (f need not be convex as is supposed in [161]). The measures (10.61) are f-divergences if, and only if,

$$\mu_n(P\|Q) = cD_n(P\|Q) + c'D_n(Q\|P).$$

In fact, since

$$\begin{aligned} bH_n(P) + b'H_n(Q) + cI_n(P\|Q) + c'I_n(Q\|P) \\ = -(b+c)\sum p_i \log p_i - (b'+c')\sum q_i \log p_i \\ + cD_n(P\|Q) - b'D_n(Q\|P), \end{aligned}$$

μ_n is an f-divergence if, and only if,

$$-(b+c)\sum p_i \log p_i - (b'+c')\sum q_i \log p_i = \sum q_i\, f(p_i/q_i)$$

for some f. This is possible if, and only if, $b+c = 0 = b'+c'$. Now the measures $\mu_n(P\|Q) = cD_n(P\|Q) + c'D_n(Q\|P)$ have the property

$$\sup_P \mu_n(P\|Q) < \infty$$

if, and only if, $c' = 0$ and consequently

$$\mu_n(P\|Q) = cD_n(P\|Q).$$

10.2g Sum Form Distance Measures

Over the years, many distance measures between discrete probability distributions have been proposed. The goal now is to give a survey of all important characterizations of sum form distance measures between discrete complete probability distributions and to point out some related functional equations [478, 151].

Almost all similarity, affinity, or distance measures $\mu_n : \Gamma_n^0 \times \Gamma_n^0 \to \mathbb{R}_+$ that have been proposed between two discrete probability distributions can be represented in the *sum form*

(10.54) $$\mu_n(P, Q) = \sum_{k=1}^{n} \phi(p_k, q_k),$$

where $\phi : I \times I \to \mathbb{R}$ is a real-valued function on a unit square or a monotonic transformation of the right side of (10.54), that is,

(10.54b) $$\mu_n(P, Q) = \psi\left(\sum_{k=1}^{n} \phi(p_k, q_k)\right),$$

where $\psi : \mathbb{R} \to \mathbb{R}_+$ is an increasing function on $\mathbb{R}$, $P, Q \in \Gamma_n^0$. The function ϕ is called a *generating function*. It is also referred to as the *kernel* of $\mu_n(P, Q)$ (Jones [397]). Some important examples of sum form distance measures between two discrete probability distributions P and Q in Γ_n^0 are:

(a) **Directed divergence** (Kannappan [431], Kannappan and Sahoo [510], Kullback and Leibler [571])

$$\phi(x, y) = x(\log x - \log y),$$
$$D_n(P, Q) = \sum_{k=1}^{n} p_k \log\left(\frac{p_k}{q_k}\right);$$

(b) **Symmetric J-divergence** (Jeffreys [393], Kannappan and Sahoo [510])

$$\phi(x, y) = (x - y)(\log x - \log y),$$
$$J_n(P, Q) = \sum_{k=1}^{n} (p_k - q_k) \log\left(\frac{p_k}{q_k}\right);$$

(c) **Hellinger coefficient** (Bhattacharyya [124], Hellinger [364])

$$\phi(x, y) = \sqrt{xy},$$
$$H_n(P, Q) = \sum_{k=1}^{n} \sqrt{p_k q_k};$$

(d) **Jeffreys distance** (Jeffreys [393])

$$\phi(x, y) = (\sqrt{x} - \sqrt{y})^2,$$
$$K_n(P, Q) = \sum_{k=1}^{n} (\sqrt{p_k} - \sqrt{q_k})^2;$$

(e) **Chernoff coefficient** (Chernoff [151], Kannappan and Sahoo [512])

$$\phi(x, y) = x^{\alpha} y^{1-\alpha},$$
$$C_{n,\alpha}(P, Q) = \sum_{k=1}^{n} p_k^{\alpha} q_k^{1-\alpha}, \quad \alpha \in I;$$

(f) **Matusita distance** (Matusita [626, 625])

$$\phi(x, y) = |x^{\alpha} - y^{\alpha}|^{\frac{1}{\alpha}},$$
$$M_{n,\alpha}(P, Q) = \sum_{k=1}^{n} |p_k^{\alpha} - q_k^{\alpha}|^{\frac{1}{\alpha}}, \quad 0 < \alpha \leq 1;$$

(g) **Divergence measure of degree** α (Aczél and Daróczy [43], Chung, Kannappan, Ng, and Sahoo [164])

$$\phi(x, y) = \frac{1}{2^{\alpha-1} - 1}[x^{\alpha} y^{1-\alpha} - x],$$
$$B_{n,\alpha}(P, Q) = \frac{1}{2^{\alpha-1} - 1}\left[\sum_{k=1}^{n}(p_k^{\alpha} q_k^{1-\alpha} - p_k)\right], \quad a \neq 1;$$

(h) **Cosine α-divergence measure** (Chung, Kannappan, Ng, and Sahoo [164])

$$\phi(x, y) = \frac{1}{2}\left[x - \sqrt{xy}\cos\left(\alpha \log\left(\frac{x}{y}\right)\right)\right],$$
$$N_{n,\alpha}(P, Q) = \frac{1}{2}\left[1 - \sum_{k=1}^{n}\sqrt{p_k q_k}\cos\left(\alpha \log\frac{p_k}{q_k}\right)\right];$$

(i) **Kagan affinity measure**

$$\phi(x, y) = \frac{(y - x)^2}{y},$$
$$A_n(P, Q) = \sum_{k=1}^{n} q_k\left[1 - \frac{p_k}{q_k}\right]^2;$$

(j) **Csiszar f-divergence measure** (Csiszar [184], Kailath [411])

$$\phi(x, y) = x f\left(\frac{x}{y}\right),$$
$$Z_{n,\alpha}(P, Q) = \sum_{k=1}^{n} p_k f\left(\frac{p_k}{q_k}\right);$$

(k) **Kullback-Leibler type f-distance measure** (Kannappan [468])

$$\phi(x, y) = x[f(x) - f(y)],$$
$$L_{n,\alpha}(P, Q) = \sum_{k=1}^{n} p_k[f(p_k) - f(q_k)];$$

(l) **Proportional distance** (Van der Pyl [812])

$$\phi(x, y) = \min\{x, y\},$$
$$X_n(P, Q) = \sum_{k=1}^{n} \min\{p_k, q_k\}.$$

(m) **Rényi's divergence measure** (Aczél and Daróczy [43])

$$\phi(x, y) = x^{\alpha} y^{1-\alpha}, \quad \psi(x) = \frac{1}{\alpha - 1} \log x,$$
$$R_{n,\alpha}(P \| Q) = \frac{1}{\alpha - 1} \log \left(\sum_{k=1}^{n} p_k^{\alpha} q_k^{1-\alpha} \right), \quad \alpha \neq 1,$$

is a monotonic transformation of sum form distance measures. Rényi's divergence measure is the logarithm of the so-called exponential entropy

$$E_{n,\alpha}(P, Q) = \left(\sum_{k=1}^{n} p_k^{\alpha} q_k^{1-\alpha} \right)^{\frac{1}{\alpha-1}}.$$

In order to derive the principle of minimum divergence axiomatically, Shore and Johnson [736] formulated a set of four axioms, namely *uniqueness, invariance, system independence*, and *subset independence.* They proved that if a functional $\mu_n : \Gamma_n^0 \times \Gamma_n^0 \to \mathbb{R}_+$ satisfies the axioms of uniqueness, invariance, and subset independence, then there exists a generating function (or kernel) $\phi : I \times I \to \mathbb{R}$ such that

$$\mu_n(P, Q) = \sum_{k=1}^{n} \phi(p_k, q_k)$$

for all $P, Q \in \Gamma_n^0$. In view of this result, one can conclude that the sum form representation above is not artificial. In most applications involving distance measures between probability distributions, one encounters the minimization of $\mu_n(P, Q)$ and the sum form representation makes problems tractable.

10.2h Kullback-Leibler Type Distance Measures

A function $\mu_n : \Gamma_n^0 \times \Gamma_n^0 \to \mathbb{R}_+$ is said to be a *distance measure of Kullback-Leibler type* [507, 468] if there exists a generating function $f : I \to \mathbb{R}$ such that

$$\mu_n(P, Q) = \sum_{k=1}^{n} p_k [f(p_k) - f(q_k)] \tag{10.54c}$$

for all $P, Q \in \Gamma_n^0$. It is easy to see that the directed divergence $D_n(P, Q)$ is a Kullback-Leibler type distance measure with the generating function $f(p) = \log p$. However, it is not obvious whether all Kullback-Leibler type distance measures are

essentially constant multiples of directed divergence. The following characterization result due to Kannappan and Sahoo [507] regarding Kullback-Leibler type distance measures says that if $n \geq 3$, then they are essentially constant multiples of directed divergence.

Result (i). *The distance measures of Kullback-Leibler type on Γ_n^0 for fixed $n \geq 3$ are of the form*

$$\mu_n(P, Q) = bD_n(P, Q),$$

where b is a nonnegative arbitrary constant and $D_n(P, Q)$ is the directed divergence between P and Q.

In view of the definition of a Kullback-Leibler type distance measure, one obtains a functional inequality

$$\sum_{k=1}^{n} p_k f(q_k) \leq \sum_{k=1}^{n} p_k f(p_k)$$

for all $P, Q \in \Gamma_n^0$ (with $n \geq 3$), which is considered in Chapter 16, on general inequalities.

Result (ii). *The distance measures of Kullback-Leibler type on Γ_2^0 with a continuous generating function are of the form*

$$\mu_2(P, Q) = \sum_{k=1}^{2} p_k \left[(1 - p_k) H\left(p_k - \frac{1}{2}\right) - (1 - q_k) H\left(q_k - \frac{1}{2}\right) \right] + \sum_{k=1}^{2} p_k \int_{q_k - \frac{1}{2}}^{p_k - \frac{1}{2}} H(t)dt,$$

where $H : \left]-\frac{1}{2}, \frac{1}{2}\right[\to \mathbb{R}$ is an arbitrary continuous, increasing odd function.

Note that, without the continuity assumption on the generating function, the general form of the distance measures of Kullback-Leibler type on Γ_2^0 (that is, on the set of Bernoulli distributions) is not known. This is due to the fact that the general solution of the functional inequality

$$x f(y) + (1 - x) f(1 - y) \leq x f(x) + (1 - x) f(1 - x)$$

for all $x, y \in I$ is unknown.

However, in general, $\mu_2(P, Q)$ includes measures other than $D_2(P, Q)$, where $D_2(P, Q)$ is the directed divergence between P and Q. For instance, $H(t) = t$ yields $\mu_2(P, Q) = \frac{1}{2} \sum_{k=1}^{2} p_k(p_k - q_k)(2 - p_k - q_k)$.

10.2i Symmetric Distance Measures

Many distance measures we listed above are not symmetric. If a measure is not symmetric, it can be easily symmetrized. For example, the divergence measure of degree α can be symmetrized by taking

$$J_{n,\alpha}(P, Q) = B_{n,\alpha}(P, Q) + B_{n,\alpha}(Q, P).$$

In explicit form, $J_{n,\alpha}$ is

$$J_{n,\alpha}(P, Q) = \frac{\sum_{k=1}^{n}[p_k^{\alpha} q_k^{1-\alpha} + q_k^{\alpha} p_k^{1-\alpha}] - 2}{2^{\alpha-1} - 1}.$$

Note that if $\alpha \to 1$, then $\lim_{\alpha \to 1} J_{n,\alpha}(P, Q) = J_n(P, Q)$. Characterizations of symmetric divergence measures of degree α on closed domains can be found in [689].

In this section, we present some characterization results from Chung, Kannappan, Ng, and Sahoo [164]. We need the following definition to state the result precisely. A sequence of measures $\{\mu_n\}$ is said to be symmetrically compositive if, for some $\lambda \in \mathbb{R}$,

$$\mu_{nm}(P * R, Q * S) + \mu_{nm}(P * S, Q * R) = 2\mu_n(P, Q) + 2\mu_m(R, S) + \lambda \mu_n(P, Q)\mu_m(R, S)$$

for all $P, Q \in \Gamma_n^0$, $S, R \in \Gamma_m^0$. If $\lambda = 0$, then $\{\mu_n\}$ is said to be symmetrically additive.

For applications of symmetric divergence to "pattern recognition" and "questionnaire analysis," the reader is referred to Ton and Gonzales [805] and Haaland, Brockett, and Levine [332], respectively.

Result (iii). *Suppose that a sequence of measures $\{\mu_n\}$ has the sum form* (10.5) *with a measurable generating function $\phi : I^2 \to \mathbb{R}$ and is symmetrically compositive for all pairs of positive integers m, n (≥ 2). Then*

$$\mu_n(P, Q) = \sum_{k=1}^{n}[p_k(a \log p_k + b \log q_k) + p_k(a \log q_k + b \log p_k)]$$

if $\lambda = 0$, or else

$$\mu_n(P, Q) = \frac{1}{\lambda}\left[\sum_{k=1}^{n}(p_k^{\alpha} q_k^{\beta} + p_k^{\beta} q_k^{\alpha}) - 2\right],$$

or

$$\mu_n(P, Q) = -\frac{2}{\lambda}\left[1 - \sum_{k=1}^{n}(p_k q_k)^{\beta} \cos\left(\alpha \log \frac{p_k}{q_k}\right)\right],$$

or

$$\mu_n(P, Q) = -\frac{2}{\lambda},$$

where a, b, α, and β are real arbitrary constants.

If the sequence $\{\mu_n\}$ is the measuring distance between P and Q, it is natural to assume that

$$\mu_n(P, P) = 0.$$

Result (iv). *A measure of distance* $\{\mu_n\}$ *satisfies the hypothesis in Result* (iii) *and* $\mu_n(P, P) = 0$ *if, and only if,*

$$\mu_n(P, Q) = a\, J_n(P, Q),$$

or

$$\mu_n(P, Q) = \frac{2^{\alpha-1} - 1}{\lambda} J_{n,\alpha}(P, Q),$$

or

$$\mu_n(P, Q) = -\frac{4}{\lambda} N_{n,\alpha}(P, Q).$$

The proof of the previous two results relies on the solutions of the functional equations

$$f(pr, qs) + f(ps, qr) = (r + s) f(p, q) + (p + q) f(r, s) \tag{10.62}$$

and

$$f(pr, qs) + f(ps, qr) = f(p, q) f(r, s) \tag{10.62a}$$

for all $p, q, r, s \in I$ (solution below).

10.2j Some Functional Equations

The solutions of (10.62) and (10.62a) were found in [164, 512] without any regularity conditions on f. A multiplicative map $M : I_0 \to \mathbb{R}$ has a unique extension to the multiplicative map $\overline{M} :]0, \infty[\to \mathbb{R}$. A logarithmic map $L :]0, 1[\to \mathbb{R}$ has a unique extension $\overline{L} :]0, \infty[\to \mathbb{R}$.

Result (v). *A function* $f : I_0^2 \to \mathbb{R}$ *satisfies the functional equation* (10.62) *for all* $p, q, r, s \in I_0$ *if, and only if,*

$$f(p, q) = p[L_1(q) - L_2(p)] + q[L_1(p) - L_2(q)],$$

where L_1, L_2 *are logarithmic on* $]0, \infty[$.

The following result yields the solution of the functional equation (10.62a), whose proof can be found in [164].

Result (vi). *Suppose* $f : I_0^2 \to \mathbb{R}$ *satisfies the functional equation* (10.62a) *for all* $p, q, r, s \in I_0$. *Then*

$$f(p, q) = M_1(p)M_2(q) + M_1(q)M_2(p),$$

where $M_1, M_2 : I_0 \to \mathbb{C}$ *are multiplicative. Further, either* M_1 *and* M_2 *are both real or* M_2 *is the complex conjugate of* M_1. *The converse is also true.*

The functional equations (10.62) and (10.62a) can be generalized to

$$f_1(pr, qs) + f_2(ps, qr) = g_1(r, s)h_1(p, q) + g_2(p, q)h_2(r, s) \tag{10.62b}$$

for all $p, q, r, s \in I_0$. Our problem is the following: Find all functions $f_1, f_2, g_1, g_2, h_1, h_2 : I_0 \to \mathbb{C}$ satisfying the functional equation (10.62b) for all $p, q, r, s \in I_0$. To solve this equation, one studies a series of functional equations that are special cases of (10.62b). These equations are the following:

$$f(pr, qs) + f(ps, qr) = (r + s) f(p, q) + (p + q) f(r, s), \quad [164]; \tag{10.62c}$$

$$f(pr, qs) + f(ps, qr) = f(p, q) f(r, s), \quad [164]; \tag{10.62d}$$

$$f(pr, qs) + f(ps, qr) = 2 f(p, q) + 2 f(r, s);$$

$$f_1(pr, qs) + f_2(ps, qr) = (r + s) f_3(p, q) + (p + q) f_4(r, s), \quad [503];$$

$$f_1(pr, qs) + f_2(ps, qr) = f_3(p, q) f_4(r, s);$$

$$f_1(pr, qs) + f_2(ps, qr) = f_3(p, q) + f_4(r, s);$$

$$f(pr, qs) + f(ps, qr) = g(r, s) f(p, q) + g(p, q) f(r, s);$$

$$f(pr, qs) + f(ps, qr) = g(r, s)h(p, q) + g(p, q)h(r, s);$$

$$f(pr, qs) + f(ps, qr) = g_1(r, s)h_1(p, q) + g_2(p, q)h_2(r, s);$$

$$f(pr, qs) - f(ps, qr) = g_1(r, s)h_1(p, q) + g_2(p, q)h_2(r, s).$$

Solutions of equations (10.62c) and (10.62d) are known if they hold for all $p, q, r, s \in I_0$ (see Results (v) and (vi)). The solutions of some of these equations on I_0 are not known.

We are not aware of any research work that characterizes all the existing sum form information measures in a systematic and unified manner. However, there are many research articles that deal with similarity, dissimilarity, affinity, or distance measures. The interested reader should refer to Basseville [105], Chung, Kannappan, and Ng [162], Van der Pyl [812], and references therein.

10.2k Weighted Entropies

The Shannon entropy gives us the measure of information as a function of the probabilities with which various events occur. But there exist many situations where it is necessary to take into account both these probabilities and some qualitative characteristics of the physical events. With this in mind, first we consider the following measures. Measures depending on the events themselves (mixed theory) will be considered in the next section.

Let (Ω, A, μ) be a probability space, and let us consider an experiment that is a finite measurable partition $\{C_1, C_2, \ldots, C_n\}$ $(n > 1)$ of Ω. The *weighted entropy* of such an experiment is defined by Belis and Guiasu [112] and in [331] as

$$H_n(P; W) = -\sum_{i=1}^{n} w_i p_i \log p_i, \tag{10.5}$$

and the weighted entropy of degree β ($\neq 1$) introduced in [258] is

$$H_n^\beta(P; W) = (2^{1-\beta} - 1)^{-1} \sum_{i=1}^{n} w_i (p_i^\beta - p_i), \tag{10.5a}$$

where $p_i = \mu(C_i)$ is the objective probability of the event C_i, $P \in \Gamma_n$, and $W = (w_1, w_2, \ldots, w_n)$ with $w_i \in \mathbb{R}_+$ the nonnegative reals.

A sequence of measures $\mu_n : \Gamma_n \times \mathbb{R}_+^n \to \mathbb{R}$ (reals) is said to have the sum property provided there exists a function $f : I \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\mu_n(P; W) = \sum_{i=1}^{n} f(p_i, w_i) \tag{10.63}$$

for all $P \in \Gamma_n$, $W \in \mathbb{R}_+^n$, where $I = [0, 1]$ and the function f is called a generating function. Evidently, the weighted entropies (10.5) and (10.5a) possess the sum property.

The weighted entropy of degree β satisfies the composition law

$$\begin{aligned} H_{mn}^\beta(P * Q, U * V) = H_n^\beta(P, U) \cdot \sum_{j=1}^{m} v_j q_j + H_m^\beta(Q, V) \cdot \sum_{i=1}^{m} u_i p_i \\ + \lambda H_n^\beta(P, U) \cdot H_m^\beta(Q, V) \end{aligned} \tag{10.63a}$$

for all $P \in \Gamma_n$, $Q \in \Gamma_m$, $U \in \mathbb{R}_+^n$, $V \in \mathbb{R}_+^m$, where $\lambda = (2^{1-\beta} - 1)^{-1}$. This property is known as the *weighted λ-additivity of H_n^β*. If $\beta \to 1$, then (10.63a) yields

$$H_{mn}(P * Q, U * V) = H_n(P, U) \cdot \sum_{j=1}^{m} v_j q_j + H_m(Q, V) \cdot \sum_{i=1}^{n} u_i p_i \tag{10.63b}$$

(which also holds for $\lambda = 0$). The property above is called the *weighted additivity of H_n^β*.

If f is Lebesgue measurable in each variable in (10.63), then $\{\mu_n\}$ is said to have *measurable sum form.* A sequence of information measures $\{\mu_n\}$ is said to be *weighted (ℓ, m)-additive of type (α, β)* if

$$\mu_{\ell m}(P * Q, U * V) = \sum_{j=1}^{m} v_j q_j^\beta \mu_\ell(P, U) + \sum_{i=1}^{\ell} u_i p_i^\alpha \mu_m(Q, V) \tag{10.63c}$$

for all $P \in \Gamma_\ell$, $Q \in \Gamma_m$, $U \in \mathbb{R}_+^\ell$, $V \in \mathbb{R}_+^m$, and $\alpha, \beta \in \mathbb{R} - \{0\}$. Notice that if $\alpha = 1 = \beta$, then (10.63c) reduces to the weighted (ℓ, m)-additivity.

In axiomatic characterizations of information measures, one identifies some information-theoretic properties (such as sum form, recursivity, additivity, additivity of type (α, β), and so forth) and other relevant characteristics (such as monotonicity, boundedness, continuity, and so forth) that restrict the information measures to

some class of admissible functional forms. The objective is to seek the representation of all sum form information measures that are weighted entropies. The weighted entropies possess the algebraic properties of "sum representation" and some form of "additivity". Characterization of the weighted entropies by these properties led to the study of many "sum form" functional equations.

The measures $\mu_n : \Gamma_n \times \mathbb{R}_+^n \to \mathbb{R}$ are called weighted β-additive (weighted additive when $\beta = 1$) provided

$$\mu_{mn}(P * Q; W * V) = \sum_{j=1}^{m} q_j v_j \cdot \mu_n(P; W) + \sum_{i=1}^{n} p_i^{\beta} w_i \cdot \mu_m(Q; V) \tag{10.63d}$$

for $P \in \Gamma_n,\ Q \in \Gamma_m,\ W \in \mathbb{R}_+^n,\ V \in \mathbb{R}_+^m$.

Note that the sum property in conjunction with the "additivity" and a regularity condition with some initial conditions characterize these measures. The properties of additivity and the sum representation led to the study of the "sum form" functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j, w_i v_j) = \sum_{j=1}^{m} q_j v_j \sum_{i=1}^{n} f(p_i, w_i) + \sum_{i=1}^{n} p_i^{\beta} w_i \sum_{j=1}^{m} f(q_j, v_j), \tag{10.64}$$

whose measurable (in each variable) solution $f : I \times \mathbb{R}_+ \to \mathbb{R}$ is given by [448]

$$f(p,u) = \begin{cases} apu \log p + bpu \log u, & \beta = 1, \\ cu(p^{\beta} - p), & \beta \neq 1, \end{cases}$$

where a, b, c are constants.

A generalization of (10.64),

$$\sum_{i=1}^{n}\sum_{j=1}^{m} h(p_i q_j, w_i v_j) = \sum_{j} q_j v_j \cdot \sum_{i} k(p_i, w_i) + \sum_{i} p_i^{\beta} w_i \cdot \sum_{j} g(g_j, v_j), \tag{10.64a}$$

treated in Kannappan [454, 478], has the measurable solutions $h, g, k : I \times \mathbb{R}_+ \to \mathbb{R}$ given by

$$\left.\begin{aligned} g(p,u) &= apu \log p + bpu \log u + c_1 pu, \\ k(p,u) &= apu \log p + bpu \log u + cpu, \\ h(p,u) &= pu(a \log p + b \log u + c + c_1), \end{aligned}\right\} \quad \beta = 1,$$

$$\left.\begin{aligned} g(p,u) &= ap^{\beta} u + bpu, \\ k(p,u) &= -bp^{\beta} u + bp + cp + d, \\ h(p,u) &= bpu + ap^{\beta} u, \end{aligned}\right\} \quad \beta \neq 1.$$

Another generalization of (10.64) is the equation

(10.64b)
$$\sum_{i=1}^{n}\sum_{i=j}^{m} f(p_iq_j, w_iv_j) = \sum_{j=1}^{m} q_j^{\alpha}v_j^{\beta}\sum_{i=1}^{n} g(p_i, w_i) + \sum_{i=1}^{n} p_i^{\gamma}w_i^{\delta}\sum_{j=1}^{m} h(q_j, v_j),$$

where $\alpha, \beta, \gamma, \delta$ are nonzero reals.

Result 10.20. (Kannappan [452]). *Let $f, g, h : I \times \mathbb{R}_+ \to \mathbb{R}$ be measurable in each variable and satisfy* (10.64b) *for a fixed pair $m, n \geq 3$. Then they are given by*

$$f(x,u) = x^{\alpha}u^{\beta}(d\log x + a\log u + b + c) + a_1x + b_1,$$
$$g(x,u) = x^{\alpha}u^{\beta}(d\log x + a\log u + b) + a_2x + b_2,$$
$$h(x,u) = x^{\alpha}u^{\beta}(d\log x + a\log u + c) + a_3x + b_3,$$

for $\alpha = \gamma,\ \beta = \delta$, and

$$f(x,u) = ax^{\alpha}u^{\beta} + bx^{\gamma}u^{\delta} + a_1x + b_1,$$
$$g(x,u) = ax^{\alpha}u^{\beta} + cx^{\gamma}u^{\delta} + a_2x + b_2,$$
$$h(x,u) = -cx^{\alpha}u^{\beta} + bx^{\gamma}u^{\delta} + a_3x + b_3$$

otherwise, with $a_1 + mnb_1 = 0 = a_2 + nb_2 = a_3 + mb_3$, where a, b, c, and d are arbitrary constants.

The sum form and weighted λ-additivity led to the study of the functional equation

(10.65)
$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_iq_j, u_iv_j)\,f(p_i, u_i)$$
$$= \sum_{j=1}^{m} v_jq_j\sum_{i=1}^{n} + \sum_{i=1}^{n} u_ip_i\sum_{j=1}^{m} f(q_j, v_j) + \lambda\sum_{i=1}^{n} f(p_i, u_i)\sum_{j=1}^{n} f(q_j, v_j),$$

where $P \in \Gamma_n^0,\ Q \in \Gamma_m^0,\ U \in \mathbb{R}_+^n,\ V \in \mathbb{R}_+^m$, and λ a real constant. For $\lambda \neq 0$, with

$$F(p,u) := pu + \lambda f(p,u),$$

(10.65) reduces to

(10.65a)
$$\sum_{i=1}^{n}\sum_{j=1}^{m} F(p_iq_j, u_iv_j) = \sum_{i=1}^{n} F(p_i, u_i)\cdot\sum_{j=1}^{m} F(q_j, v_j),$$

for all $P \in \Gamma_n^0,\ Q \in \Gamma_m^0,\ U \in \mathbb{R}_+^n$, and $V \in \mathbb{R}_+^m$, whose general solution $f : I_0 \times \mathbb{R}_+ \to \mathbb{R}$ (holding for a fixed pair $m, n\ (\geq 3)$) is given by [503]

$$F(p,u) = A(p) + c, \quad \text{for } p \in I_0,\ u \in \mathbb{R}_+,$$

with condition $(A(1) + mnc) = (A(1) + nc)(A(1) + mc)$, where A is additive and c is a constant.

A sequence of information measures $\{\mu_n\}$ is said to be *weighted* (ℓ, m)*-additive of type* (α, β) [506] if

$$\mu_{\ell m}(P * Q, U * V) = \sum_{j=1}^{m} v_j q_j^{\beta} \cdot \mu_\ell(P,U) + \sum_{i=1}^{\ell} u_i p_i^{\beta} \mu_m(Q,V)$$

for all $P \in \Gamma_\ell^0,\ Q \in \Gamma_m^0,\ U \in \mathbb{R}_+^\ell,\ V \in \mathbb{R}_+^m$, and $\alpha, \beta \in \mathbb{R} - \{0\}$.

From the sum representation and the weighted (ℓ, m)-additivity of type (α, β) there results the functional equation

(10.64c)

$$\sum_{i=1}^{\ell}\sum_{j=1}^{m} f(p_i q_j, u_i v_j) = \sum_i u_i p_i^{\alpha} \cdot \sum_j f(q_j, v_j) + \sum_j v_j q_j^{\beta} \cdot \sum_i f(p_i, u_i),$$

the general solution of which (10.64c), when $\ell = 2$, and $m = 3$, is given by [506]

$$f(p,u) = \begin{cases} A(p) + au(p^{\alpha} - p^{\beta}) & \text{if } \alpha \neq \beta, \\ A(p) + L(p)up^{\alpha} + L_1(u)up^{\alpha} & \text{if } \alpha \neq \beta, \end{cases}$$

where L is logarithmic on I_0, L_1 is logarithmic on $\mathbb{R}_+$, and A is additive on $\mathbb{R}$.

Result 10.21. [506]. *Let* $\mu_n : \Gamma_n^0 \times \mathbb{R}_+^n \to \mathbb{R}$ $(n = 2, 3, \ldots)$ *have the sum form and satisfy the weighted* $(2, 3)$*-additivity of type* (α, β)*, where* $\alpha, \beta \in \mathbb{R} - \{0, 1\}$*. Then*

$$\mu_n(P,U) = \begin{cases} a \sum_{i=1}^{n} u_i (p_i^{\alpha} - p_i^{\beta}) & \text{if } \alpha \neq \beta, \\ \sum_{i=1}^{n} u_i p_i^{\alpha} L(p_i) + \sum_{i=1}^{n} p_i^{\alpha} u_i L_1(u_i) & \text{if } \alpha = \beta, \end{cases}$$

where $L : I_0 \to \mathbb{R}$, $L_1 : \mathbb{R}_+ \to \mathbb{R}$ *are logarithmic functions and* a *is a constant.*

The measures obtained contain some of the known weighted information measures.

10.3 Mixed Theory of Information—Inset Measures

In the last section, we considered measures (weighted) that depend not only on the probability but also on some qualitative characteristics of the events. Finally, in this section, we consider measures that depend not only on the probability of the events but also on the events themselves that were initiated and developed by Aczél and Daróczy [42, 51]. The resulting functional equations involve set-theoretic parts that

are difficult to solve. Now we present a brief survey of this mixed theory with some applications (see [42, 57, 51, 446, 22, 19, 23, 520, 246, 725, 633, 50, 453, 521, 239, 236, 459]).

Measures of information or uncertainty depend in the classical information theory *solely* on the probabilities of the events (an event may be the receipt of a message, an answer on a questionnaire, a gamble, an outcome of an experiment, etc.), *not* on the events themselves [43]. In the nonprobabilistic theory of information, it is assumed that the measure is determined solely by the events themselves. Now, a mixed theory of information is proposed, where measures of information may depend both on the events and their probabilities.

One of the aims of the mixed theory of information is to find and characterize inset measures such as *inset entropies*, *inset derivations*, and other similar measures of information having certain useful properties analogous to those in the probabilistic theory of information (classical information theory). There is an obvious advantage in allowing the measures of information to depend also on the events and not just their probabilities. To this end, here information measures known as "entropy of a randomized system of events" or "inset entropy" are defined and studied as below.

A nontrivial example of a previously known information measure that fits better into the new theory than into the classical one is provided later. It is shown how the inset entropies arise from entirely different assumptions in the theory of gambles. The theory of gambles deals with the problem of characterizing the preferences of an individual who selects one gamble out of a set of gambles. A *gamble* is a lottery, bet, or risky situation with various (specified) possible mutually exclusive and exhaustive outcomes, each outcome having a (specified) surjective probability that it will have occurred by some definite future time. The possible outcomes of a gamble are called *prospects*. The traditional solution of predicting the choice is for the individual to assign a utility number to each outcome of the gamble, multiply the utility number by the probability of the outcome, add the products to give the expected utility, and then choose the gamble with the largest expected utility. This method is called the *Expected Utility Rule*. If the function that computes the expected utility of the gamble from the assigned utility numbers is generalized, then these functions that give the new rule and still satisfy the basic properties of the gambles are to be determined. It turns out that all possible functions for the Generalized Utility Rule under the assumptions are given by the inset entropies or by the inset entropies of degree β (β (> 0) $\neq 1$).

Notation and Definition

Let B be a ring of sets (containing with any two sets their union and difference and thus also their intersection and the 0-set) with its elements the "events"

$$\Omega_n = \{X = (x_1, x_2, \ldots, x_n) \mid x_i \cap x_j = 0 \text{ for } i \neq j;\ x_1 \in B;\ i, j = 1, 2, \ldots, n\}.$$

Then $(X, P) \in \Omega_n \times \Gamma_n$ is called a randomized system of events.

Inset Entropy. An entropy of a randomized system of events, an *inset entropy* for short, is a sequence of mappings $\mu_n : \Omega_n \times \Gamma_n \to \mathbb{R}$ ($n = 2, 3, \ldots$; $\mathbb{R}$ the set of real numbers).

Various systems of postulates were used to characterize the measures: Shannon's entropy, entropy of degree $\beta \neq 1$, directed and generalized directed divergences, etc.

Here we intend to consider their generalizations to inset entropy, inset entropy of degree β, inset derivation, etc.

(i) **Symmetry.** An inset entropy μ_m is m-symmetric ($m \geq 2$) if

$$\mu_m \begin{pmatrix} x_1, x_2, \ldots, x_m \\ p_1, p_2, \ldots, p_m \end{pmatrix} = \mu_m \begin{pmatrix} x_{\alpha(1)}, x_{\alpha(2)}, \ldots, x_{\alpha(m)} \\ p_{\alpha(1)}, p_{\alpha(2)}, \ldots, p_{\alpha(m)} \end{pmatrix}$$

for all $(X, P) \in \Omega_m \times \Gamma_m$ and all permutations α on $\{1, 2, \ldots, m\}$. Symmetry means that the uncertainty given by the measure does not depend on the labelling of events.

(ii) **β-recursivity.** The sequence μ_n is β-recursive if, for all $n > 2$ and all randomized systems of events $(X, P) \in \Omega_n \times \Gamma_n$, it satisfies

$$\mu_n \begin{pmatrix} x_1, x_2, \ldots, x_n \\ p_1, p_2, \ldots, p_n \end{pmatrix} = \mu_{n-1} \begin{pmatrix} x_1 \cup x_2, x_3, \ldots, x_n \\ p_1 + p_2, p_3, \ldots, p_n \end{pmatrix} + (p_1 + p_2)^\beta \mu_2 \begin{pmatrix} x_1 & x_2 \\ \frac{p_1}{p_1+p_2} & \frac{p_2}{p_1+p_2} \end{pmatrix}$$

with the convention $0^\beta \cdot \begin{pmatrix} x_1 & x_2 \\ 0/0 & 0/0 \end{pmatrix} := 0$.

(ii′) **Recursive.** The sequence μ_n is said to be *recursive* if $\beta = 1$ in (ii).
Recursivity shows how splitting the event $x_1 \cup x_2$ into the two events x_1 and x_2 changes the uncertainty.

(iii) **Measurability.** An inset entropy is measurable if the function

$$t \to \mu_2 \begin{pmatrix} x_1 & x_2 \\ 1-t & t \end{pmatrix}$$

is measurable on $]0, 1[$ for all fixed $(x_1, x_2) \in \Omega_2$.

(iv) A deviation of a randomized system of events, an *"inset deviation"* for short, is a sequence $K_n : \Omega_n \times \Gamma_n \times \Gamma_n \to \mathbb{R}$ ($n = 2, 3, \ldots$).

10.3.1 Characterizations

The definitions above lead to the following characterization results [42, 51] regarding inset entropy, inset entropy of degree ($\beta \neq 1$), inset deviation, etc.

Result 10.22. (Aczél and Daróczy [42]). *The sequence $\mu_n : \Omega_n \times \Gamma_n \to \mathbb{R}$ ($n = 2, 3, \ldots$) is recursive* (ii′), 3*-symmetric, and measurable if, and only if, there exists a constant $c \in \mathbb{R}$ and a function $g : B \to \mathbb{R}$ such that*

$$\mu_n(X, P) = g\left(\bigcup_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} p_i g(x_i) - c\sum_{i=1}^{n} p_i \log p_i \tag{10.66}$$

for all $(X, P) \in \Omega_n \times \Gamma_n$ $(n \geq 2)$ *with the convention* $0 \cdot \log 0 := 0$.

Remark. The last member in (10.66) is, of course, the Shannon entropy H_n. If $\bigcup_{i=1}^{n} x_i$ is the whole space Ω (the certain event) (i.e., if the system $x_1, x_2, \ldots, x_n$ is complete), then $g(\cup x_i) = b$ is a constant and letting $h(x) = b - g(x)$ in (10.66) gives

$$\mu_n(X, P) = \sum_{i=1}^{n} p_i h(x_i) - c\sum_{i} p_i \log p_i. \tag{10.66a}$$

Thus, in the case of complete systems of events, the general recursive, 3-symmetric, and measurable inset entropies are sums of the expected value of an arbitrary random variable and of an arbitrary constant multiple of the Shannon entropy.

Result 10.23. (Aczél and Kannappan [51]). *There exists a function* $k : B \to \mathbb{R}$ *such that*

$$\mu_n(X, P) = \sum_{i=1}^{n} k(x_i) p_i^{\beta} - k\left(\bigcup_{i=1}^{n} x_i\right) \tag{10.67}$$

for all $(X, P) \in \Omega_n \times \Gamma_n$ *with* $\beta \neq 0, 1$ *and with the convention* $0^{\beta} := 0$ *if, and only if, the sequence* $\mu_n : \Omega_n \times \Gamma_n \to \mathbb{R}$ $(n \geq 2)$ *is* 3*-symmetric and* β*-recursive* (ii).

Remark. If k is constant ($a(2^{1-\beta} - 1)^{-}1$, say), then (10.67) is a constant multiple of the entropy of degree β, $H_n^{\beta}(P)$. If we define $g(x) := a(2^{1-\beta} - 1) - k(x)$, then (10.67) goes over into

$$\mu_n(X, P) = g\left(\bigcup_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} p_i^{\beta} g(x_i) + a H_n^{\beta}(P). \tag{10.67a}$$

As $\beta \to 1$, (10.67a) tends to $g(\cup x_i) - \sum p_i g(x_i) + a H_n(P)$, the result obtained in Result 10.22 [42], under the measurability condition, in the case $\beta = 1$, where H_n is the Shannon entropy.

Remark. The main step to obtain (10.66), (10.66a), and (10.67) is to solve the functional equation

$$f_1(s) + (1 - s)^{\beta} f_2\left(\frac{t}{1 - s}\right) = f_3(t) + (1 - t)^{\beta} f_4\left(\frac{s}{1 - t}\right)$$

(see [442, 489, 495]), a generalization of the fundamental equation of information that is derived from the 3-symmetry and recursivity (β-recursivity) of μ_n.

10.3.1.1 Characterization of Inset Deviation Entropies

To illustrate the interplay of sets in functional equations, we prove Theorem 10.24, which follows.

An inset deviation is *recursive* if, for all $n > 2$ and all randomized systems of events

$$\begin{pmatrix} x_1, \ldots, x_n \\ p_1, \ldots, p_n \\ q_1, \ldots, q_n \end{pmatrix} \in \Omega_n \times \Gamma_n \times \Gamma_n,$$

it satisfies

$$K_n \begin{pmatrix} x_1, x_2, \ldots, x_n \\ p_1, p_2, \ldots, p_n \\ q_1, q_2, \ldots, q_n \end{pmatrix} = K_{n-1} \begin{pmatrix} x_1 \cup x_2, x_3, \ldots, x_n \\ p_1 + p_2, p_3, \ldots, p_n \\ q_1 + q_2, q_3, \ldots, q_n \end{pmatrix} + (p_1 + p_2) K_2 \begin{pmatrix} x_1 & x_2 \\ p_1/(p_1 + p_2) & p_2/(p_1 + p_2) \\ q_1/(q_1 + q_2) & q_2/(q_1 + q_2) \end{pmatrix}$$

with the convention

$$0 \cdot K_2 \begin{pmatrix} x_1 & x_2 \\ 0/0 & 0/0 \\ 0/0 & 0/0 \end{pmatrix} = 0.$$

The sequence $\{K_n\}$ is *m-symmetric* ($m \geq 2$) if

$$K_m \begin{pmatrix} x_1, \ldots, x_m \\ p_1, \ldots, p_m \\ q_1, \ldots, q_m \end{pmatrix} = K_m \begin{pmatrix} x_{r(1)}, \ldots, x_{r(m)} \\ p_{\alpha(1)}, \ldots, p_{\alpha(m)} \\ q_{\alpha(1)}, \ldots, q_{\alpha(m)} \end{pmatrix}$$

for all

$$\begin{pmatrix} x_1, \ldots, x_m \\ p_1, \ldots, p_m \\ q_1, \ldots, q_m \end{pmatrix} \in \Omega_m \times \Gamma_m \times \Gamma_m$$

and all permutations α on $\{1, 2, \ldots, m\}$ (meaning simply that the uncertainty does not depend on the labelling of events). Finally, our inset deviation is *measurable* if the function

$$(t, u) \to K_2 \begin{pmatrix} x_1 & x_2 \\ 1 - t & t \\ 1 - u & u \end{pmatrix}$$

is measurable in each variable t and u on $J = ([0, 1] \times]0, 1[) \cup \{(0, 0)\} \cup \{(1, 1)\}$ for each fixed $(x_1, x_2) \in \Omega_2$. We will prove the following theorem.

Theorem 10.24. (Kannappan [446]). *The sequence $K_n : \Omega_n \times \Gamma_n^2 \to \mathbb{R}$ ($n \geq 2$) is recursive, 3-symmetric, and measurable if, and only if, there exists a function $g : B \to \mathbb{R}$ such that*

$$K_n\begin{pmatrix} x_1,\dots,x_n \\ p_1,\dots,p_n \\ q_1,\dots,q_n \end{pmatrix} = g\left(\bigcup_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} p_i g(x_i) - a\sum_{i=1}^{n} p_i \log p_i + c\sum_{i=1}^{n} p_i \log q_i \tag{10.68}$$

for all

$$\begin{pmatrix} x_1,\dots,x_n \\ p_1,\dots,p_n \\ q_1,\dots,q_n \end{pmatrix} \in \Omega_n \times \Gamma_n \times \Gamma_n,$$

where a and c are arbitrary constants.

Remark. We follow the conventions $0\cdot\log 0 = 0$; whenever $q_i = 0$, the corresponding $p_i = 0$.

Proof. It is evident that any inset deviation given by (10.68) with arbitrary $a, c \in \mathbb{R}$ and $g : B \to \mathbb{R}$ is recursive, symmetric, and measurable. Now we prove the converse.

We introduce a function $F : \Omega_2 \times J \to \mathbb{R}$ by

$$F(x_1, x_2; t, u) = K_2\begin{pmatrix} x_1 & x_2 \\ 1-t & t \\ 1-u & u \end{pmatrix}. \tag{10.69}$$

Then, for each fixed $(x_1, x_2) \in \Omega_2$, F is measurable in each variable t and u.

Recursivity for $n = 3$ and 3-symmetry has two important consequences. On the one hand, it shows that K_2 is symmetric, too, which by (10.69) yields

$$F(x_1, x_2; t, u) = F(x_2, x_1; 1-t, 1-u)$$

for all $(x_1, x_2) \in \Omega_2$, $(t, u) \in J$. On the other hand, they give

$$\begin{aligned} &K_2\begin{pmatrix} x_1 \cup x_2 & x_3 \\ 1-t & t \\ 1-u & u \end{pmatrix} + (1-t)K_2\begin{pmatrix} x_1 & x_2 \\ 1-s/(1-t) & s/(1-t) \\ 1-v/(1-u) & v/(1-u) \end{pmatrix} \\ &= K_3\begin{pmatrix} x_1 & x_2 & x_3 \\ 1-t-s & s & t \\ 1-u-v & v & u \end{pmatrix} \\ &= K_3\begin{pmatrix} x_1 & x_3 & x_2 \\ 1-t-s & t & s \\ 1-u-v & u & v \end{pmatrix} \\ &= K_2\begin{pmatrix} x_1 \cup x_3 & x_2 \\ 1-s & s \\ 1-v & v \end{pmatrix} + (1-s)K_2\begin{pmatrix} x_1 & x_3 \\ 1-t/(1-s) & t/(1-s) \\ 1-u/(1-v) & u/(1-v) \end{pmatrix}. \end{aligned}$$

In other words, F given by (10.69) satisfies the functional equation

$$\begin{aligned} F(x_1 \cup x_2, x_3; t, u) + (1-t)F\left(x_1, x_2; \frac{s}{1-t}, \frac{v}{1-u}\right) \\ = F(x_1 \cup x_3, x_2; s, v) + (1-s)F\left(x_1, x_3; \frac{t}{1-s}, \frac{v}{1-v}\right), \end{aligned} \tag{10.70}$$

for all $(x_1, x_2, x_3) \in \Omega_3$, $(t, u), (s, v) \in ([0, 1] \times]0, 1[) \cup \{(0, 0)\}$ with $0 \le t + s \le 1$, $0 \le u + v \le 1$.

For fixed $(x_1, x_2, x_3) \in \Omega_3$ and u, v in $]0, 1[$, (10.70) reduces to the equation

$$F_1(t) + (1-t)F_2\left(\frac{s}{1-t}\right) = F_3(s) + (1-s)F_4\left(\frac{t}{1-s}\right) \tag{10.70a}$$

for $t, s \in]0, 1[$ with $0 \le t + s \le 1$, with the notation

$$\left.\begin{aligned} &F_1(t) = F(x_1 \cup x_2, x_3; t, u),\ F_3(s) = F(x_1 \cup x_3, x_2; s, v) \\ &F_2(t) = F\left(x_1, x_2; t, \frac{v}{1-u}\right),\ F_4(s) = F\left(x_1, x_3; s, \frac{u}{1-v}\right) \end{aligned}\right\}. \tag{10.70b}$$

The general solutions, measurable on $]0, 1[$, have been determined for this equation (10.70a) in [489], from which and from (10.70b) we obtain F_2, F_3, and F_4, which we do not need here, and

$$\begin{aligned} F_1(t) &= F(x_1 \cup x_2, x_3; t, u) \\ &= -a(x_1 \cup x_2, x_3; u)(t \log t + (1-t)\log(1-t)) \\ &\quad + b(x_1 \cup x_2, x_3; u)t + c(x_1 \cup x_1 \cup x_2, x_3; u), \end{aligned}$$

so that

$$\begin{aligned} F(x_1, x_3; t, u) = -a(x_1, x_3; u)(t \log t + (1-t)\log(1-t)) \\ + b(x_1, x_3; u)t + c(x_1, x_3; u), \\ (t \in]0, 1[,\ u \in]0, 1[,\ (x_1, x_2) \in \Omega_2). \end{aligned} \tag{10.71}$$

If we substitute (10.71) and (10.70) and compare the terms containing $(1-t)\log(1-t)$ and $s \log s$ on the left- and right-hand sides, we obtain

$$\left.\begin{aligned} a(x_1 \cup x_2, x_3; u) &= a(x_1, x_2; v)/(1-u) \\ a(x_1 \cup x_3, x_2; v) &= a(x_1, x_2; v)/(1-u) \end{aligned}\right\} \tag{10.72}$$

for $u, v, u + v \in]0, 1[$.

From (10.72), we have $a(x_1 \cup x_2, x_3; u) = a(x_1 \cup x_3, x_2; v)$ for $u, v, u+v \in]0, 1[$; that is, $a(x_1 \cup x_2, x_3; u)$ is independent of u. Thus,

$$a(x_1 \cup x_2, x_3; u) = a(x_1, x_3).$$

Now use (10.72) again to get

$$a(x_1 \cup x_3, x_2) = a(x_1, x_2) = a(x_1 \cup x_2, x_3),$$

which, first setting $x_1 = 0$, gives

$$a(x_3, x_2) = a(0, x_2) = \alpha(x_2), \quad \text{say,}$$

and then yields $\alpha(x_2) = \alpha(x_3)$, where α is a constant, so that

$$a(x, y; u) = a$$

is a constant. Thus (10.71) becomes

$$F(x_1, x_3; t, u) = -a(t \log t + (1-t)\log(1-t)) + b(x_1, x_3; u)t + c(x_1, x_3; u). \tag{10.71a}$$

We substitute (10.71a) into (10.70), and the comparison of the coefficients of t and the constant terms (terms free from t and s) yields

$$b(x_1 \cup x_2, x_3; u) - c\left(x_1, x_2; \frac{v}{1-u}\right) = b\left(x_1, x_3; \frac{u}{1-v}\right) \tag{10.72a}$$

and

$$c(x_1 \cup x_2, x_3; u) + c\left(x_1, x_2; \frac{v}{1-u}\right) = c\left(x_1, x_3; \frac{u}{1-v}\right) + c(x_1 \cup x_3, x_2; v), \tag{10.72b}$$

respectively.

For fixed $(x_1, x_2, x_3) \in \Omega_3$, by introducing the functions

$$f(u) = (b+c)(x_1 \cup x_2, x_3; u), h(u) = (b+c)(x_1, x_3; u)$$

and

$$k(u) = c(x_1 \cup x_3, x_2; u), \quad (u \in]0, 1[),$$

from (10.72a) and (10.72b) there results a Pexider equation,

$$f(u) = h\left(\frac{u}{1-v}\right) + k(v), \quad \text{for } u, v \in]0, 1[$$

(that is, $f(wz) = h(w) + k(1-z)$, with $w = \frac{u}{1-v}$, $z = 1-v$).

The measurability of the functions implies

$$k(v) = d_1 \log(1-v) + d_2,$$

where d_1 and d_2 are functions of $x_1 \cup x_3$, and x_2; that is,

$$c(x_1 \cup x_3, x_2; u) = d_1(x_1 \cup x_3, x_2) \log(1-u) + d_2(x_1 \cup x_3, x_2),$$

or

$$c(x_1, x_2; u) = d_1(x_1, x_2) \log(1-u) + d_2(x_1, x_2). \tag{10.72c}$$

The substitution of (10.72c) into (10.72b) leads to

$$d_1(x_1, x_2) = d_1(x_1, x_3),\ d_1(x_1 \cup x_2, x_3) = d_1(x_1, x_2)$$

and

$$d_2(x_1 \cup x_2, x_3) + d_2(x_1, x_2) = d_2(x_1 \cup x_3, x_2) + d_2(x_1, x_3),$$

the former yielding $d_1(x_1, x_2)$ a constant, c (say), and the latter for $x_1 = 0$ yielding, with the notation $g(x) = d_2(0, x)$,

$$d_2(x_2, x_3) - d_2(x_3, x_2) = d_2(0, x_3) - d_2(0, x_2) = g(x_3) - g(x_2). \tag{10.72d}$$

Now, the symmetry of F and (10.71a) give

$$c(x_1, x_2; u) = b(x_2, x_1; 1-u) + c(x_2, x_1; 1-u),$$

which, together with (10.72c) and (10.72d), yields

$$b(x_2, x_1; 1-u) = c \log \frac{1-u}{u} + g(x_2) - g(x_1)$$

or

$$b(x_1, x_2; u) = c \log \frac{u}{1-u} + g(x_1) - g(x_2). \tag{10.73}$$

Finally, use (10.72a) and (10.72b) to get

$$c(x_1, x_2; u) = c \log(1-u) + g(x_1 \cup x_2) - g(x_1). \tag{10.73a}$$

Thus, from (10.69), (10.71a), (10.73), and (10.73a), we obtain

$$\begin{aligned} K_2 \begin{pmatrix} x_1 & x_2 \\ 1-u & t \\ 1-u & u \end{pmatrix} &= F(x_1, x_2; t, u) \\ &= -a(t \log t + (1-t)\log(1-t)) \\ &\quad + c(t \log u + (1-t)\log(1-u)) \\ &\quad + (g(x_1) - g(x_2))t + g(x_1 \cup x_2) - g(x_1) \end{aligned} \tag{10.74}$$

for $(x_1, x_2) \in \Omega_2,\ t \in [0, 1[,\ u \in]0, 1[$.

An examination at the boundary points $(0, 0)$, $(1, 1)$, $(1, u)$, $(u \in]0, 1[)$ with the help of (10.70) reveals that F satisfying (10.70) indeed has the form (10.74) for all $(x_1, x_2) \in \Omega_2,\ (t, u) \in J$.

This shows that (10.68) holds for $n = 2$. Suppose it is true for $n - 1$. Then, by the recursivity and (10.73),

$$\begin{aligned}
K_n\begin{pmatrix} x_1, x_2, \ldots, x_n \\ p_1, p_2, \ldots, p_n \\ q_1, q_2, \ldots, q_n \end{pmatrix} &= K_{n-1}\begin{pmatrix} x_1 \cup x_2, x_3, \ldots, x_n \\ p_1 + p_2, p_3, \ldots, p_n \\ q_1 + q_2, q_3, \ldots, q_n \end{pmatrix} \\
&\quad + (p_1 + p_2)K_2\begin{pmatrix} x_1 & x_2 \\ p_1/(p_1 + p_2) & p_2/(p_1 + p_2) \\ q_1/(q_1 + q_2) & q_2/(q_1 + q_2) \end{pmatrix} \\
&= g(x_1 \cup x_2 \cdots \cup x_n) - (p_1 + p_2)g(x_1 \cup x_2) \\
&\quad - \sum_{i=3}^{n} p_i g(x_i) - a\sum_{i=3}^{n} p_i \log p_i - a(p_1 + p_2)\log(p_1 + p_2) \\
&\quad + c\sum_{i=3}^{n} p_i \log q_i + c(p_1 + p_2)\log(q_1 + q_2) \\
&\quad + (p_1 + p_2)\left[g(x_1 \cup x_2) - \frac{p_1}{p_1 + p_2}g(x_1) - \frac{p_2}{p_1 + p_2}g(x_2) \right. \\
&\quad - a\frac{p_1}{p_1 + p_2}\log\frac{p_1}{p_1 + p_2} - a\frac{p_2}{p_1 + p_2}\log\frac{p_2}{p_1 + p_2} \\
&\quad \left. + c\frac{p_1}{p_1 + p_2}\log\frac{q_1}{q_1 + q_2} + c\frac{p_2}{p_1 + p_2}\log\frac{q_2}{q_1 + q_2} \right] \\
&= g\left(\bigcup_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} p_i g(x_i) - a\sum_{i=1}^{n} p_i \log p_i + c\sum_{i=1}^{n} p_i \log q_i
\end{aligned}$$

for all $(x_1, \ldots, x_n) \in \Omega_n$, $(p_1, \ldots, p_n)$, $(q_1, \ldots, q_n) \in \Omega_n$; that is, (10.70) holds also for n. This completes the proof. □

Remarks. The third member, $\sum p_i \log p_i$ in (10.70), is, of course, the *Shannon entropy*. The fourth term, $\sum p_i \log q_i$ in (10.70), is the *inaccuracy*. For $c = a$, the third and the fourth terms, $\sum p_i \log \frac{p_i}{q_i}$ in (10.70), are the *directed divergence*. When g is constant, the right side of (10.70) becomes

$$a\sum p_i \log p_i + b\sum p_i \log q_i + c,$$

where a, b, c are constants, which include the above-mentioned measures (see [489]). For $c = 0$, the right side of (10.70) becomes (the inset entropy)

$$g\left(\bigcup_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} p_i g(x_i) - A\sum_{i=1}^{n} p_i \log p_i,$$

the result found in [42].

10.4 Applications

10.4.1 Continuous Shannon Measure and Shannon-Wiener Inset Information

The usual measure

$$-\int_u^v P(t)\log P(t)dt \tag{10.75}$$

for uncertainty for continuous probability distributions (P is the probability frequency function) is, contrary to one's first impression, not the limit of the Shannon entropy for discrete distributions

$$-\sum_{i=1}^{n} p(t_i)\log p(t_i). \tag{10.75a}$$

It is the limit of $-\sum P(q_i)\log P(q_i)(t_i - t_{i-1}),\ (q_i \in]t_{i-1}, t_i[)$; that is, with the appropriate choice of q_i $(i = 1, 2, \ldots, n)$ and with the distribution function $F(F' = P,\ F(u) = 0,\ F(v) = 1)$, the limit of

$$-\sum_{i=1}^{n}(F(t_i) - F(t_{i-1}))\log\frac{F(t_i) - F(t_{i-1})}{t_i - t_{i-1}}. \tag{10.75b}$$

If, as usual, $F(t_i) - F(t_{i-1}) = p_i$ is interpreted as the probability belonging to the interval $X_i =]t_{i-1}, t_i[,\ i = 1, \ldots, n,\ \bigcup_{i=1}^{n} X_i =]u, v[= U$, then (10.75b) goes over into an inset entropy

$$-\sum p_i \log p_i + \sum_{i=1}^{n} p_i \log \ell(X_i) \tag{10.76}$$

(where $\ell(X_i)$ is the length of X_i).

Contrary to (10.75a), the amount (10.76) is not necessarily nonnegative since (10.75a) may be positive for some probability distributions and negative for others. For $n = 1,\ X_1 = U =]0, 1[$, (10.76) reduces to

$$\log \ell(U) \tag{10.76a}$$

as a measure of uncertainty if we know only that the value of the random variable falls into U but don't know its probability distribution. If we know the distribution, the uncertainty reduces to (10.76). The difference between these two uncertainties is the amount,

(10.77)

$$S_n\begin{pmatrix} x_1, x_2, \ldots, x_n \\ p_1, p_2, \ldots, p_n \end{pmatrix} = \log \ell\left(\bigcup_{i=1}^{n} X_i\right) + \sum p_i \log p_i - \sum p_i \log \ell(X_i)$$
$$= \sum p_i \log \frac{p_i \ell(U)}{\ell(X_i)},$$

of information gained from the probability distribution. By Shannon's inequality, (10.77) is nonnegative. Even more importantly, putting $p_i = 1,\ p_j = 0\ (i \neq j)$ in (10.77), we get

$$S_n \begin{pmatrix} x_1, \ldots, x_i, \ldots, x_n \\ 0, \ldots, 1, \ldots, 0 \end{pmatrix} = -\log \frac{\ell(X_i)}{\ell(U)} \tag{10.77a}$$

as the measure of information gained from the knowledge that the value of the random variable lies in the subinterval X_i of U. But (10.77a) is precisely the measure of information introduced by Wiener, which together with Shannon's measures (10.75) and (10.75a) was at the source of information theory in 1948. For a characterization of "Shannon-Wiener Inset Information" (10.77), refer to [22].

10.4.2 Theory of Gambles

Consider a gamble G with independent outcomes $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$. Then $X = (x_1, \ldots, x_n) \in \Omega_n,\ P = (p_1, \ldots, p_n) \in \Gamma_n$, and denote

$$G = (X, P) = \begin{pmatrix} x_1, \ldots, x_n \\ p_1, \ldots, p_n \end{pmatrix}.$$

Let U be the function that assigns utility numbers to the x_i; that is, $U : B \to \mathbb{R}$. Then the Expected Utility Rule can be written as

$$U(G) = \sum_{i=1}^{n} p_i U(x_i). \tag{10.78}$$

The Expected Utility Rule (EUR) for gamble is isomorphic to the ordinary rule for computing a marginal probability. If we were able to find a new utility rule for gambles, then this would presumably be isomorphic to a new rule for computing marginal probabilities.

What utility of gamble rules are of interest? It is proposed that a utility rule for gambles should have a simple additive symmetry in order to be "interesting". This rather weak condition, with boundary conditions for degenerate gambles, is shown to imply that the utility rule for a gamble should be a member of a specific class of functions. One member of this class is the EUR, and another is the "EUR plus Entropy".

$$U(X, P) = \sum_{i=1}^{n} (p_i U(x_i) - b p_i \log p_i), \tag{10.79}$$

where b is a positive constant. These utility rules were previously known. The remaining, previously unknown, members of this class have the form

$$U(X, P) = \sum_{i=1}^{n} p_i^c U(x_i) + \beta \left(\sum_{i=1}^{n} p_i^c - 1 \right) \quad (c \neq 1). \tag{10.80}$$

Inspection of (10.78) and (10.80) suggests a generalization of the utility rule to the form

$$U(X, P) = \sum_{i=1}^{n} f(U(x_i), p_i), \tag{10.81}$$

where $f(U, p)$ is a continuous and "differentiable" function of U and p. Equation (10.81) has the desirable attribute of symmetry, so that the right-hand side of (10.81) is unaffected by any interchange of subscripts among the n pairs $(U(x_i), p_i)$. There are other functions, not of the form (10.81), that also have this type of symmetry. One, quite obviously, is the rule

$$U(X, P) = \prod_{i=1}^{n} f(U(x_i), p_i). \tag{10.82}$$

Note, however, that (10.82) can be converted into the form (10.81) by a suitable monotonically increasing transformation. This permits us to concentrate our attention on rules of the form (10.81). There are two elementary properties that we would like a utility rule to have. First, if a prospect is impossible (i.e., has probability equal to zero), then we should be able to ignore it, so that

$$U(x_1, \ldots, x_{n-1}, x_n; p_1, \ldots, p_{n-1}, 0) = U(x_1, \ldots, x_{n-1}, p_1, \ldots, p_{n-1}).$$

This, in (10.81), implies that

$$f(U(x_n), 0) = 0. \tag{10.83}$$

Second, if a prospect is sure (i.e., has probability equal to one), then there is really no gamble at all, so we should have

$$U(x_1; 1) = U(x_1).$$

This, in (10.81), implies

$$f(U(x_1), 1) = U(x_1). \tag{10.83a}$$

We also make use of the *compound gamble rule*,

$$\begin{aligned} x_1 &= (y_1, \ldots, y_m; q_1, \ldots, q_m) \\ &\quad U[(y_1, y_2, \ldots, y_m; q_1, \ldots, q_m), x_2, \ldots, x_n; p_1, p_2, \ldots, p_n] \\ &= U(y_1, \ldots, y_m, x_2, \ldots, x_n; p_1 q_1, \ldots, p_1 q_m, p_2, \ldots, p_n), \end{aligned}$$

for all $m \geq 1,\ n \geq 2$.

There should be a nontrivial dependence of the rule (10.81) on $U(x)$. So, f should not assign the same expected utility to prospects that have been assigned different utility numbers. Thus f must be nonconstant in its first variable in its domain when its second variable is fixed. A certain amount of regularity is required for f. From these considerations is obtained the following result.

Result 10.25. (Meginnis [633]). *Let U be a utility function defined on B, $U : B \to \mathbb{R}$ and let f be a function defined by $f : \mathbb{R} \times [0,1] \to \mathbb{R}$ that is nonconstant in its first variable when its second variable is fixed; i.e., $f(\cdot, p)$ is nonconstant and the second partial derivative in its first variable exists and is continuous. If U is recursively defined by* (10.81)*, satisfying* (10.83) *and* (10.83a)*, then the function f satisfies*

$$\begin{aligned} &f(U(x_1), p_1) + f(U(x_2), p_2) \\ &\quad = f\left[\left\{ f\left(U(x_1), \frac{p_2}{p_1+p_2}\right) + f\left(U(x_2), \frac{p_2}{p_1+p_2}\right)\right\}, p_1+p_2\right] \end{aligned} \tag{10.84}$$

and U is given by

$$U(X,P) = \begin{cases} \sum_{i=1}^{n} [p_i U(x_i) + a p_i \log p_i], & c = 1, \\ \sum_{i} \left[p_i^c U(x_i) - \frac{a}{c-1}(p_i - p_i^c)\right], & c \neq 1. \end{cases} \tag{10.85}$$

10.4.3 Recursive Inset Entropies of Multiplicative Type

At the Twenty-first International Symposium on Functional Equations, in 1983, J. Aczél presented a list of unsolved problems in the theory of functional equations. Now we present the solution of Aczél's Problem 10, described as follows.

Let B be a ring of subsets of a given set $\Omega = \cup B$. Let $B^* = B \backslash (\emptyset\}$, and for $n = 2, 3, \ldots$, let

$$\mathcal{D}_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in B^*,\ x_i \cap x_j = \emptyset \text{ for all } i \neq j\}.$$

Let k be any fixed positive integer, and

$$I_0^k =]0,1[^k, \quad \mathcal{D} = \{(p,q) \mid p, q, p+q \in I\}.$$

A map $\varphi : \mathcal{D}_2 \times I_0^k \to \mathbb{R}$ (the reals) is called a (*k-dimensional*) *inset information function of multiplicative type* on the open domain if, for some *multiplicative* map $M : I_0^k \to \mathbb{R}$ (i.e., $M(pq) = M(p)M(q)$), φ satisfies the equation

$$\begin{aligned} &\varphi(x_1 \cup x_2, x_3, p) + M(1-p)\varphi\left(x_1, F; \frac{q}{1-p}\right) \\ &\qquad = \varphi(x_1 \cup x_3, x_2; q) + M(1-q)\varphi\left(x_1, G; \frac{p}{1-q}\right) \end{aligned} \tag{10.86}$$

for all $(x_1, x_2, x_3) \in \mathcal{D}_3$ and all $(p,q) \in \mathcal{D}$. To avoid triviality, we assume $\mathcal{D}_3 \neq \emptyset$. The problem posed by Aczél is to find the general solution of (10.86), the *multidimensional fundamental equation of inset information of multiplicative type*.

Equation (10.86) and various special cases of it have a long history, to which many authors have contributed. The solution to this problem leads to an axiomatic

characterization of measures of inset information $I_n(x_1, \dots, x_n, \overline{p}_1, \dots, \overline{p}_n)$ that have the representations [239]

$$f(x_1) + \sum_2^n g(x_i) - f\left(\bigcup_1^n x_i\right) + L(\overline{p}_1) + \sum_1^n \lambda(x_i, \overline{p}_i) \quad \text{(for } M = 1\text{)},$$

$$f(x_1)M(\overline{p}_1) + \sum_2^n g(x_i)M(\overline{p}_i) - f\left(\bigcup_1^n x_i\right) + \sum_1^n L(\overline{p}_i)M(\overline{p}_i) \quad \text{(for } M \text{ additive)},$$

$$f(x_1)M(\overline{p}_1) + \sum_2^n g(x_i)M(\overline{p}_i) - f\left(\bigcup_1^n x_i\right) \quad \text{(for } M \neq 1 \text{ and not additive)}.$$

Here L is logarithmic, λ is additive in the events and logarithmic in the probabilities, and f and g are arbitrary functions.

A key step in the process is to solve the equation

$$J(x_1, x_2, x_3) + J(x_1, x_2) = J(x_1 \cup x_3, x_2) + J(x_1, x_3)$$

for disjoint triples x_1, x_2, x_3 of nonempty sets in B (a cocycle equation). A new construction was developed to handle the case where B happens to be an algebra.

Branching Inset Information Measures (IIM)

Definition 10.26. An inset information measure (I_n), $I_n : \mathcal{D}_n \times I_0 \to \mathbb{R}$, has the branching property if there is a function $\Delta : \mathcal{D}_2 \times I_0^2 \to \mathbb{R}$ such that

$$I_n \begin{pmatrix} x_1, \dots, x_n \\ p_1, \dots, p_n \end{pmatrix} = I_{n-1} \begin{pmatrix} x_1 \cup x_2, x_3, \dots, x_n \\ p_1 + p_2, p_3, \dots, p_n \end{pmatrix} + \Delta \begin{pmatrix} x_1 & x_2 \\ p_1 & p_2 \end{pmatrix} \tag{10.87}$$

for all $(X, P) \in \mathcal{D}_n \times \Gamma_n$ $(n = 3, 4, \dots)$,

- is 3-semisymmetric if

$$I_3 \begin{pmatrix} x_1 & x_2 & x_3 \\ p_1 & p_2 & p_3 \end{pmatrix} = I_3 \begin{pmatrix} x_1 & x_3 & x_2 \\ p_1 & p_3 & p_2 \end{pmatrix}, \quad (X, P) \in \mathcal{D}_3 \times \Gamma_3,$$

- is 4-semisymmetric if

$$I_4 \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ p_1 & p_2 & p_3 & p_4 \end{pmatrix} = I_4 \begin{pmatrix} x_1 & x_3 & x_2 & x_4 \\ p_1 & p_3 & p_2 & p_4 \end{pmatrix}$$

  for all $(X, P) \in \mathcal{D}_4 \times \Gamma_4$, and
- is ℓ-symmetric for some $\ell \in Z_+^*$, $\ell \geq 3$, if

$$I_\ell \begin{pmatrix} x_1, \dots, x_\ell \\ p_1, \dots, p_\ell \end{pmatrix} = I_t \begin{pmatrix} x_{\pi(1)}, \dots, x_{\pi(\ell)} \\ p_{\pi(1)}, \dots, p_{\pi(\ell)} \end{pmatrix}, \quad (X, P) \in \mathcal{D}_1 \times \Gamma_1,$$

for all permutations π on $\{1, 2, \dots, \ell\}$.

The branching property is one of the most fundamental axioms for any information measure having the following interpretation. In an experiment X with possible outcomes $x_1, \ldots, x_n$ and corresponding probabilities $p_1, \ldots, p_n$, the left-hand side of (10.87) is the uncertainty of X. The right-hand side contains as first summand the uncertainty of the experiment Y with outcomes $x_1 \cup x_2, x_3, \ldots, x_n$ and corresponding probabilities $p_1 + p_2, p_3, \ldots, p_n$. So we are uncertain whether x_1 or x_2 in the outcome will occur. To overcome the difference of the uncertainties of X and Y, a third experiment C will be performed to decide whether x_1 or x_2 will occur. This is taken into account by the second summand Δ on the right-hand side of (10.87): Δ depends only on x_1, x_2 and p_1, p_2, which is a weak assumption.

It is interesting that it is sufficient to use only the branching property and the 3-semisymmetry and 4-semisymmetry to get an important characterization of an IIM (I_n).

Assuming that (I_n) satisfies (10.87), the 4-semisymmetry of (I_n) leads to the so-called fundamental equation of IIM with the branching property

$$\Delta \begin{pmatrix} x_1 \cup x_2 & x_3 \\ p+q & r \end{pmatrix} + \Delta \begin{pmatrix} x_1 & x_2 \\ p & q \end{pmatrix} = \Delta \begin{pmatrix} x_1 \cup x_3 & x_2 \\ p+r & q \end{pmatrix} + \Delta \begin{pmatrix} x_1 & x_3 \\ p & r \end{pmatrix}$$

for all $(p, q, r) \in \mathcal{D}_3$ and all (x_1, x_2, x_3) for which there exists an x_4 with $(x_1, x_2, x_3, x_4) \in \mathcal{D}_4$ (a cocycle equation).

Result 10.27. (Kannappan and Sander [521]). *Let B be a ring of sets. An* IIM (I_n) *on $\mathcal{D}_n \times \Gamma_n$ is 3-semisymmetric and 4-semisymmetric and has the branching property if, and only if, there exist maps $a : \mathcal{D}_1''' \times I_0^k \to \mathbb{R}$, $b : \mathcal{D}_1' \times I_0^k \to \mathbb{R}$, and $c : \mathcal{D}_1''' \to \mathbb{R}$ such that*

$$I_n(X, P) = a(x_1, p_1) \sum_{i=2}^{n} b(x_i, p_i) + c\left(\bigcup_{i=1}^{n} x_i\right)$$

for all $(X, P) \in \mathcal{D}_n \times \Gamma_n$, $n \in \mathbb{N}$, $n \geq 2$.

If, in addition, (I_n) is 3- and 4-symmetric, then (I_n) with

$$I_n(X, P) = \sum_{i=1}^{n} b(x_i, p_i)$$

has the so-called sum form property with generating function b.

Remember that nearly all known information measures have this sum form property. Thus it is surprising that this result holds without any regularity condition. The proof is rather long and is based on many earlier works.

11

Abel Equations and Generalizations

In this chapter, Abel's equation, exponential iteration, associative and commutative equations, trigonometric equations, and systems of equations are treated. Generalizations and connections to information measures are treated.

Hilbert, in his famous address to the International Congress of Mathematicians held in Paris in 1900 [372], posed many unsolved problems. The second part of his fifth problem is devoted in general to functional equations and in particular to functional equations treated by Abel, who was the first to treat functional equations systematically. He presented a general method of solving equations of quite general forms. For the most part, he solved functional equations by reducing them to differential equations. Hilbert posed the question as follows:

> Moreover, we are thus led to the wide and interesting field of functional equations which have been heretofore investigated usually only under the assumption of the differentiability of the functions involved. In particular the functional equations treated by Abel (*Oeuvres*, vol. 1, pp. 1, 61, 389) with so much ingenuity ... and other equations occurring in the literature of mathematics, do not directly involve anything which necessitates the requirement of the differentiability of the accompanying functions. ... In all these cases, then, the problem arises: *In how far are the assertions which we can make in the case of differentiable functions true under proper modifications without this assumption*?

This chapter is devoted to the study of the equations considered by Abel and some generalizations applied in particular to information (theory) measures.

Abel had four publications and three manuscripts [1, 2, 3, 4, 5, 6, 7] containing functional equations; see the nice, informative survey article by Aczél [33] giving a status report. Abel presented a general method of solving quite general functional equations by reducting them to differential equations. Keeping in mind Hilbert's question, we will see how recent results succeeded in reducing the regularity suppositions to weaker regularity conditions or nothing at all.

Indeed, differentiability conditions used by Abel have been successfully replaced by much weaker conditions. Many researchers have treated various functional

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equations without any regularity assumptions. Now we deal in depth with the equations treated by Abel.

Some of the equations considered by Abel are:

$$\text{(AFE1)} \quad f(z+w) = f(z)\,f(w), \quad f : \mathbb{C} \to \mathbb{C} \quad \text{(exponential equation)},$$

$$\text{(AFE2)} \quad f(\phi(x)) = f(x) + 1, \quad f : (a,b) \to \mathbb{R} \quad \text{(iteration equation)},$$

$$\begin{aligned}\text{(AFE3)} \quad & F(x,F(y,z)) = F(z,F(x,y)) \qquad \text{(associative-commutative}\\ & = F(y,F(z,x)) = F(x,F(z,y)) \qquad \text{equation)},\\ & = F(z,F(y,x)) = F(y,F(x,z)).\end{aligned}$$

$F : I \times I \to I,\ I \subseteq \mathbb{R}$ is a proper interval.

$$\text{(AFE4)} \quad f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right),\ x,y \in \mathbb{R} \quad \text{(arctan equation)},$$

$$\begin{aligned}\text{(AFE5)} \quad & \phi(x+y) = \phi(x)\,f(y) + f(x)\phi(y),\\ & \phi,\ f : \mathbb{R} \to \mathbb{R},\end{aligned}$$

$$\begin{aligned}\text{(AFE6)} \quad & \psi(x+y) = g(xy) + h(x-y),\\ & \psi, g, h : \mathbb{R} \to \mathbb{R},\end{aligned}$$

$$\begin{aligned}\text{(AFE7)} \quad & \phi(x) + \phi(y) = \psi(xf(y) + yf(x)),\\ & f : \mathbb{R} \to \mathbb{R},\ \phi : \mathbb{R}^2 \to \mathbb{R},\end{aligned}$$

is a system of functional equations, and

$$\text{(AFE8)} \quad \phi(x+y)\phi(x-y) = \phi(x)^2 f(y)^2 - \phi(y)^2 f(x)^2,$$

$$\text{(AFE8a)} \quad f(x+y)\,f(x-y) = f(x)^2 f(y)^2 - c^2\phi(x)^2\phi(y)^2, \quad f,\phi : \mathbb{R}^2 \to \mathbb{R},$$

$$\begin{aligned}\text{(AFE9)} \quad & \psi\left(\frac{x}{1-x}\cdot\frac{y}{1-y}\right) = \psi\left(\frac{y}{1-x}\right) + \psi\left(\frac{x}{1-y}\right)\\ & \qquad - \psi(x) - \psi(y) - \log(1-x)\log(1-y),\end{aligned}$$

$$\text{(AFE9a)} \quad \psi(x) + \psi(1-x) = c - \log x \log(1-x), \quad \text{for } x,y \in]0,1[$$

(again a system).

11.1 Solutions of Abel Equations

To obtain the solutions of these equations, we treat the equations in the order listed above.

11.1.1 (AFE1)—Exponential Equation $f(z+w) = f(z)\,f(w)$

Abel justified "Newton's binomial series" by using the exponential equation (AFE1). Abel solved this equation by assuming continuity and not differentiability. This

equation is solved under the weaker assumption of measurability (Kac [407], Aczél [33]). Obviously, f is zero everywhere or nowhere (see Theorem 1.36). We consider only the nontrivial solution. To start with, restrict f to $\mathbb{R}$; that is, $f : \mathbb{R} \to \mathbb{C}$. Clearly

$$|f(x+y)| = |f(x)|\,|f(y)|, \quad \text{for } x, y \in \mathbb{R}, \tag{11.1}$$

and

$$\chi(x+y) = \chi(x)\chi(y), \qquad \text{for } x, y \in \mathbb{R}, \tag{11.2}$$

where $\chi(x) = \frac{f(x)}{|f(x)|}$ with $|\chi(x)| = 1$; that is, $\chi : \mathbb{R} \to \mathbb{C}$ is a measurable character (it satisfies (E)). Since χ is bounded, χ is Lebesgue integrable. Integrate (11.2) over a to c with $\int_a^c \chi(y)dy = d \neq 0$ (otherwise $\chi = 0$ a.e.) to get

$$\chi(x) \cdot \int_a^c \chi(y)dy = \int_a^c \chi(x+y)dy;$$

that is,

$$\chi(x) = \frac{1}{d}\int_{x+a}^{x+c} \chi(t)dt \quad (x+y=t).$$

Since χ is integrable, the right side is a continuous function of x and so is the left side χ. Since χ is continuous, it is differentiable. Differentiate (11.2) with respect to y and then put $y = 0$ to have

$$\chi'(x) = a\chi(x) \quad \text{and then} \quad \chi(x) = ae^{bx}, \quad \text{for } x \in \mathbb{R},$$

where $a, b \in \mathbb{C}$. This χ satisfies (11.2) provided $a = 1$ and $|\chi(x)| = 1$ implies $b = i\beta,\ \beta \in \mathbb{R}$. Thus

$$\chi(x) = e^{i\beta x}$$

for some real β (see Theorem 1.38 and Hille and Phillips [376]).

As to the measurable solution of (11.1), it is known (see also Theorem 1.37) that

$$|f(x)| = e^{\alpha x} \quad \text{for } x \in \mathbb{R} \text{ and some } \alpha \in \mathbb{R}.$$

Now,

$$f(z) = f(x+iy) = f(x) \cdot f(iy) = e^{\alpha x + i\beta x} \cdot e^{c_1 y + ic_2 y} = e^{\gamma z + \delta \overline{z}}$$

for some $c_1, c_2 \in \mathbb{R},\ \gamma, \delta \in \mathbb{C}$ (see also Result 1.41 and [44]). Thus, the measurable solution $f : \mathbb{C} \to \mathbb{C}$ satisfying (AFE1) or (E) is given by $f(z) = e^{cz + d\overline{z}}$, where $c, d \in \mathbb{C}$.

For the general solution $f : \mathbb{R} \to \mathbb{C}$ of (E), see Result 1.40.

11.1.2 (AFE2)—Iteration Equation $f(\phi(x)) = f(x) + 1$

The literature of this iteration equation is large (see [554, 560]). We quote here two results.

Result 11.1. (Kuczma [552]). *Let $f, \phi :]a, b[\to \mathbb{R}$ satisfy* (AFE2). *Suppose f is concave, $a < f(x) < x$ in $]a, b[$, $f'(x) > 0$, and $\lim_{x \to a+0} f'(x) = 1$. Equation* (AFE2) *has a convex solution ϕ in $]a, b[$. Further, this solution is unique up to an additive constant and is given by*

$$\phi(x) = \lim_{n \to \infty} \frac{f^n(x) - f^n(x_0)}{f^{n+1}(x_0) - f^n(x_0)},$$

where x_0 is an arbitrary element in $]a, b[$ and $f^n(x)$ stands for the nth iterate of $f(x)$. Also, the regular iterates of $f(x)$ are given by $f^u(x) = \phi^{-1}(\phi(x) + u)$, where ϕ is given as above.

11.1.3 (AFE3)—Associative, Commutative Equations

Abel solved this system of functional equations by reducing them to partial differential equations; that is, effectively using differentiability. The associativity equation

$$F(x, F(y, z)) = F(F(x, y), z), \quad \text{for } x, y, z \in I', \tag{11.3}$$

where I' is a real interval, has been completely solved under continuity and cancellativity conditions without assuming differentiability or commutativity (see [12]; here again the literature is voluminous). Here we quote a result in Paganoni Marzegalli [658].

Result 11.2. *Let I' be a proper real interval and $F : I'^2 \to I'$ be continuous, satisfy* (11.3)*, and be cancellative* ($(\exists\, y \in I' : F(x_1, y) = F(x_2, y) \Rightarrow x_1 = x_2$ *and* $\exists\, x \in I' : F(x, y_1) = F(x, y_2) \Rightarrow y_1 = y_2$). *Then there is an $f : I' \to \mathbb{R}$ that is continuous and strictly monotonic such that*

$$F(x, y) = f^{-1}(f(x) + f(y)) \quad \text{for } x, y \in I'. \tag{11.3a}$$

The converse also holds.

Remark. Using cancellativity and $F(x, F(y, z)) = F(x, F(z, y))$, we get the commutativity $F(y, z) = F(z, y)$, and then from $F(x, F(y, z)) = F(z, F(x, y))$ we get the associativity (11.3). Thus this system is referred to as the associative-commutative equation.

For the solution of the commuting system

$$F(x, F(y, z)) = F(y, F(x, z)) \quad \text{for } x, y \in S,\ z \in T,$$

where S, T are compact under continuity of $F : S \times T \to T$ and weak transitivity (see [65]), the result is as follows. If S is a commutative monoid, with c as unit, under an operation o, there is a continuous surjection $f : S \to T$ such that $F(x, z) = f(x)oz$ (actually $f(x) = F(x, c)$).

11.1.4 (AFE4)—Arctan Equation

This equation has no continuous solution on $\mathbb{R}$ except $f \equiv 0$. There exist local solutions and global but multivalued solutions (see [536, 602]).

11.1.5 (AFE5)—Trig Equation $\phi(x+y) = \phi(x)f(y) + \phi(y)f(x)$

Abel solved (AFE5) without using derivatives. Note that this equation determines two unknown functions. The general solution of (AFE5) is obtained in Chung, Kannappan, and Ng [162] without using any regularity conditions (see Corollary 3.56a and [474]).

Let S be a semigroup (not necessarily commutative) and F a field. Suppose ϕ ($\neq 0$) and $f : S \to F$ satisfy (AFE5). Then there exist exponentials E, E_1 and additive A such that

$$\phi = a(E_1 - E), \quad f = \frac{1}{2}(E_1 + E); \quad \phi = EA, \quad f = E.$$

Regular (measurable, continuous) solutions of (AFE5) for $\phi, f : \mathbb{R} \to \mathbb{R}$ are

$$\phi(x) = a(e^{c_1 x} - e^{cx}), \quad f(x) = \frac{1}{2}(e^{c_1 x} + e^{cx}); \quad \phi(x) = dxe^{cx}, \quad f(x) = e^{cx},$$

where a, c, c_1, d are constants.

An equation equivalent to the cosine equation (C) was solved by Abel in [1] using derivatives. For the general solution of (C) on groups without using any regularity conditions, see Theorem 3.21 [424].

11.1.6 (AFE6)—$\psi(x+y) = g(xy) + h(x-y)$

Abel has given the differentiable solution [1]. The general solution of this equation is obtained for $\psi, g, h : F \to G$, where F is a field and G an Abelian group. While investigating the dependence of the quadratic difference $f(x+y) + f(x-y) - 2f(x) - 2f(y)$ (which will be considered in Chapter 13) on the product xy and some generalizations, the authors came across (AFE6). First we give some preliminaries.

Suppose $\psi, g, h : \mathbb{R} \to \mathbb{R}$ satisfy (AFE6). Then

$$g(x^2) - g(0) = h(2x) - h(0) = \psi(2x) - \psi(0) \tag{11.4}$$

$$= g(0) - g(-x^2) \tag{11.4a}$$

$$= g(x^2 - a^2) + g(a^2) - 2g(0) \tag{11.4b}$$

holds for $x, a \in \mathbb{R}$. Also, $\psi(0) = g(0) + h(0)$.

Set $y = x$ in (AFE6) to have

$$\psi(2x) = g(x^2) + h(0).$$

Put $x = 0$ and $y = 0$ separately in (AFE6) to get

$$\psi(x) = g(0) + h(x); \quad \psi(y) = g(0) + h(-y) \quad \text{for } x, y \in \mathbb{R},$$

showing thereby that h and thus ψ are even.

So, $h(2x) - h(0) = \psi(2x) - g(0) - h(0) = g(x^2) - g(0)$, which is (11.4). Further, $y = -x$ in (AFE6) yields

$$g(0) = g(-x^2) + h(2x) = g(x^2) - g(0) + h(0) + g(-x^2)$$

or

$$g(x^2) - g(0) = g(0) - g(-x^2),$$

which is (11.4a).

Finally, for any $a \in \mathbb{R}$, replace x by $x + a$ and y by $x - a$ in (AFE6) to obtain

$$\psi(2x) = g(x^2 - a^2) + h(2a);$$

that is,

$$g(x^2) - g(0) + \psi(0) = g(x^2 - a^2) + g(a^2) - g(0) + h(0),$$

which is (11.4a).

Note that (11.4), (11.4a), and (11.4b) hold on a field to an Abelian group. Now we consider (AFE6) from Z_7 to Z_7 and prove the following theorem.

Theorem 11.3. *Let $\psi, g, h : Z_7 \to Z_7$ and satisfy* (AFE6)*. Then the functions are given by*

$$(11.5) \quad \begin{cases} g(x) = g(0) + x(g(1) - g(0)) = g(0) + A(x), \\ \psi(x) = g(2x^2) - g(0) + \psi(0) = A(2x^2) + \psi(0) = A\left(\dfrac{x^2}{4}\right) + \psi(0), \\ h(x) = g(2x^2) - g(0) + h(0) = A\left(\dfrac{x^2}{4}\right) + h(0), \end{cases}$$

for $x \in Z_7$, where $A : Z_7 \to Z_7$ is additive.

Proof. Since the domain is small, the solution can be easily handled by direct computation. Any additive $A : Z_7 \to Z_7$ is of the form $A(x) = xA(1)$. Take $A(1) = g(1) - g(0)$; that is, $A(x) = (g(1) - g(0))x$, $x \in Z_7$. Let $x = 3$, $a = 6$ in (11.4b) to have

$$g(2) - g(0) = 2(g(1) - g(0));$$

that is,

$$g(2) = g(0) + 2(g(1) - g(0)).$$

With $x = 3$, (11.4a) gives

$$g(2) - g(0) = g(0) - g(-2);$$

that is,

$$g(-2) = g(5) = 2g(0) - g(2) = g(0) - 2(g(1) - g(0)) \\ = g(0) + 5(g(1) - g(0)).$$

Put $x = 6$ in (11.4a) to have

$$g(1) - g(0) = g(0) - g(-1);$$

that is,

$$g(6) = g(0) + 6(g(1) - g(0)).$$

In (11.4b), take $x = 3$ and $a = 2$. Then

$$g(2) - g(0) = g(5) + g(4) - 2g(0)$$

so that

$$g(4) = g(0) + 4(g(1) - g(0)).$$

Finally, in (11.4b), take $x = 2$ and $a = 1$. Then

$$g(4) - g(0) = g(3) + g(1) - g(0)$$

gives

$$g(3) = g(0) + 3(g(1) - g(0)).$$

Thus

$$g(x) = g(0) + x(g(1) - g(0)) = g(0) + A(x) \quad \text{for } x \in Z_7.$$

From (11.4), we get

$$h(2x) = A(x^2) + h(0) \quad \text{or} \quad h(x) = A\left(\frac{x^2}{4}\right) + h(0) \quad (\text{replace } x \text{ by } 4x)$$

since $\frac{1}{4} = 2$, and

$$\psi(2x) = A(x^2) + \psi(0) = A\left(\frac{x^2}{4}\right) + \psi(0) \quad \text{for } x \in Z_7.$$

This proves the result. □

Indeed, solutions of (AFE6) from $\mathbb{R} \to \mathbb{R}$ or $F \to G$ are of the form (11.5). These solutions can be obtained by reducing them either to the Pexider equation (PA) or the additive equation (A). Note that this equation determines three unknown functions g, h, ψ.

Result 11.3a. [33]. *Suppose* $g, h, \psi : \mathbb{R} \to \mathbb{R}$ *satisfy* (AFE6)*. Then*

$$\text{(11.5a)} \qquad \begin{cases} g(y) = g(\{y\}) + [y](g(1) - g(0)), \\ h(y) = g\left(\left\{\dfrac{y^2}{4}\right\}\right) + \left[\dfrac{y^2}{4}\right](g(1) - g(0)) - g(0) + h(0), \\ \psi(y) = g\left(\left\{\dfrac{y^2}{4}\right\}\right) + \left[\dfrac{y^2}{4}\right](g(1) - g(0)) - g(0) + \psi(0), \end{cases}$$

for $y \in \mathbb{R}$, *where* $[y]$ *and* $\{y\}$ *are the integral and fractional parts of* $y \in \mathbb{R}$, *respectively. Further, the general solution of* (AFE6) *is given by* (11.5)*.*

Proof. The proof of (11.5a) is from Kannappan and Bunder [481]. First we will show that

$$g(x^2) = g(\{x^2\}) + [x^2](g(1) - g(0)) \quad \text{for } x \in \mathbb{R}.$$

Note that $x = [x] + \{x\}$ for $x \in \mathbb{R}$.

In (AFE6), replace x by $x + 1$ and y by $x - 1$ to get

$$\psi(2x) = g(x^2 - 1) + h(2),$$

which, by using (11.4), gives

$$g(x^2) - g(0) + \psi(0) = g(x^2 - 1) + g(1) - g(0) + h(0)$$

or

$$g(x^2) = g(x^2 - 1) + g(1) - g(0).$$

If $x^2 - 1 < 0$, then

$$g(x^2) = g(\{x^2\}) + 0 \cdot (g(1) - g(0)).$$

Otherwise (for $x^2 - 1 \geq 0$),

$$\begin{aligned} g(x^2) &= g(x^2 - 2) + 2(g(1) - g(0)) \\ &= g(x^2 - 3) + 3(g(1) - g(0)), && \text{if } x^2 - 2 \geq 0, \\ &= g(x^2 - [x^2]) + [x^2](g(1) - g(0)) \\ &= g(\{x^2\}) + [x^2](g(1) - g(0)), && \text{for } x \in \mathbb{R}. \end{aligned}$$

From (11.4a) and (11.4b), we obtain (take $a = 1$)

$$\begin{aligned} g(-x^2) &= -g(x^2 - 1) - g(1) + 3g(0) \\ &= g(1 - x^2) - (g(1) - g(0)). \end{aligned}$$

If $1 - x^2 > 0$, then

$$g(-x^2) = g(\{-x^2\}) - (g(1) - g(0)).$$

Otherwise, for $1 - x^2 < 0$,

$$\begin{aligned} g(-x^2) &= g(2 - x^2) - 2(g(1) - g(0)) \\ &= g([x^2] - x^2) - [x^2](g(1) - g(0)) \\ &= g(\{-x^2\}) + [-x^2](g(1) - g(0)) \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

Thus,

$$g(y) = g(\{y\}) + [y](g(1) - g(0)) \quad \text{for } y \in \mathbb{R}.$$

By replacing x by $\frac{y}{2}$ in (11.4), we obtain $h(y)$ and $\psi(y)$ in (11.5a). This proves (11.5a). □

This equation was studied in Roscáu [709] and Sablik [720], and the measurable solution is obtained in [720]. Now we determine the general solution of (AFE6) without assuming any regularity condition.

Note that (11.4a) is the same as $g(x) - g(0) = g(0) - g(-x)$ for $x \in \mathbb{R}$. From (11.4) and (AFE6), we have

$$g\left(\frac{(x+y)^2}{4}\right) + h(0) = g(xy) - g\left(-\frac{(x-y)^2}{4}\right) + \psi(0) \quad \text{for } x, y \in \mathbb{R};$$

that is,

$$g(u + v) = g(u) - g(-v) + g(0)$$

by the transformation $xy = u$ and $\frac{(x-y)^2}{4} = v$ for $u, v \in \mathbb{R}$. Note that the equation above is valid for all $u, v \in \mathbb{R}$ with v and $u + v = \frac{(x+y)^2}{4} \geq 0$. Hence

$$g(u + v) = g(u) + g(v) - g(0) \quad \text{for } u, v \in \mathbb{R} \text{ with } v, u + v \geq 0.$$

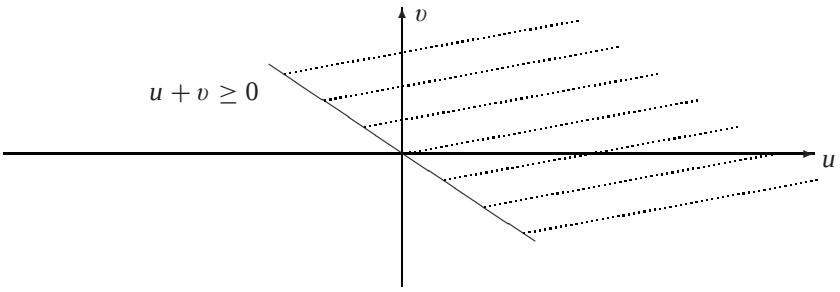
Define

$$A(x) := g(x) - g(0) \quad \text{for } x \in \mathbb{R}. \tag{11.6}$$

Then the equation above becomes

$$A(u + v) = A(u) + A(v) \tag{11.7}$$

for all $u, v \in \mathbb{R}$, with $v, u + v \geq 0$. Now we extend A to all of $\mathbb{R}^2$.



Choose $u, v \in \mathbb{R}$ such that $u + v < 0$. Then there exists a real number $x \geq 0$ such that $x + u \geq 0$ and $x + u + v \geq 0$.

By using (11.7), consider

$$\begin{aligned} A(x) + A(u + v) &= A(x + u + v) \\ &= A(x + u) + A(v) \\ &= A(x) + A(u) + A(v). \end{aligned}$$

Hence we have

$$A(u + v) = A(u) + A(v),$$

for all $u, v \in \mathbb{R}$, with $u + v < 0$. Thus A is additive on $\mathbb{R}$.

Therefore, by (11.6), we obtain

$$g(x) = A(x) + \alpha,$$

where $\alpha = g(0)$. Using (11.4), we get

$$\psi(x) = A\left(\frac{x^2}{4}\right) + \alpha + \beta \quad \text{and} \quad h(x) = A\left(\frac{x^2}{4}\right) + \beta,$$

where $\beta = h(0)$. This proves (11.5) and the result. □

Corollary. *If any one of g, h, ψ in* Result 11.3a *is measurable* (*or continuous*), *then*

$$g(x) = cx + d, \quad h(x) = \frac{cx^2}{4} + b, \quad \psi(x) = \frac{cx^2}{4} + d + b, \quad \text{for } x \in \mathbb{R},$$

where b, c, d are constants.

(AFE6) on a Field

The general solution of equation (AFE6) for maps $\psi, g, h : F \to G$, where F is a field belonging to a certain class $\mathfrak{a}$ and G is an Abelian group, is obtained. The class $\mathfrak{a}$ includes all finite fields, the rationals Q, and fields F with the property that for each $x \in F$ either x or $-x$ is a square. This includes $\mathbb{R}$ and $\mathbb{C}$. The functional equation (AFE6) is closely linked to the functional equation

$$f(x^2 - y^2) = f(x^2) - f(y^2) \quad \text{for } x, y \in F, \ f : F \to G.$$

For many fields, this forces f to be a morphism from the additive group of F into G.

Result 11.3b. [146]. *Let F belong to class* $\mathfrak{a}$, char $F \neq 2$, *and $F \neq z_3, z_5$. Then the general solution of* (AFE6) *for $\psi, g, h : F \to G$ is given by* (11.5) *for some additive A on F and arbitrary constants $g(0)$ and $g(1)$ in G.*

11.1.7 (AFE7)—$\phi(x) + \phi(y) = \psi(xf(y) + yf(x))$

Abel obtained the differentiable solution by repeated differentiation [1], where ϕ, f, ψ are unknown functions mapping reals into reals (without precisely stating the domains of these functions). This is one of the places where Abel noted the remarkable fact that one functional equation determines several unknown functions. The conclusion he drew from this fact is equally remarkable:

> Thus, it is generally possible to find all the functions by means of a single equation. It follows that such an equation can exist only very seldom. Indeed, since the form of an arbitrary function appearing in the given conditional equation, by virtue of the equation itself, has to be dependent on the forms of the others, it is obvious that, in general, one cannot assume any of these functions to be given. Thus, for example, the above equation could not be satisfied if $f(x)$ had any other form than that which was found.

It is not quite clear why Abel singled out this equation. He started, however, from the solutions

$$f(x) = \frac{x}{2}, \qquad \psi(x) = \phi(x) = \log x,$$
$$f(x) = \sqrt{1 - x^2}, \quad \psi(x) = \phi(x) = \arcsin x.$$

These are solutions only on subsets of the real plane (the first on $\mathbb{R}^2_+ = \{(x, y) \mid x > 0, y > 0\}$, the second on $\{(x, y) \mid x, y \in [-1, 1]\}$ (there are also problems with the functions being multivalued in the second solution). So it is preferable to find the general (at least continuous) solutions on such subsets (say, on regions).

Continuous Solutions of (AFE7)

Let I' be an interval in $\mathbb{R}$ containing 0, and ψ, ϕ, f are real functions. For $x, y \in$ dom f, define

$$A_f(x, y) = xf(y) + yf(x)$$

and

$$A'_f(x) = A_f(x, x) = 2xf(x),$$

the diagonalization of A_f.

The case $f(0) = 0$ is not of much interest. It leads to the solution f arbitrary (continuous) for $x \neq 0$, ψ on $A_f(I', I')$ is constant, and ϕ is constant. An interesting case is $f(0) \neq 0$. With $F(x) = f(x)f(0)^{-1}$, $\delta(x) = \psi(f(0)x) - \psi(0)$, (AFE7) can be reduced to

$$\delta(x) + \delta(y) = \delta(xF(y) + yF(x)) = \delta(A_F(x, y)),$$

and this to

$$F(A_F(x, y)) = F(x)F(y) + cxy,$$

which is pointed out to be essentially the associative equation (11.3) for

$$A_F(x, y) = xF(y) + yF(x)$$

and that if assumed for all $x, y \in \mathbb{R}$ and if $c > 0$, this equation is equivalent to the "Baxter equation"

$$h(A_h(x, y) - xy) = h(x)h(y).$$

For linear operators on Banach algebra, the equation above defines the Baxter (or summation) operators connected to queuing theory [635].

Result 11.4. (Sablik [718]). *Let $I' \subseteq \mathbb{R}$ be an interval containing 0. Let $\psi : A_f(I' \times I') \to \mathbb{R}$ and $\phi : I \to \mathbb{R}$ be functions and assume that f, ϕ, ψ are continuous solutions of* (AFE7)*. Then* (AFE7) *has a list* (*seven*) *of lengthy, complicated solutions to be reproduced.*

Corollary. [718]. *If either ϕ or ψ in* Result 11.4 *is locally bounded from above* (*or below*)*, then the assertions of* Result 11.4 *hold.*

In [718], the author obtains the solution of (AFE7) when 0 is not in the domain I', assuming instead that 0 is a value of f; that is, $0 \in f(I')$.

11.1.8 System of Equations (AFE8) and (AFE8a)

$$\phi(x + y)\phi(x - y) = \phi(x)^2 f(y)^2 - \phi(y)^2 f(x)^2,$$
$$f(x + y) f(x - y) = f(x)^2 f(y)^2 - c\phi(x)^2\phi(y)^2.$$

In an unfinished manuscript, Abel reduces the system of functional equations (AFE8), (AFE8a) to differential equations of fourth order assuming $f, \phi : \mathbb{C} \to \mathbb{C}$ are analytic. These equations are related to elliptic functions. In [348], entire solutions of (AFE8) alone are determined.

Result 11.5. *Suppose $\phi, f : \mathbb{C} \to \mathbb{C}$ are entire functions satisfying* (AFE8)*. Then the solutions are given by*

(i) $\phi = 0$, f *arbitrary;*
(ii) $\phi(x) = a_1 e^{a_2 x^2}\sigma(x)$, $f(x) = \pm e^{a_2 x}\sigma_i(x)$, $(i = 1, 2)$ σ_i *cosine functions in the theory of elliptic functions;*
(iii) $\phi(x) = a_1 x e^{a_2 x^2}$, $f(x) = \pm e^{a_2 x^2}$;
(iv) $\phi(x) = a_1 e^{a_2 x^2} \sin a_3 x$, $f(x) = \pm e^{a_2 x^2}$;
(v) $\phi(x) = a_1 e^{a_2 x^2} \sin a_3 x$, $f(x) = \pm e^{a_2 x^2} \cos x_3 x$,

for $x \in \mathbb{C}$, a_1, a_2, a_3 are constants, and σ, σ_i are elliptic functions.

Now we discuss continuous solutions of the system.

Result 11.6. (Bonk [127]). *Suppose $\phi, g : \mathbb{R} \to \mathbb{C}$ are functions that satisfy*

$$\phi(u+v)\phi(u-v) = \phi(u)^2 g(v) - \phi(v)^2 g(u), \quad u, v \in \mathbb{R}^n. \tag{11.8}$$

If ϕ is continuous, it is one of the following functions:

$$\phi(u) = ae^{q(u)}\sigma(\alpha(u)),$$
$$\phi(u) = ae^{q(u)}\sin\alpha(u),$$
$$\phi(u) = ae^{q(u)}\alpha(u) \quad \text{for } u \in \mathbb{R}^n,$$

where σ denotes the elliptic function, $\alpha, q : \mathbb{R}^n \to \mathbb{C}$ are given by

$$\alpha(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i, \quad c_i \in \mathbb{C}$$

(*an $\mathbb{R}_-$ linear map*),

$$q(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} c_{ij} \cdot x_i x_j, \quad c_{ij} \in \mathbb{C},$$

(*a complex-valued quadratic form on $\mathbb{R}^n$*), *and a is a constant.*

Given $\phi \neq 0$, the function g is not uniquely determined by the functional equation (11.8). *If $c_1, c_2 \in \mathbb{C}$ and $u_0 \in \mathbb{R}^n$ are suitably chosen,*

$$g(u) = c_1\phi(u)^2 + c_2\phi(u+u_0)\phi(u-u_0), \quad u \in \mathbb{R}^n.$$

Continuous Solutions of the System

As an application of Result 11.6, we get the continuous solution of the system (AFE8), (AFE8a).

Corollary 11.6a. *Suppose $\phi : \mathbb{R}^n \to \mathbb{C}$ and $f : \mathbb{R}^n \to \mathbb{C}$ are functions and $c \in \mathbb{C}$ is a constant such that the equalities* (AFE8) *and* (AFE8a) *hold for all $u, v \in \mathbb{R}^n$. If $\phi \not\equiv 0$ is continuous, then ϕ is one of the functions listed in* Result 11.6 *and f and c^2 are*

(i) $f(u) = \pm\exp(q(u))\sigma_\lambda(\alpha(u))$, $c^2 = (e_\lambda - e_\mu)(e_\lambda - e_\nu)/a^4$, $\{\lambda, \mu, \nu\} = \{1, 2, 3\}$,
(ii) $f(u) = \pm\exp(q(u))\cos(\alpha(u))$, $c^2 = 1/a^4$ *or* $f(u) = \pm\exp(q(u))$, $c = 0$,
(iii) $f(u) = \pm\exp(q(u))$, $c = 0$,

in the cases (i), (ii), and (iii) *of* Result 11.6, *respectively.*

Conversely, every triple ϕ, f, c^2, as given above, satisfies the functional equations (AFE8), (AFE8a).

If $\phi \equiv 0$, then the system (AFE8), (AFE8a) reduces to

$$f(u+v)\,f(u-v) = f(u)^2\,f(v)^2, \quad u, v \in \mathbb{R}^n. \tag{11.9}$$

The continuous solutions $f : \mathbb{R}^n \to \mathbb{C}$ of (11.9) are given by

$$f(u) = ae^{q(u)}, \quad \text{for } u \in \mathbb{R}^n,$$

where $q : \mathbb{R}^n \to \mathbb{C}$ is the complex-valued quadratic form

$$q(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j,$$

and a_{ij}, a are constants.

11.1.9 (AFE9) and (AFE9a)

Finally, we consider the system (AFE9), (AFE9a), which connects to information measures that will be dealt with in the next section.

$$\text{(AFE9)} \qquad \psi\left(\frac{x}{1-x} \cdot \frac{y}{1-y}\right) = \psi\left(\frac{y}{1-x}\right) + \psi\left(\frac{x}{1-y}\right) - \psi(x) - \psi(y) - \log(1-x)\log(1-y),$$

$$\text{(AFE9a)} \qquad \psi(x) + \psi(1-x) = c - \log x \log(1-x), \quad x, y \in]0, 1].$$

There are so many functional equations that are connected to characterization of various measures of information and in particular with Shannon's entropy (see Chapter 10). One of the equations one comes across in the characterization of Shannon's entropy is the fundamental equation of information (FEI).

First we give some notation. Denote $I = [0, 1]$, $I_0 =]0, 1[$, $J_2 = I_0^2 \cup \{(0, 0), (1, 1)\}$, $J_3 = I_0^3 \cup \{(0, 0, 0), (1, 1, 1)\}$, $I(x) = \int_\alpha^x \left(\frac{\log t}{1-t} + \frac{\log(1-t)}{t}\right) dt$, for $x \in I_0$, and α a fixed element in I_0.

Abel [1] considered the series

$$\psi(x) = x + \frac{x^2}{2^2} + \cdots + \frac{x^n}{n^2} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n^2},$$

which converges for $|x| \leq 1$ in connection with (AFE9), (AFE9a),

$$\psi'(x) = 1 + \frac{x}{2} + \cdots + \frac{x^{n-1}}{n} + \cdots = -\frac{\log(1-x)}{x}.$$

So,

$$\psi(x) = -\int \frac{\log(1-x)}{x} dx.$$

He has given so many properties of ψ and values of ψ at a few points 1, $\frac{1}{2}$, etc.,

$$\psi(1-x) = \int \frac{\log x}{1-x} dx,$$

so that

$$\psi(x) + \psi(1-x) = c - \log x \log(1-x),$$

which is (AFE9a) (this can be obtained from (AFE9) by putting $y = 1 - x$ and $c = \psi(1)$ when 1 is in the domain), and

$$c = \psi(1) = \sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6},$$

$$\psi(x) + \psi(-x) = \frac{1}{2}\psi(x^2),$$

so that

$$\psi(-1) = -\frac{\pi^2}{12} = \sum_1^\infty \frac{(-1)^n}{n^2},$$

$$\psi\left(\frac{1}{2}\right) = \frac{\pi^2}{6} - (\log 2)^2.$$

It appears from Ramanujan's notebooks [677, 341, 342] that he had found the values of

$$F(x) = \int_0^x \frac{\log(1-t)}{t} dt = \sum_1^\infty \frac{x^n}{n^2}$$

[1] for the special values $x = \frac{1}{2}, 2, \frac{1}{2}(\sqrt{5} - 1)$ given by Rogers's [707] result that

$$F(x) + F(y) - F(xy) - F\left(\frac{x(1-y)}{1-xy}\right) - F\left(\frac{y(1-x)}{1-xy}\right)$$

is an elementary function without the knowledge of these writers. Ramanujan discovered that F satisfies the functional equation

$$F(x) + F(y) + F(1-xy) + F\left(\frac{1-y}{1-xy}\right) + F\left(\frac{1-x}{1-xy}\right) = 3F(1).$$

This was rediscovered by Rogers [707, 342]. This is to be found in [1]. In [535], the author reduces the system (AFE9), (AFE9a) to the equation

$$F(x) + F(y) + F(1-xy) + F\left(\frac{1-y}{1-xy}\right) + F\left(\frac{1-x}{1-xy}\right) = 0 \tag{11.10}$$

for $x, y \in I$ or I_0, which is known as the Abel functional equation. The solution of (11.10) connected to (FEI) is in the following theorem.

Theorem 11.7. [33, 206]. *The general integrable solution $F : I \to \mathbb{R}$ of* (11.10) *is*

$$F(x) = cI(x), \quad \text{for } x \in I, \tag{11.10a}$$

where c is a constant.

Proof. The solution is obtained by showing first that F is continuous, then differentiable, and then has derivatives of all orders. By using the derivative of F, (11.10) can be reduced to (FEI) for $x, y \in I_0$, the fundamental equation, satisfied by $f(x) = x(1-x)F'(x)$. This connects to the Shannon entropy.

To deduce the continuity and from it the differentiability of F from integrability, integrate (11.10) with respect to y from 0 to 1 and obtain

$$F(x) = -\int_0^1 F(y)dy - \frac{1}{x}\int_{1-x}^1 F(t)dt \\ - (1-x)\int_0^1 \frac{F(t)}{(1-xt)^2}dt - \frac{1-x}{x}\int_{1-x}^1 \frac{F(t)}{t^2}dt.$$

Since the right side is continuous in x on $]0, 1[$, so is the left side; that is, $F(x)$ is continuous. Then the right side is differentiable, and so is the left side. It is easy to see that F has derivatives of all orders (we need only up to order 3). Differentiate (11.10) with respect to x for $x, y \in I_0$, and get

$$F'(x) - yF'(1-xy) - \frac{1-y}{(1-xy)^2}F'\left(\frac{1-x}{1-xy}\right) + \frac{y(1-y)}{(1-xy)^2}F'\left(\frac{1-y}{1-xy}\right) = 0,$$

which goes over to

$$f(x) - f\left(\frac{1-x}{1-xy}\right) - \frac{1-x}{1-xy}f(1-xy) + xf\left(\frac{1-y}{1-xy}\right) = 0,$$

where

$$f(x) = x(1-x)F'(x) \quad \text{for } x \in I_0. \tag{11.11}$$

With the substitution $u = 1-x,\ v = 1 - \frac{1-x}{1-xy}$, this equation transforms to (FEI),

$$f(1-u) + (1-u)f\left(\frac{v}{1-u}\right) = f(1-v) + (1-v)f\left(\frac{u}{1-v}\right), \tag{11.12}$$

for $u, v, u+v \in I_0$.

Set $y = 0$ in (11.10) to have

$$F(x) + F(1-x) = -F(0) - 2F(1),$$

which, after differentiation, yields

$$F'(x) = F'(1-x) \quad \text{for } x \in I_0.$$

Thus (11.11) shows that

$$f(x) = f(1-x) \quad \text{for } x \in I_0$$

(f is symmetric), and (11.12) yields (FEI).

The general measurable (integrable, continuous) solution of (FEI) is given by (see Aczél and Daróczy [43], Lajko [598], and Kannappan and Ng [489])

$$f(x) = cs(x) + ax + b \quad \text{for } x \in I_0. \tag{11.13}$$

Now, since $f(x) = f(1-x)$, because F is defined at 0 and 1, $a = 0 = b$ and (11.11) results in (11.10a).

This proves the result. □

Remark 11.7a. If the domain of F is $]0, 1[$, then $a \neq 0,\ b \neq 0$ in (11.13). Substitution of (11.13) in (11.12) yields $b = -2a$, so that (11.11) gives

$$F(x) = cI(x) - a \log \frac{x^2}{1-x},$$

and for $x \in I_0$ (linking F to the dilogarithm (Kiesewetter [535])) (11.10) has more solutions on I_0 than on I (*on smaller domains, a functional equation may have more solutions*).

Remark 11.7b. The Abel functional equation (11.10) is connected to (FEI) and to Shannon entropy.

In the next section, we study generalizations to (11.10) and connections to other information measures.

Problem 1. The general solution of (11.10) is not known. Determine the general solution of (11.10) (without using any regularity condition) on $[0, 1]$ or $]0, 1[$.

Summary. Either general solutions (without any regularity assumption) or solutions with weaker regularity conditions such as integrability, measurability, etc., of Abel's equations are obtained, eliminating differentiability and thus answering the second part of Hilbert's Fifth Problem.

11.2 Generalizations and Information Measures

In Chapter 10, we have seen that the well-known measures of information, such as the Shannon entropy, the directed divergence, and the inaccuracy, are characterized through sets of axioms with the help of several functional equations. One of the functional equations that arises in these characterizations is the fundamental equation of information theory. In the previous section, we have connected them to the Abel equation (11.10). Now we treat many generalizations of (11.10) and link them to other measures of information. We start off with the first generalization,

$$F(u)+G(v)+H\left(\frac{1-u}{1-uv}\right)+K(1-uv)+P\left(\frac{1-u}{1-uv}\right)=0, \tag{11.14}$$

for $u, v \in I$, where $F, G, H, K, P : I \to \mathbb{R}$, which is a generalization of (11.10), can be solved by reducing it again to (FEI).

Theorem 11.8. (Kannappan [440]). *Let $F, G, H, K, P : I \to \mathbb{R}$ satisfy* (11.14)*. If any one of the functions is Lebesgue integrable, then so are the remaining functions. Further, all these functions are differentiable in I_0, and the solution of* (11.14) *is given by*

$$\begin{aligned}
F(x) &= aI(x)+c_1,\\
G(x) &= F(x)+c_2,\\
H(x) &= -F(1-x)+c_3,\\
K(x) &= -F(1-x)+c_4,\\
P(x) &= -F(1-x)+c_5,
\end{aligned}$$

for $x \in I_0$, where $\alpha \in I_0$ and the c_i's are constants.

Proof. The solution of (11.14) is obtained first by eliminating four of the five unknown functions and rewriting it as an equation of the form (11.10) and then reducing it to the fundamental equation (FEI).

Let $F, G, H, K, P : [0,1] \to \mathbb{R}$ satisfy (11.14). $v = 1$ in (11.14) gives

$$K(u) = -F(1-u) - G(1) - H(1) - P(0), \tag{11.15a}$$

and $v = 0$ in (11.14) yields

$$H(u) = -F(1-u) - G(0) - K(1) - P(1). \tag{11.15b}$$

Now $u = 1$ in (11.14) and (11.15a) gives

$$G(v) = F(v) + G(1) + H(1) + P(0) - F(1) - H(0) - P(1), \tag{11.15c}$$

and use (11.15c) and (11.14) with $u = 0$ to get

$$\begin{aligned}
P(v) = -F(1-v) - G(1) - 2H(1) - P(0)\\
+ F(1) + H(0) + P(1) - F(0) - K(1).
\end{aligned} \tag{11.15d}$$

Now (11.15a) with $u = 1$ and (11.15b) with $u = 1$ give

$$-F(0) = K(1) + G(1) + H(1) + P(0) = H(1) + G(0) + K(1) + P(1). \tag{11.16}$$

With the use of (11.15a), (11.15b), (11.15c), (11.15d), and (11.16), (11.14) reduces to

$$F(u)+F(v)-F\left(\frac{u(1-v)}{1-uv}\right)-F(uv)-F\left(\frac{v(1-u)}{1-uv}\right)+F(0)=0. \tag{11.15}$$

Define

$$N(x) = F(x) - F(0), \quad x \in [0, 1]. \tag{11.17}$$

Then (11.15) and (11.17) give

$$N(u) + N(v) - N\left(\frac{u(1-v)}{1-uv}\right) - N(uv) - N\left(\frac{v(1-u)}{1-uv}\right) = 0, \tag{11.18}$$

an equation encountered before.

Suppose any one of the functions, say F, is Lebesgue integrable. Then K, H, G, and P are also Lebesgue integrable and N is also integrable.

Integrating equation (11.18) with respect to v between 0 and 1, we get

$$N(u) = -\int_n^1 N(v)dv + \frac{1-u}{u}\int_0^u \frac{N(t)}{(1-t)^2}dt - \frac{1}{u}\int_0^u N(t)dt - (1-u)\int_0^1 \frac{N(t)}{(1-u+ut)^2}dt.$$

It is evident that N is continuous on $]0, 1[$ and also that N is differentiable in $]0, 1[$. Thus N has derivatives of all orders (with the argument as in Theorem 11.7).

Differentiating (11.18) with respect to u, we obtain

$$N'(u) - \frac{1-v}{(1-uv)^2}N'\left(\frac{u(1-v)}{1-uv}\right) - vN'(uv) + \frac{v(1-v)}{(1-uv)^2}N'\left(\frac{v(1-u)}{1-uv}\right) = 0 \tag{11.19}$$

for $0 < u,\ v < 1$.

Define

$$f(x) = x(1-x)N'(x) \tag{11.20}$$

for $x \in]0, 1[$. Note that $N'(x) = F'(x)$. Now (11.19) and (11.20) yield

$$f(u) - f\left(\frac{u(1-v)}{1-uv}\right) - \frac{1-u}{1-uv}f(uv) + uf\left(\frac{v(1-u)}{1-uv}\right) = 0$$

for $u, v \in]0, 1[$, which, by the substitution

$$x = 1-u, \quad y = 1 - \frac{1-u}{1-uv},$$

reduces to

$$f(1-x) - f(y) - (1-y) - f\left(1 - \frac{x}{1-y}\right) + (1-x)f\left(1 - \frac{y}{1-x}\right) = 0, \tag{11.21}$$

for $0 < x,\ y < 1$, with $x + y < 1$. For $y = x$, from (11.21) we get

$$f(x) = f(1-x) \quad \text{for } 0 < x < \frac{1}{2}. \tag{11.22}$$

Hence (11.22) holds for all $x \in]0, 1[$, so that (11.21) can be rewritten as

$$f(x) + (1-x)\, f\left(\frac{y}{1-x}\right) = f(y) + (1-y)\, f\left(\frac{x}{1-y}\right) \tag{FEI}$$

for $x, y, x + y \in]0, 1[$.

As N' is continuous, f is defined by (11.20). Then it follows from [43, 489] that $f(x) = cs(x)$ and that

$$N'(x) = c\left[\frac{\log x}{1-x} + \frac{\log(1-x)}{x}\right]$$

for $x \in]0, 1[$, so that from (11.17) it follows that $F(x) = cI(x)$ for $x \in [0, 1]$, where c is an arbitrary constant. The functions K, H, G, and P can be determined from (11.15a), (11.15b), (11.15c), and (11.15d), respectively, using $F(x)$. This proves the theorem that describes the solution of (11.14). □

Remark. If F, G, H, K, P are defined only on I_0, there exist more solutions of (11.14) [441].

For example,

$$\begin{aligned} F(u) &= a \log \frac{u^2}{1-u} = G(u), \\ H(u) &= a \log \frac{(1-u)^2}{u} = P(u), \\ K(u) &= -2a \log \frac{u}{(1-u)^2}, \end{aligned}$$

and

$$\begin{aligned} F(u) &= a \log u^2(1-u) = G(u), \\ H(u) &= a \log \frac{u}{(1-u)^2} = P(u), \\ K(u) &= -2a \log u, \end{aligned}$$

as well as

$$F(u) = \log \frac{u^2}{1-u} = G(u) = H(u) = P(u) = K(u),$$

for $u \in I_0$, are solutions of equation (11.14).

Problem. Determine the regular solution of (11.14) on I_0 and the general solutions on I.

Now we consider other generalizations of (11.10) linked to other measures of information and (FEI). First we treat the functional equation

$$F(u,s)+F(v,t)+F\left(\frac{1-u}{1-uv},\frac{1-s}{1-st}\right)+F(1-uv,1-st)+F\left(\frac{1-v}{1-uv},\frac{1-t}{1-st}\right)=0, \tag{11.23}$$

which is a two-dimensional generalization of (11.10).

Result 11.9. (Kannappan [447]). *Let $F : J_2 \to \mathbb{R}$ be Lebesgue integrable in the first variable and satisfy* (11.23)*. Then $F(x,y) = aI(x) + L(y)\log x - L(1-y)\log(1-x) + P(y)$, where P satisfies* (11.10) *and L satisfies the logarithmic equation*

$$L(1-xy) = L(1-x) + L(1-y). \tag{11.24}$$

Further, if F is Lebesgue integrable in the second variable also, then F is given by

$$F(x,y) = aI(x) + bI(y) + c_1 \log x \log(1-y) - c_1 \log(1-x)\log y - c_2 \log\frac{y^2}{1-y}$$

for $(x,y) \in J_2$.

The solution is obtained by first showing that integrability implies continuity, which in turn implies the partial derivative, and then showing that F has partial derivatives of all orders with respect to the first variable. Then reduce (11.23) to the two-variable fundamental equation (with $f(x,y) = x(1-x)F_1(x,y)$)

(10.49)
$$f(x,y) + (1-x)\,f\left(\frac{u}{1-x},\frac{v}{1-y}\right) = f(u,v) + (1-u)\,f\left(\frac{x}{1-u},\frac{y}{1-v}\right),$$

which links to the directed divergence and the inaccuracy [43, 535].

Next we consider a generalization of (11.23),

$$F(u,s)+G(v,t)+H\left(\frac{1-u}{1-uv},\frac{1-s}{1-st}\right)+K(1-uv,1-st)+P\left(\frac{1-v}{1-uv},\frac{1-t}{1-st}\right)=0, \tag{11.25}$$

where $F, G, H, K, P : J_2 \to \mathbb{R}$.

Result 11.10. (Kannappan [461]). *Suppose $F, G, H, K, P : J_2 \to \mathbb{R}$ satisfy* (11.25) *and any one of the functions is integrable with respect to the first variable. Then the remaining functions are also integrable in the first variable and all have partial derivatives of all orders with respect to the first variable and*

$$F(x,y) = aI(x) + L(y)\log x - L(1-y)\log(1-x) + \beta(y) + c$$

for $x, y \in I_0$, where L satisfies (11.24) *and β satisfies an equation of type* (11.10)*. If the functions are integrable also with respect to the second variable, then*

$$F(x,y) = aI(x) + bI(y) + c_1[\log x \log(1-y) - \log(1-x)\log y] - c_2 \log \frac{y^2}{1-y} + c_3$$

and the other functions differ from it by constants.

Remark.

$$F(u,s) = a\log\frac{u^2}{1-u} + b\log\frac{s^2}{1-s} = G(u,s),$$
$$H(u,s) = a\log\frac{u^2}{1-u} + b\log\frac{(1-s)^2}{s} = P(u,s),$$
$$K(u,s) = a\log\frac{u^2}{1-u} + 2b\log\frac{s}{(1-s)^2},$$

etc., is a solution of (11.25) on I_0^2 (on smaller domains, a functional equation may have more solutions).

Problem. Determine the regular solution of (11.25) on I_0^2.

Finally, we consider the three-dimensional version of (11.10), namely

(11.26)
$$F(u,\gamma,t) + F(v,s,w) + F\left(\frac{1-u}{1-uv}, \frac{1-\gamma}{1-\gamma s}, \frac{1-t}{1-tw}\right) + F(1-uv, 1-\gamma s, 1-tw) + F\left(\frac{1-v}{1-uv}, \frac{1-s}{1-\gamma s}, \frac{1-w}{1-tw}\right) = 0,$$

and

(11.27)
$$F(u,\gamma,t) + G(v,s,w) + H\left(\frac{1-u}{1-uv}, \frac{1-\gamma}{1-\gamma s}, \frac{1-t}{1-tw}\right) + K(1-uv, 1-\gamma s, 1-tw) + P\left(\frac{1-v}{1-uv}, \frac{1-s}{1-\gamma s}, \frac{1-w}{1-tw}\right) = 0,$$

where $F, G, H, K, P : J_3 \to \mathbb{R}$.

Result 11.11. [461]. *Suppose F defined on $J_3 \to \mathbb{R}$ is Lebesgue integrable in each variable and satisfies* (11.26)*. Then F has partial derivatives of all orders and is given by*

$$F(u,v,w) = aI(u) + bI(v) + CI(w) + C_1(\log(1-v)\log u - \log v \log(1-u)) + C_2(\log(1-w)\log v - \log w \log(1-v)) + C_3(\log(1-u)\log w - \log(1-w)\log u)$$

for $u, v, w \in I_0$.

The solution is obtained by connection to the fundamental equation (of three variables)

$$f(x,y,z)+(1-x)\,f\left(\frac{u}{1-x},\frac{v}{1-y},\frac{w}{1-z}\right) = f(u,v,w)+(1-u)\,f\left(\frac{x}{1-u},\frac{y}{1-v},\frac{z}{1-w}\right), \tag{10.59}$$

and thereby it connects to the generalized directed divergence.

Now we will determine the integrable solution of (11.27).

Result 11.12. [461]. *Let $F, G, H, K, P : J_3 \to \mathbb{R}$ satisfy* (11.27) *with F integrable in the first argument. Then all are integrable in the first variable and are given by*

$$\begin{aligned}
F(x,y,z) &= aI(x)+L(y,z)\log x - L(1-y,1-z)\log(1-x)+\beta(y,z)+C_1,\\
H(x,y,z) &= aI(x)+L(y,z)\log x - L(1-y,1-z)\log(1-x)\\
&\qquad -\beta(1-y,1-z)+C_2,\\
K(x,y,z) &= aI(x)+L(y,z)\log x - L(1-y,1-z)\log(1-x)\\
&\qquad -\beta(1-y,1-z)+C_3,\\
G(x,y,z) &= aI(x)+L(y,z)\log x - L(1-y,1-z)\log(1-x)+\beta(y,z)+C_4,
\end{aligned}$$

and

$$\begin{aligned}
P(x,y,z) &= aI(x)+L(y,z)\log x - L(1-y,1-z)\log(1-x)\\
&\qquad -\beta(1-y,1-z)+C_5,
\end{aligned}$$

for $x, y, z \in I_0$, where C_i are arbitrary constants and L satisfies

$$L(1-xy,1-uv)=L(1-x,1-u)+L(1-y,1-v)$$

and β is a solution of

$$\begin{aligned}
d(r,t)+d(s,w)-d\left(\frac{r(1-s)}{1-rs},\frac{t(1-w)}{1-tw}\right)-d(rs,tw)\\
-d\left(\frac{s(1-r)}{1-rs},\frac{w(1-t)}{1-tw}\right)+C_6=0,
\end{aligned}$$

for $r, s, t, w \in I_0$, which is a generalization of (11.10).

Remark. It is easy to check that

$$F(x,y,z)=a\log\frac{x^2}{1-x}+b\log\frac{y^2}{1-y}+C\log\frac{z^2}{1-z}$$

is a solution of (11.26) on I_0^3.

Also

$$F(x_1, x_2, x_3) = \sum_{i=1}^{3} a_i \log \frac{x_i^2}{1 - x_i} = G(x_1, x_2, x_3),$$

$$H(x_1, x_2, x_3) = \sum_{i=1}^{2} a_i \log \frac{x_i^2}{1 - x_i} + a_3 \log \frac{(1 - x_3)^2}{x_3} = P(x_1, x_2, x_3),$$

$$K(x_1, x_2, x_3) = \sum_{i=1}^{2} a_i \log \frac{x_i^2}{1 - x_i} + 2a_3 \log \frac{x_3}{(1 - x_3)^2}$$

is a solution of (11.27) on I_0^3.

Problem. Determine the general solution of (11.26) and the regular solution of (11.27) on I_0^3.

12

Regularity Conditions— Christensen Measurability

Regularity conditions and Christensen measurability and its applications are treated in this chapter.

Conditions such as boundedness, monotonicity, measurability, continuity at a point, continuity, the Baire property, integrability, differentiability, and analyticity, for example, are called regularity conditions. To solve functional equations, it was customary to assume a rich regularity property like differentiability and reduce a functional equation to a differential equation and solve it. The trend for quite some time has been to solve functional equations under weaker regularity conditions like integrability or measurability or no regularity condition at all (solve algebraically).

In connection with his fifth problem, Hilbert [372] commented that although the method of reduction to a differential equation from differentiability makes it easier to solve functional equations, the question is, is the differentiability assumption necessary? And how far are the assertions that we can make in the case of differentiable functions true under proper modifications without this assumption? (See Aczél [33] and Chapter 11.) This second part of the problem draws attention to what are now known as regularity problems. The main idea used in these investigations is to assume weak regularity properties of the unknown function(s), say monotonicity, measurability, continuity, etc., and to obtain strong regularity conditions, for example integrability, continuity, differentiability, or analyticity. Results of this kind are called regularity theorems for functional equations (Járai [390]). First we illustrate these with specific examples, mostly from Cauchy equations. Then we present some general theory. This chapter ends with Christensen measurability.

Illustrative Examples

Differentiability

Let $f : \mathbb{R} \to \mathbb{R}$.

(i) Suppose f is differentiable and satisfies (A). Differentiate with respect to y to get

$$f'(x + y) = f'(y)$$

so that $f'(x) = c = f'(0)$ and $f(x) = cx + d$. Since $f(0) = 0$, $f(x) = cx$.

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(ii) Let f be a nonconstant differentiable solution of (E). Differentiate with respect to y to have

$$f'(x+y) = f(x)f'(y)$$

so that $f'(x) = cf(x),\ c = f'(0)$. If $c = 0$, then $f'(x) = 0$ or $f(x) =$ constant, which is not the case. Then we see that f has derivatives of all orders and $f^{(n)}(x) = c^n f(x)$. Note that f is never zero and $f(x) = e^{cx}$.

(iii) Suppose $f : \mathbb{R}_+^* \to \mathbb{R}$ is a nonzero differentiable solution of (L). As before, differentiating with respect to y, we get

$$xf'(xy) = f'(y)$$

or

$$xf'(x) = c = f'(1).$$

$c = 0$ would imply $f =$ constant. Then f has derivatives of all orders and $f(x) = c\log x$.

(iv) Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a nonconstant differentiable solution of (M). Again differentiating with respect to y, we obtain

$$xf'(xy) = f(x)f'(y);$$

that is,

$$xf'(x) = cf(x),\ c = f'(1).$$

$c = 0$ would imply $f(x) =$ constant. Then f has derivatives of all orders and $f(x) = x^c$ (note that $f(x) \neq 0$ for $x \neq 0$).

(v) Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable solution of (A). Integrating (A) with respect to x in an interval $[c, d]$, we have

$$(d-c)f(y) = \int_c^d f(x+y)dx - \int_c^d f(x)dx = \int_{c+y}^{d+y} f(u)du + c_1.$$

Thus f is continuous and then f is differentiable. Use (i) (see Hille [374], Aczél [12], etc.).

(vi) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a nonzero integrable solution of (E). Note that f is never zero and positive. Integrating with respect to x over $[c, d]$, we get

$$\int_c^d f(x+y)dx = f(y)\int_c^d f(x)dx;$$

that is,

$$bf(y) = \int_{c+u}^{d+u} f(u)du.$$

Hence, as before, f is differentiable, and use (ii) (see also Dhombres [218]).

(vii) [44]. The general solution bounded, from one side, on a set of positive measure $f : \mathbb{R}^* \to \mathbb{R}$ of (L) is given by $f(x) = c \log |x|,\ x \in \mathbb{R}^*$.

(viii) [44]. The general solution f of (E) on $\mathbb{R}$ in the class of functions bounded from above on an interval or on a set of positive measure is given by $f(x) = e^{cx}$ for $x \in \mathbb{R},\ (\mathbb{R}^*_+)$.

(ix) [44]. Nonzero $f : \mathbb{R}^*_+ \to \mathbb{R}$ continuous at a point or bounded on a set of positive measure satisfying (M) is given by $f(x) = x^c$ for $x > 0$.

(x) For similar results about (A), see Theorems 1.1 and 1.2.

(xi) Every $f : [1, \infty[\to \mathbb{R}$ of (L) bounded on $[1, a]$ is $c \log x$.

(xii) Suppose f on $\mathbb{R}$ is a measurable solution of (A). Then f is continuous (see Theorem 1.1, [44, 426, 218]). Every measurable solution f on $\mathbb{R}$ of (E) is continuous (Hewitt and Ross [369]). All measurable solutions f of (C) on $\mathbb{R}$ are continuous (see [424, 44], Remark 3.19).

We encountered many functional equations in information theory in Chapter 10. Every solution of (FEI) that is measurable or has the Baire property is continuous and has derivatives of all orders. Also, integrable solutions of (FEI) are continuous and infinitely differentiable. Measurable solutions f_i $(i = 1 \text{ to } 4) :]0, 1[\to \mathbb{R}$ that satisfy

$$f_1(x) + (1 - x) f_2\left(\frac{y}{1 - x}\right) = f_3(y) + (1 - y) f_4\left(\frac{x}{1 - y}\right)$$

for $x, y \in]0, 1[$ with $x + y \in]0, 1[$ are continuous and infinitely often differentiable (Ebanks, Sahoo, and Sander [245]).

Another equation one comes across is the sum form functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij}(x_i y_j) = \sum_{i=1}^{n} g_i(x_i) + \sum_{j=1}^{m} h_j(y_j),$$

$(x_i) = x \in \Gamma_n,\ y = (y_j) \in \Gamma_m$.

Let $f_{ij}, g_i, h_j :]0, 1[\to \mathbb{R}$ $(i = 1, 2,\ j = 1, 2)$ satisfy the functional equation above. If f_{ij}, h_j, g_i are measurable, they are continuous and differentiable (see [245, 218, 452], Chapter 11).

If $f_{ij} = g_i = h_j = f$ is measurable, then f is continuous [448, 198, 44].

(xiii) Let $F : \mathbb{R}^2 \to \mathbb{C}$ be a nonconstant solution of the equation

$$F(x, y)F(u, v) = F(xu - yv, xv + yu + yv)$$

measurable on a set of positive Lebesgue measure.
Then f is continuous and given by

$$F(x, y) = \begin{cases} (x + jy)^n e^{c \log(x^2 + xy + y^2)} & \text{for } (x, y) \neq (0, 0), \\ 0 & \text{for } x = 0 = y, \end{cases}$$

where $j = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. If $F : \mathbb{R}^2 \to \mathbb{R}$, then

$$F(x,y) = \begin{cases} (x^2+y^2+xy)^b & \text{for } (x,y) \neq (0,0), \\ 0 & \text{for } x = 0 = y \end{cases}$$

(see [313, 37]).

(xiv) In Chapter 7, we introduced the difference operator Δ_h,

$$\Delta_h^1 f(x) = \Delta_h f(x) = f(x+h) - f(x),$$
$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh).$$

If $f : \mathbb{R} \to \mathbb{R}$ is a measurable function such that $\Delta_h^{n+1} f(x) = 0$ almost everywhere (a.e.) on $\mathbb{R}$ for $n \in Z_+^*$ and $h \in H$ a suitable subset of $\mathbb{R}$, then f is equal a.e. on $\mathbb{R}$ to a polynomial of degree at most n. In particular, every measurable polynomial function $f : \mathbb{R} \to \mathbb{R}$ is a polynomial.

(xv) It is not that all measurable solutions of a functional equation are continuous; continuous solutions are differentiable, bounded above or positive solutions are continuous, etc.

Let $f : [0, \infty[\to \mathbb{R}$ such that $f(0) = 1,\ f(x) = 0$, for $x \neq 0$. Then f satisfies (E) and is measurable but not continuous; $f(0) = 0$ and $f(x) = 1$ for $x \neq 0$, and $x \in \mathbb{R}$ is a measurable solution of (M) but not continuous.

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x|$. Then f satisfies (M) and is continuous but not differentiable.

The Aczél-Benz functional equation [38]

$$f(x + f(y)) = f(x) + f(x + y - f(x)), \quad \text{where } f : \mathbb{R} \to \mathbb{R},$$

has the continuous solution

$$f(x) = \frac{1}{2}(x \pm |x|),$$

which is not differentiable.

Let X be a linear space over Q endowed with a topology such that X is a separable topological space with the mapping $(\lambda, x, z) \to \lambda x + z$ continuous in each variable for $\lambda \in Q,\ x, z \in X$.

Let Y be a linear space over Q that is endowed with a topology such that Y is a separable and metrizable topological space with the mapping $(\mu, y, w) \to \mu y + w$ jointly continuous with respect to the triple $(\mu, y, w) \in Q \times Y \times Y$.

Let $\mathcal{M}$ be a σ-algebra of subsets of X. A function $f : X \to Y$ is $\mathcal{M}$-measurable if and only if $f^{-1}(V) \in \mathcal{M}$ for every open set $V \subseteq Y$.

Let $\mathcal{N}$ be a proper σ-ideal of $\mathcal{M}$. For $c \in \mathcal{M}\backslash\mathcal{N},\ c_1 \subseteq X$ dense in X with $c + c_1 \in \mathcal{M},\ X\backslash(c + c_1) \in \mathcal{N}$.

Let H be a countable, dense subsemigroup of X.

Result 12.0. (Kuczma [549]). *Suppose K is a countable, dense vector subspace of X over Q with the following property.*

Let $H \notin \mathcal{N}$ be a subgroup of X such that $H_0 = H \cap K$ is dense in X; $\lambda P + z \in \mathcal{N},\ \lambda R + z \in \mathcal{M}$ for every $\lambda \in Q,\ z \in X,\ P \in \mathcal{N},\ R \in \mathcal{M}$; and $\mathcal{M}$ contains all Borel subsets of X. If an $\mathcal{M}$-measurable function $f : X \to Y$ satisfies the condition

$$\Delta_h^{n+1} f(x) = 0 \quad \text{for all } x \in X,\ h \in H,$$

then f is a continuous polynomial function of order n.

If an $\mathcal{M}$-measurable function $f : X \to Y$ satisfies for every $h \in H$ the condition $\Delta_h^{n+1} f(x) = 0,\ \mathcal{N}_-$ (a.e.), then there exists a continuous polynomial function $\phi : X \to Y$ of order n such that $f(x) = \phi(x),\ \mathcal{N}_-$ (a.e.).

Result 12.0a. *Every Lebesgue measurable or Baire measurable polynomial $f : \mathbb{R} \to \mathbb{R}$ (of order n) is continuous and hence is a real polynomial (of degree at most n).*

12.1 Some General Results

The following results show that, under quite general circumstances, measurability of a solution implies continuity [390].

Baire Category

Definitions. A subset F of a topological space X is of *first category* if F can be represented as a countable union of *nowhere dense* sets (i.e., sets having closure with empty interior); otherwise F is said to be of *second category.*

A topological space X is called a *Baire space* if the intersection of any countable class of dense open sets is dense. The theorem of Baire states that locally compact spaces and complete metric spaces are Baire spaces.

Let F be a subset of the topological space X. We say that F is *of second category at a point $x \in X$* if $F \cap V$ is of second category for every neighbourhood V of x. Let $D(F)$ denote the set of all points $x \in X$ for which F is of second category at x. Then $D(F) = \emptyset$ if and only if F is of first category. Moreover, $D(E)$ is closed and the set $F \setminus D(F)$ is of first category.

We will say that $F \subset X$ *has the Baire property* if there exists an open set V such that the symmetric difference $F \Delta V$ is of first category. All subsets of X having Baire property form a σ-algebra. Of course this σ-algebra contains Borel sets, the members of the smallest σ-algebra containing all open sets. A set $F \subset X$ has the Baire property if and only if $F \setminus D(F)$ is of first category. The set F has the Baire property if and only if each point x of X has an open neighbourhood U such that $U \cap F$ has the Baire property in X.

Baire Property of Functions. The *function f* has the *Baire property on E* if the domain of f contains F except for a set of first category, the range of f is in a

topological space Y, and $F \cap f^{-1}(V)$ has the Baire property in X for every open subset V of Y. We simply say that f has the Baire property if it has the Baire property on X.

This definition is very similar to the definition of a Borel function. A function f mapping some subset of a topological space X into another topological space Y is called a *Borel function* if for each open subset V of Y the set $f^{-1}(V)$ is a Borel subset of X.

The properties of functions having the Baire property are very similar to the properties of measurable functions.

Result 12.1. (Járai [390]). *Let X be a completely regular space and X_0 a σ-compact completely regular space, and let X_i $(i = 1, 2, \ldots, n)$ be completely regular spaces with countable bases. Let Y_i $(i = 1, 2, \ldots, n)$ and Y be locally compact spaces and T be an arbitrary topological space. Let ν be a Radon measure on Y and let μ_i $(i = 1, 2, \ldots, n)$ be Radon measures on Y_i. Suppose that $\nu(Y) < \infty$, $\mu_i(Y_i) < \infty$ $(i = 1, 2, \ldots, n)$, and let C be a subset of $T \times Y$. Consider the functions $f : T \to X$, $f_0 : Y \to X_0$, $f_i : Y_i \to X_i$, $h : C \times X_0 \times X_1 \times \cdots \times X_n \to X$, $g_i : C \to Y_i$ $(i = 1, 2, \ldots, n)$. Let t_0 be a fixed element of T, and suppose that the following conditions hold.*

For each $(t, y) \in C$,

$$f(t) = h(t, y, f_0(y), f_1(g_1(t, y)), \ldots, f_n(g_n(t, y))); \tag{12.1}$$

h is continuous; the function f_i is μ_i-measurable on Y_i $(i = 1, 2, \ldots, n)$; g_i is continuous on C $(i = 1, 2, \ldots, n)$; C_t is measurable and there exists an $\eta > 0$ such that $\nu(C_t \cap C_{t_0}) \geq \eta$ for each $t \in T$; and for each $\varepsilon > 0$ there exists a $\delta > 0$ for which $\mu_i(g_{i,t}(F)) \geq \delta$ whenever $F \subset C_t$, $\nu(F) \geq \varepsilon$, $t \in T$, and $1 \leq i \leq n$. Then f is continuous at the point t_0.

Result 12.2. [390]. *Let X be a completely regular space, X_0 a σ-compact completely regular space, X_i $(i = 1, 2, \ldots, n)$ completely regular spaces having countable bases, and T an arbitrary topological space. Let Y be an open subset of $\mathbb{R}^k$, Y_i an open subset of $\mathbb{R}^{r_i}$ $(i = 1, 2, \ldots, n)$, and C an open subset of $T \times Y$. Consider the functions $f : T \to X$, $f_0 : Y \to X_0$, $g_i : C \to Y_i$, $f_i : Y_i \to X_i$ $(i = 1, 2, \ldots, n)$, and $h : C \times X_0 \times X_1 \times \cdots \times X_n \to X$. Suppose that $t_0 \in T$ and the following conditions are satisfied.*

For each $(t, y) \in C$, (12.1) holds; h is continuous; f_i is λ measurable on the subset C_1 of Y_i $(i = 1, 2, \ldots, n)$; g_i is continuous on C $(i = 1, 2, \ldots, n)$; $\lambda^k\left(\bigcup_{i=1}^{n} g_{i,t_0}^{-1}(C_i)\right) > 0$; and the partial derivative $\frac{\partial g_i}{\partial y}$ is continuous and has rank r_i for each $(t, y) \in C$ $(i = 1, 2, \ldots, n)$.

Then f is continuous on a neighbourhood of t_0.

Result 12.3. [390]. *Let T, Y, X_i, Y_i $(i = 1, 2, \ldots, n)$ be topological spaces, X_0 a σ-compact space, and X a completely regular space. Let $C \subset T \times Y$, $C_i \subset X_i$ $(i = 1, 2, \ldots, n)$, and $t_0 \in T$, and consider the functions $f : T \to X_n$, $f_0 : Y \to X_0$,*

$g_i : C \to X_i,\ f_i : X_i \to Y_i\ (i = 1, 2, \ldots, n),\ h : C \times X_0 \times X_1 \times \cdots \times X_n \to X$. *Suppose that the following conditions hold.*

For each $(t, y) \in C$, *we have* (12.1)*;* h *is continuous;* f_i *has the Lusin-Baire property on the subset* C_i *of* X_i $(i = 1, 2, \ldots, n)$; g_i *is continuous* $(i = 1, 2, \ldots, n)$; *there exist sets* V *and* K *such that* $V \times K \subset C$, V *is open,* $t_0 \in V$, K *is of second category and has the Baire property, and*

$$K \subset \bigcap_{i=1}^{n} g_{i,t_0}^{-1}(C_i);$$

and if F *is a second category subset of* Y *and* $F \subset K$, *then* $g_{i,t}(F)$ *is a second category subset of* X_i *whenever* $1 \le i \le n$ *and* $t \in V$.

Then f *is continuous on a neighbourhood of* t_0.

Problem 12.4. [29]. Let T and X be open subsets of $\mathbb{R}^3$ and $\mathbb{R}^m$, respectively, and let C be an open subset of $T \times T$. Let $f : T \to X$, $g_i : C \to T$ $(i = 1, 2, \ldots, n)$, and $h : C \times X^{n+1} \to X$ be functions. Suppose that

$$f(t) = h(t, y, f(y), f(g_1(t, y)), \ldots, f(g_n(t, y))) \tag{12.2}$$

whenever $(t, y) \in C$; h is analytic; g_i is analytic; and for each $t \in T$ there exists a y for which $(t, y) \in C$ and $\frac{\partial g_i}{\partial y}(t, y)$ has rank s $(i = 1, 2, \ldots, n)$.

Is it true that every f that is measurable or has the Baire property is analytic?

The complete answer to this problem is unknown [390].

Result 12.5. [390]. *In* Problem 12.4, *if* h *is continuous and the functions* g_i *are continuously differentiable, then every Lebesgue measurable solution* f *is continuous.*

Proof. This is a consequence of Result 12.2 if we apply it locally.

Result 12.6. [390]. *In* Problem 12.4, *if the functions* h *and* g_i *are* ℓ *times continuously differentiable, then every locally Lipschitz solution* f *is* ℓ *times continuously differentiable* $(1 \le \ell < \infty)$.

Result 12.7. [390]. *In* Problem 12.4, *if the functions* h *and* g_i *are* $\max\{2, \ell\}$ *times continuously differentiable and there exists a compact subset* C *of* T *such that for each* $t \in T$ *there exists a* $y \in T$ *satisfying* $g_i(t, y) \in C$, *then every Lebesgue measurable solution* f *is* ℓ *times continuously differentiable* $(1 \le \ell \le \infty)$.

Proof. It follows from Results 12.5 and 12.6.

Result 12.8. [390]. *In* Problem 12.4, *if* $m = 1$ *and* $n = 1$ *and the function* h *is* p *times and the function* g_1 *is* $\max\{2, p\}$ *times continuously differentiable, then every Lebesgue measurable solution* f *is* p *times continuously differentiable* $(1 \le p \le \infty)$.

Proof. The statement is a consequence of Results 12.5 and 12.6.

Result 12.9. [390]. *In* Problem 12.4, *if equation* (12.2) *has the special form*

$$f(t) = \sum_{i=1}^{n} h_i(t, y, f(g_i(t, y))),$$

where the functions $h_i : C \times X \to \mathbb{R}^m$ *are* p *times and the functions* g_i *are* $\max\{2, p\}$ *times continuously differentiable, then every Lebesgue measurable solution* f *is* p *times continuously differentiable* $(1 \leq p \leq \infty)$.

12.2 Applications

A fairly typical situation is that measurability implies continuity.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a solution of (A). With substitution $t = x + y$, (A) becomes $f(t) = f(y) + f(t - y)$. If f is measurable on a set of positive measure, f is continuous by Result 12.2, and then from Result 12.9 it follows that f is infinitely differentiable.
2. Let $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^{n+2} \to \mathbb{R}$ satisfy

$$f(x + y) = h(x, y, f(x), f(y)), \quad x, y \in \mathbb{R}^n.$$

   With substitution $t = x + y$, the equation becomes

$$f(t) = h(t - y, y, f(t - y), f(y)),$$

   where h is given and f is unknown.

   Every solution f that is measurable on a set of positive measure is continuous by Result 12.2 and, from Result 12.8, has infinite differentiability of f.
3. The Pexider equation and its generalization.

   Problem 12.10. (Smajdor [756]). Let f, g, and h be real functions and $H : \mathbb{R}^2 \to \mathbb{R}$ a continuous function, and suppose that

$$f(x + y) = H(g(x), h(y)) \tag{12.3}$$

   whenever $x, y \in \mathbb{R}$. Are the following statements true?

   - If g and h have the Baire property, then f is continuous.
   - If g and h are Lebesgue measurable on a set of positive measure, then f is continuous.
   - If g or h is Lebesgue measurable, then f is continuous.

   By applying the results above, the assertions above are true [390].
4. Now we consider a result in [386]. Let S be a locally compact space on which is defined a positive Radon measure μ satisfying certain mild restrictions. There is a composition law $(s, t) \to s \circ t$ defined for $(s, t) \in C \subset S \times S$, and this mapping is continuous in C. Let S_f, S_g, S_h be three Hausdorff spaces and let G be a mapping of $S_g \times S_h$ into S_f that is continuous on every compact set $K \subset S_g \times S_f$.

Let f, g, h be three mappings of S into S_f, S_g, S_h, respectively. Suppose g and h are μ-measurable and that

$$f(s \circ t) = G[g(s), h(s)] \quad \text{for } (s,t) \in C. \tag{12.3a}$$

Then f is continuous on S. This theorem applies to a large number of functional equations commonly considered (see [374]).

We saw that any continuous solution of (A) is also differentiable, and it obviously admits derivatives of all orders. This does not imply, however, that in the complex plane a continuous solution of (A) is necessarily analytic. Indeed,

$$f(x+iy) = ax + by$$

is a continuous solution, and this is analytic if, and only if, $b = ia$. The same phenomenon holds for other equations of the type

$$f(z_1 \circ z_2) = G(f(z_1),\ f(z_2)], \quad z_1, z_2 \in \mathbb{C}.$$

On the other hand, if $G(u, v)$ is a symmetric analytic function of u, v and if the rule of composition is symmetric and analytic, then analytic solutions may be expected.

Let us add some remarks pertaining to the case

$$f(x+y) = G[f(x),\ f(y)] \tag{12.3b}$$

with a symmetric function G holomorphic in some domain $C = C_0 \times C_0$ in $\mathbb{C}^2$. Let a be a root of the equation

$$G(a,a) = a, \quad a \in C_0.$$

We exclude the case in which this equation is an identity or has no solutions in C_0. Equation (12.3b) has constant solutions, $f(x) \equiv a$, where a is any root of $G(a,a) = a$. Suppose now that $f(x)$ is a solution of (12.3b) defined and continuous in some interval $[0, w]$ with $f(0) = a$, $f(x) \not\equiv a$. Suppose that $f(x) \in C_0$ and that

$$G'_u(a,a) = 1.$$

Then it may be shown that $f(x)$ has derivatives of all orders in $[0, w]$ and the higher right-hand derivatives at $x = 0$ are uniquely determined by $f(0)$ and $f'(0)$. This follows from results in [234], where the theorem is proved assuming that $f(x)$ has values in a Banach algebra. It follows that if f and g are two continuous solutions of (12.3b) and if $f(0) = g(0)$, $f'(0) = g'(0)$, then f and g are identical in $[0, w]$. This does not exclude the possibility that these solutions may be extended to the complex plane in two different ways, both extensions satisfying (12.3b). At most, one of these extensions can be analytic.

The case where G is algebraic and $x \circ y = x + y$ has been examined in great detail. Here the analytic solutions have been known since Weierstrass. Continuous solutions of a real variable have been studied by many authors. Here it is

found that any continuous solution is piecewise analytic. If $f(x)$ is a continuous solution, then there is an analytic function $g(z)$ that is algebraic in z or e^{az} or $C(bz)$ and constants α_k such that in the kth interval $f(x) = g(x + \alpha_k)$.

5. The cosine equation.
Let G be a locally compact group, H be a topological ring with a countable base, and $f : G \to H$ be a solution of (C). With the substitution $t = xy^{-1}$, (C) is $f(t) = 2f(ty)f(y) - f(ty^2)$. When G is a locally Euclidean group (i.e., a Lie group), every Lusin measurable solution of (C) or solutions with the Lusin-Baire property are continuous [390].
6. The Pexider equation (PA), Jensen equation (J), Hosszu equation,

$$f(x + y - xy) + f(xy) = f(x) + f(y), \quad x, y \in \mathbb{R}, \tag{12.4}$$

$$f(xy) + g(x(1 - y)) + h((1 - x)(1 - y)) + k((1 - x)y) = 0, \tag{10.39c}$$

etc., are of the form (12.2). Measurability of f implies differentiability of order ∞ (see also [756]).
7. Abel's or dilogarithm equation.
Locally integrable or measurable solutions or those having the Baire property of Theorem 11.7 are continuous and have derivatives of all orders (use Result 12.9) ([233, 33, 440, 460]).
8. Equation from the spectral theory of random fields.
Measurable solutions of the equation

$$f(x)(1 + f(2a)) = f(a)(f(x + a) + f(x - a)), \quad x, a \in \mathbb{R},$$

are continuous and hence locally bounded [390].
9. Equation of the duplication of the cube [71, 390].
Let I be an open interval of positive length in $\mathbb{R}$ and $f : I \to \mathbb{R}^*$ be measurable on a subset of I with positive measure and satisfying

$$\begin{aligned} &f(x)f(px + (1 - p)y) + f(y)f((1 - p)x + py) \\ &\quad = f(px + (1 - p)y)^2 + f((1 - p)x + py)^2. \end{aligned}$$

Then f is continuous.
10. Translation equation (Zdun [840]).
Let X be a compact metric space $f :]0, \infty[\times X \to X$, and suppose that f satisfies the translation functional equation

$$f(u + v, x) = f(u, f(v, x)), \quad u, v \in]0, \infty[, \ x \in X, \tag{12.5}$$

and f is measurable with respect to the first variable and continuous with respect to the second one. Then f is continuous.

Guzik extended this result proving a "measurability implies continuity" type result for the more general equation

$$f(u + v, x) = \sum_{i=1}^{n} h_i(a_i(v, x), f_i(u, b_i(v, x))), \quad u, v \in]0, \infty[, \ x \in X.$$

12.3 Christensen Measurability

Background

It is well known that any Lebesgue measurable solution $f : \mathbb{R} \to \mathbb{R}$ satisfying Cauchy functional equations (A) and (E) has to be continuous. This has been proved by many authors (see Chapter 1). This type of result ("measurability implies continuity") is very typical; indeed it occurs with respect to many well-known and important functional equations (e.g., the functional equation characterizing polynomials (Chapter 7), trigonometric functional equations (Chapter 3), and Jensen's equation (Chapter 1)). In the majority of cases, the property "measurability implies continuity" is shared by solutions defined on a locally compact Abelian group; in that case, measurability is understood in the sense of a completed Haar measure. However, in many instances, local compactness turns out to be a pretty restrictive requirement; in particular, it eliminates all infinite-dimensional Banach spaces. However, in the early 1970s, it was shown in Christensen [159] that the concept of a zero set in the sense of a Haar measure can be generalized to the case of any Abelian Polish topological group. It turns out that in many respects these groups behave like locally compact groups, and various basic theorems (suitably modified) from abstract harmonic analysis carry over to this case (see [173]).

We now give some notation and terminology. Let $X(+)$ denote an Abelian Polish group and let $\mathcal{D}(X)$ be the σ-field of all Borel subsets of X. By $\mathcal{M}_\mu(X)$ we denote the completion of $\mathcal{B}(X)$ with respect to a given measure $\mu : \mathcal{B}(X) \to [0, \infty]$. The members of the σ-field

$$\mathcal{M}(X) := \cap\{\mathcal{M}_\mu(X) : \mu \text{ is a probability measure on } \mathcal{B}(X)\}$$

are called universally measurable sets. Put $\mathcal{H}_0(X) := \{C \in \mathcal{M}(X) :$ for some probability measure μ on $\mathcal{M}(X)$ one has $\mu(C + x) = 0$ for all $x \in X\}$; any element C of $\mathcal{H}_0(X)$ is called a Haar zero set in X, and a corresponding probability measure is named the testing measure for C.

Let $\mathcal{C}_0(X) := \{B \subset X :$ there exists an $A \in \mathcal{H}_0(X)$ such that $B \subset A\}$ and $\mathcal{C}(X) := \{C \cup B : C \in \mathcal{M}(X)$ and $B \in \mathcal{C}_0(X)\}$ [279].

Any set $C \in \mathcal{C}_0(X)$ is called a Christensen zero set, whereas the phrase "$C \subset X$ is Christensen measurable" will stand for "$C \in \mathcal{C}(X)$". Therefore, any Christensen measurable set is a union of a universally measurable set and a Christensen zero set; the latter is simply any subset of a Haar zero set.

A mapping f from X into a topological space Y is said to be Christensen measurable provided that $f^{-1}(U) \in \mathcal{C}(X)$ for any open set $U \subset Y$.

Needless to say, the notion of a Christensen measurable mapping reduces to the classical Haar measurability in the case where the topological group considered happens to be locally compact.

A presentation of several new results in the spirit "Christensen measurability implies continuity" valid for solutions to various classical equations and inequalities is presented in [308].

Christensen Zero Sets and the Ideals

Let $G(+)$ be a group (not necessarily commutative). A nonempty family $\mathcal{T} \subset 2^G \setminus \{G\}$ is called a proper linearly invariant set ideal (respectively σ-ideal) if, and only if, $\mathcal{T}$ is closed under finite (respectively σ-finite) set-theoretic unions, hereditary with respect to descending inclusions, and such that $x - C \in \mathcal{T}$ provided that $x \in G$ and $C \in \mathcal{T}$.

The family of all first category subsets of a second category topological group and the family of all sets with zero Haar measure is an Abelian locally compact group yielding the pattern examples of proper linearly invariant (abbreviated to p.l.i. in what follows) σ-ideals (see, for example, Dhombres and Ger [221] and Kuczma [558]).

For $\mathcal{T}$ a p.l.i. ideal (σ-ideal) in a group G, define

$$\pi(\mathcal{T}) := \{N \subset G^2 : N \subset (G \times U) \cup (U \times G) \text{ for some } U \in \mathcal{T}\}$$

and

$$\Omega(\mathcal{T}) := \{N \subset G^2 : M_x \in \mathcal{T} \text{ for } \mathcal{T} - \text{almost all } x \in G\}.$$

Result 12.11. (Ger [308]). *In any Abelian Polish group, the family of all Christensen zero sets forms a p.l.i. σ-ideal.*

Christensen Measurability Implies Continuity

Now, we may ask whether Christensen measurability provides the results in the spirit described above, i.e., measurability implies continuity. Actually, it does. This may also be considered a good argument that the notions presented above are well motivated.

Result 12.12. [279]. *Any Christensen measurable homomorphism from an Abelian Polish group into another one is continuous.*

Result 12.13. [279]. *Let X be a real linear Polish space and let $f : X \to \mathbb{R}$ be a Jensen-convex functional; i.e.,*

$$\text{(JC)} \qquad f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \quad \text{for all } x, y \in X.$$

If f is Christensen measurable, then it is continuous (*and hence also convex*).

Result 12.14. [297]. *Let C be a nonempty open and convex subset of a real linear Polish space $X(+;\cdot)$ and let $S \subset X$ be a cone such that $S \cup (-S) \cup \{0\} = X$. Finally, let $Y(\|\cdot\|)$ be a Banach lattice with partial order $\leq$. If $f : C \to Y$ is a Christensen measurable mapping such that for some positive integer n one has*

$$\sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} f(x + jh) \geq 0$$

for all $x \in C$ and $h \in S$ for which $x + (n+1)h \in C$, then f is continuous.

Result 12.15. [294]. *If $Y(\|\cdot\|)$ is a separable (real or complex) Banach space and $H : Y \times Y \to Y$ is such that*

$$\|H(s_1,t) - H(s_2,t)\| \geq \varrho(\|s_1 - s_2\|)w(t), \quad s_1, s_2, t \in Y,$$

where $w : Y \to (0,\infty)$ and $\varrho : [0,\infty) \to [0,\infty)$ are certain functions with $\varrho(0) = 0$, $\varrho(t) > 0$ for $t \in (0,\infty)$, ϱ increasing, and if X is an Abelian Polish group, then any Christensen measurable solution to (12.4) *is continuous. If, moreover, the composition $w \circ f$ is bounded away from zero, then f is uniformly continuous.*

12.4 Functional Equations (Characterizing) from Trigonometric Functions

Theorem 12.16. (Ger [308]). *Let X be an Abelian Polish group. Then any Christensen measurable solution $f : X \to \mathbb{C}$ to the d'Alembert equation* (C) *is continuous.*

Proof. From [424, 419], it follows that either $f = 0$ or f is of the form

$$f(x) = \frac{1}{2}\left(h(x) + \frac{1}{h(x)}\right), \quad x \in X, \tag{12.6}$$

where $h : X \to \mathbb{C}\backslash\{0\}$ is an exponential function (E).

From the proof in [424], two cases arise:

(i) $f(x) \subset \{-1, 1\}$. Then $h := f$.
(ii) $f(x_0)^2 \neq 1$ for some $x_0 \in X$. Then

$$h(x) := f(x) + \frac{1}{\sqrt{f(x_0)^2 - 1}}[f(x + x_0) - f(x)f(x_0)], \quad x \in X$$

(no matter which of the two roots is fixed). Therefore, if f is Christensen measurable, then so is h. Applying Result 12.12, we infer that h is continuous. Consequently, in view of (12.6), f is continuous, which was to be proved.

Result 12.17. [308]. *If X is an Abelian connected Polish group and $f : X \to \mathbb{R}$ is a Christensen measurable and bounded nonzero solution of* (C), *then f is a real part of a continuous character of the group X.*

Proof. We have $f(0) = 0$ or $f(0) = 1$. The first possibility is excluded since it forces f to be zero, which contradicts the assumption. Therefore, $f(0) = 1$. From [424], f has the form (12.6), where h is a homomorphism of X. By virtue of Theorem 12.16, f is continuous, and no matter which of the two cases (i) or (ii) from the proof occurs, this forces h to be continuous. Since f is supposed to be real, we get

$$f(x) = \frac{1}{2}\left(h(x) + \frac{1}{h(x)}\right) = \frac{1}{2}\left(\overline{h(x)} + \frac{1}{\overline{h(x)}}\right), \quad x \in X.$$

Obviously, $\overline{h}$ is a homomorphism as well; hence f admits two representations of the form (12.6). A repeated appeal to [424] yields

(iii) $\overline{h} = h$

or

(iv) $\overline{h} = \frac{1}{h}$.

Assume (iii). Then h is real and, since

$$1 = f(0) = \frac{1}{2}\left(h(0) + \frac{1}{h(0)}\right),$$

one has $h(0) = 1 > 0$. Now, the continuity of h and the connectedness of X imply that $h(x) > 0$ for all $x \in X$. Consequently, $f = 1$ and the assertion holds true in a trivial manner.

Assume (iv). Then $|h| = 1$; i.e., h is a continuous character of the group X. Moreover,

$$f = \frac{1}{2}\left(h + \frac{1}{h}\right) = \frac{1}{2}(h + \overline{h}) = Reh,$$

and the proof is complete. □

Remark 12.17a. The assumption of the connectedness of X may be replaced by the requirement of 2-divisibility of X. In fact, the only place we have used the assumption that X is connected was to show that h is positive (in case (iii)). If X is 2-divisible, then the positivity of h may be derived as follows:

$$h(x) = h\left(2 \cdot \frac{1}{2}x\right) = h\left(\frac{1}{2}x\right)^2 > 0, \quad x \in X$$

($\frac{1}{2}x$ need not be unique).

Remark 12.17b. In [652], the result above was proved under the additional strong assumptions that (a) f is continuous and (b) X is locally compact (see Chapter 3).

Let us now consider the sine functional equation (S). The behaviour of its solutions is somewhat more troublesome than that of solutions of (C). In particular, additive functions are solutions to (S) (see Chapter 3, Theorem 3.43). Moreover, some further technical difficulties occur in the case where the domain of the function considered is not 2-divisible.

Result 12.18. [308]. *Let X be a uniquely 2-divisible Abelian Polish group. Then any Christensen measurable solution $f : X \to \mathbb{C}$ of* (S) *is continuous.*

Proof. Any solution $f : X \to \mathbb{C}$ of (S) is either additive (i.e., satisfies Cauchy's equation (A)) or of the form

$$f(x) = c\left(h(x) - \frac{1}{h(x)}\right), \quad x \in X, \tag{12.7}$$

where $h : X \to \mathbb{C}^*$ is an exponential function (E).

In the first case, it suffices to apply Result 12.12. If the latter case occurs, then additionally, for an $a \in X$ such that $f(a) \neq 0$, the function

$$g(x) := \frac{1}{2f(a)}[f(x+a) - f(x-a)], \quad x \in X, \tag{12.8}$$

has the form

$$g(x) = \frac{1}{2}\left(h(x) + \frac{1}{h(x)}\right), \quad x \in X. \tag{12.9}$$

Therefore, g yields a Christensen measurable solution to equation (C), whence, in view of Theorem 12.16, g is continuous. This forces h to be continuous (see the proof of Theorem 12.16). An appeal to formula (12.7) completes the proof. □

Result 12.19. [308]. *If X is a uniquely* 2*-divisible Abelian Polish group and $f : X \to \mathbb{R}$ is a Christensen measurable and bounded solution of* (S)*, then f is proportional to the imaginary part of a continuous character of the group X.*

Proof. Let $f : X \to \mathbb{R}$ be a Christensen measurable and bounded solution of (S). Without loss of generality, we may assume that $f \neq 0$ (the zero function is the imaginary part of the trivial character $h(x) = 1$, $x \in X$). By virtue of Result 12.18, f is continuous. Moreover, f is nonadditive; in fact, otherwise, being bounded, f would have to be zero, contrary to the assumption. Consequently, the function g defined by (12.8) is continuous and has the form (12.9) with some continuous exponential function $h : X \to \mathbb{C}^*$, whereas f is given by (12.9). An appeal to [424] allows one to state that for any other exponential function $k : X \to \mathbb{C}^*$ such that

$$g(x) = \frac{1}{2}\left(k(x) + \frac{1}{k(x)}\right), \quad x \in X,$$

one has

$$k = h \quad \text{or} \quad k = \frac{1}{h}.$$

Thus, if

$$f(x) = d\left(k(x) - \frac{1}{k(x)}\right), \quad x \in X,$$

for some exponential function k, then

$$(k = h \text{ and } d = c) \quad \text{or} \quad \left(k = \frac{1}{h} \text{ and } d = -c\right).$$

Since f is real, we obtain the equality

$$f(x) = c\left(h(x) - \frac{1}{h(x)}\right) = \overline{c}\left(\overline{h(x)} - \frac{1}{\overline{h(x)}}\right),$$

valid for all $x \in X$. Thus

$$(\overline{h} = h \text{ and } \overline{c} = c) \quad \text{or} \quad \left(\overline{h} = \frac{1}{h} \text{ and } \overline{c} = -c\right).$$

In the first case, both h and c are real. In view of the 2-divisibility of X, we infer that h is strictly positive (cf. Remark 12.17a). Since f is bounded, so is g, on account of (12.8). Proceeding as in the proof of Result 12.18, we get $h = 1$, which leads to a contradiction ($f = 0$). Therefore, the other possibility must occur; i.e.,

$$\overline{h} = \frac{1}{h} \quad \text{and} \quad \overline{c} = -c.$$

This means that h is a (continuous) character of the group X and $c = \beta i$ for some $\beta \in \mathbb{R}$. Now,

$$f(x) = i\beta(h(x) - \overline{h(x)}) = -2\beta \operatorname{Im} h(x), \quad x \in X,$$

which was to be proved. □

Remark. In the proofs of Theorem 12.16 to Result 12.19, the Christensen measurability assumption has been used indirectly by appealing to Result 12.12. Thus, for any kind of measurability forcing the continuity of group homomorphism, these results remain valid as well.

Christensen Almost Additivity

P. Erdős [260] asked whether a function $f : \mathbb{R} \to \mathbb{R}$ fulfilling (A) almost everywhere with respect to Lebesgue plane measure has to coincide with a true additive function almost everywhere in the sense of Lebesgue linear measure. An affirmative answer to that question was given in [406] and [136]. This question was next raised and answered in connection with various other functional equations and inequalities (see [558, 221, 304]). All these results are established in terms of abstract p.l.i. ideals having the "Fubini property". In particular, the following two results are special cases of the results obtained in [136] and [331].

Result 12.20. [308]. *Let X be an Abelian Polish group and let Y be any group* (*not necessarily commutative*). *If $f : X \to Y$ is a mapping such that*

$$f(x + y) = f(x) + f(y) \quad \textit{for } \Omega(\mathcal{C}_0(X)) \quad \textit{—almost all } (x, y) \in X^2,$$

then there exists exactly one homomorphism $a : X \to Y$ such that

$$f(x) = a(x) \quad \textit{for } \mathcal{C}_0(X) \quad \textit{—almost all } x \in X.$$

Result 12.21. [308]. *Let X and Y be as above. If $f : X \to Y$ is a mapping such that*

$$f(x + y) = f(x) + f(y) \quad \textit{for } \pi(\mathcal{C}_0(X) \quad \textit{—almost all } (x, y) \in X^2,$$

then f is a homomorphism; in other words, if relation (A) *holds for all $x, y \in X^*$, where $U \in \mathcal{C}_0(X)$, then* (A) *is satisfied unconditionally in X^2.*

Similar results may also be obtained for polynomial functions, d'Alembert's and sine functional equations, functional equations of Mikusiński, the Pexider equation, and Jensen convex functions.

13

Difference Equations

In this chapter, we deal with several difference equations arising from the well-known Cauchy, Pexider, quadratic, cosine, and other functional equations. Cauchy difference, difference that depends on the product, the Pompeiu functional equation and its generalizations, quadratic difference, and Pexider difference are treated.

13.1 Cauchy Difference

Let $f : G \to H$, where G is an Abelian group and H is a divisible Abelian group. Then

$$\text{(CD)} \qquad f(x+y) - f(x) - f(y), \quad \text{for } x, y \in G,$$

is called the *Cauchy difference.* It is easy to verify that

$$\text{(13.1)} \qquad \phi(x,y) = f(x+y) - f(x) - f(y), \quad \text{for } x, y \in G,$$

satisfies the so-called *cocycle equation*

$$\text{(13.2)} \qquad \phi(x,y) + \phi(x+y,z) = \phi(x,y+z) + \phi(y,z), \quad \text{for } x, y, z \in G,$$

which has been studied by many authors, including G. Galbura, Th. Anghelutza, M. Ghermanescu, I. Stamate [764], M. Hosszú [380], T.M.K. Davison [209], M. Kurepa [243], J. Erdős [259], and J.D. Aczél [395, 509]. The equation (13.2) has occurred in different fields, including homological algebra, the Dehan theory of polyhedra, statistics (Daykin and Eliezer [216]), and information theory. It was proved in Kannappan and Sahoo [509] that all differentiable solutions of (13.2) are of the form (13.1).

Put $y = 0$ in (13.2) to get

$$\text{(13.3)} \qquad \phi(x,0) = \phi(0,z) = \text{constant} = c \quad \text{for } x, z \in G.$$

Partially differentiate (13.2) with respect to z. Then

$$\phi_2(x+y,z) = \phi_2(x,y+z) + \phi_2(y,z).$$

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Set $z = 0$ to get $\phi_2(x + y, 0) = \phi_2(x, y) + \phi_2(y, 0)$, which after integrating with respect to y yields

$$\phi(x, y) = g(x + y) - g(y) + h(x).$$

Letting $y = 0$ in this equation, we have

$$\phi(x, 0) = c = g(x) - g(0) + h(x);$$

that is,

$$\begin{aligned}\phi(x, y) &= g(x + y) - g(y) - g(x) + c + g(0)\\ &= f(x + y) - f(y) - f(x),\end{aligned}$$

where $f(x) = g(x) - c - g(0)$, which is (13.1).

Result 13.1. [259]. *A solution of* (13.2) *is of the form* (13.1) *if and only if* ϕ *is a symmetric function, i.e.,*

(13.1a) $$\phi(x, y) = \phi(y, x) \quad \textit{for } x, y \in G.$$

Based on this, the following result is proved that at the same time weakens the condition of differentiability of ϕ to continuity.

Result 13.2. [259]. *Any continuous solution* $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *of* (13.2) *is of the form* (13.1).

Proof. The proof presented here is due to J. Aczél. From Result 13.1, it is enough to show that ϕ is symmetric when ϕ is continuous.

Define $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

(13.4) $$g(x, y) = \phi(x, y) - \phi(y, x) \quad \text{for } x, y \in \mathbb{R}.$$

We will show that $g = 0$. Obviously

(13.4a) $$g(x, y) = -g(y, x) \quad \text{for } x, y \in \mathbb{R}.$$

Interchanging y and x and y and z separately in (13.2), we have

$$\begin{aligned}\phi(y, x) + \phi(y + x, z) &= \phi(y, x + z) + \phi(x, z),\\ \phi(x, z) + \phi(x + z, y) &= \phi(x, z + y) + \phi(z, y).\end{aligned}$$

Adding these two equations and subtracting (13.2), we obtain

(13.5) $$g(x + z, y) = g(x, y) + g(z, y) \quad \text{for } x, y, z \in \mathbb{R}.$$

For fixed y, (13.5) is an additive equation in the first variable. Since ϕ is continuous, so is g, and by Theorem 1.1 we get

$$g(x, y) = c(y)x \quad \text{for some } c(y).$$

Using (13.4a), we have

$$c(y)x = -c(x)y \quad \text{or} \quad \frac{c(x)}{x} = -\frac{c(y)}{y} \quad \text{for } x, y\ (\neq 0) \in \mathbb{R}.$$

Thus $c(y) = 0$ and $g(x, y) = 0$ for $y \neq 0$. But, by (13.4) and (13.3), $g(x, y) = 0$ for $y = 0$ also.

Hence $g(x, y) = 0$ for all $x, y \in \mathbb{R}$, resulting in ϕ being symmetric by (13.4). This proves the result as stated.

It is proved in Jessen, Karpf, and Thorup [395] that $\phi : G \times G \to H$ defined by (13.1) is characterized by the system of functional equations (13.2) and (13.1a); that is, by the cocycle equation and symmetry. The general solution of (13.2) is obtained in the following remark (see also Hosszú [380]). This equation plays a central role in the theory of functional equations of several variables and has many applications (e.g., in the theory of group extensions).

Remark 13.3. Suppose $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a solution of (13.2). Then $\phi(x, y) = S(x, y) + B(x, y)$ for $x, y \in \mathbb{R}$, where S is a symmetric solution of (13.2) and B is a biadditive, antisymmetric ($B(y, x) = -B(x, y)$) solution of (13.2). Note that $S(x, y)$ has the form (13.1).

Define

$$S(x, y) = \frac{1}{2}(\phi(x, y) + \phi(y, x)),$$
$$B(x, y) = \frac{1}{2}(\phi(x, y) - \phi(y, x)), \quad \text{for } x, y \in \mathbb{R}.$$

It is easy to verify that S and B satisfy (13.2). Now we show that B is additive in the first variable.

$$\begin{aligned}
2B(x + y, z) &= \phi(x + y, z) - \phi(z, x + y) \\
&= \phi(x, y + z) + \phi(y, z) - \phi(x, y) \\
&\qquad - [\phi(z + x, y) + \phi(z, x) - \phi(x, y)] \quad \text{by (13.2)} \\
&= \phi(x, y + z) + \phi(y, z) - [\phi(x + z, y) + \phi(z, x)] \\
&= \phi(x, y + z) + \phi(y, z) - [\phi(x, y + z) + \phi(z, y) - \phi(x, z)] - \phi(z, x) \\
&\qquad \text{(interchange } y \text{ and } z \text{ in (13.2))} \\
&= \phi(x, z) - \phi(z, x) + \phi(y, z) - \phi(z, y) \\
&= 2B(x, z) + 2B(y, z);
\end{aligned}$$

that is, B is additive in the first variable. So, B is biadditive since it is antisymmetric.

If ϕ is not symmetric, (13.2) may have a solution different from (13.1).

Remark 13.4. [259]. That (13.1) is not the general solution of (13.2) can be seen in the following example.

Let $\{h_1, h_2, h_\nu\ (\nu \in \Gamma \text{ index set})\}$ form a Hamel basis of real numbers. Let $x, y, z \in \mathbb{R}$ with

$$x = r_1h_1 + s_1h_2 + \sum_{\nu\in\Gamma} r_\nu h_\nu,$$
$$y = r_2h_1 + s_2h_2 + \sum_{\nu\in\Gamma} r'_\nu h_\nu,$$
$$z = r_3h_1 + s_3h_2 + \sum_{\nu\in\Gamma} r''_\nu h_\nu,$$

where $r_i,\ s_i\ (i = 1, 2, 3),\ r_\nu,\ r'_\nu$, and r''_ν are rationals; all are zeros but for a finite number.

Define

$$\phi(x, y) = r_1s_2 - r_2s_1.$$

Evidently, ϕ satisfies (13.1) but is not a symmetric function, so ϕ cannot be of the form (13.1).

Theorem 13.5. (Swiatak [784]). *The necessary and sufficient condition for the equation*

$$\phi(x, y) = f(x + y) - g(x), \quad \text{for } x, y \in G, \tag{13.6a}$$

admits a solution $f, g : G \to H$ such that the function $\phi : G \times G \to H$ fulfills the equation

$$\phi(x, y + z) + \phi(y, x) = \phi(y, x + z) + \phi(x, y), \quad x, y, z \in G. \tag{13.6}$$

Proof. Evidently, (13.6a) satisfies (13.6). Conversely, let (13.6) hold. Setting $y = 0$ in (13.6), we get

$$\begin{aligned}\phi(x, z) &= \phi(0, x + z) + \phi(x, 0) - \phi(0, x)\\ &= f(x + z) - u(x),\end{aligned}$$

where $f(x) = \phi(0, x),\ u(x) = \phi(0, x) - \phi(x, 0)$. This proves the result. □

Theorem 13.6. *The necessary and sufficient condition for the equation*

$$\phi(x, y) = f(x + y) - g(x) - h(y), \quad \text{for } x, y \in G, \tag{13.7a}$$

to have a solution $f, g, h : G \to H$ is that the function $\phi : G \times G \to H$ satisfies the relation

$$\begin{aligned}&\phi(z, x + y) - \phi(x, y + z) + \phi(x, y) - \phi(y, z) - \phi(0, x + y)\\ &\quad + \phi(0, y + z) + \phi(y, 0) - \phi(0, y) - \phi(z, 0) + \phi(0, z) = 0\end{aligned} \tag{13.7}$$

for $x, y, z \in G$. Further, if (f, g, h) and (f_1, g_1, h_1) are solutions of (13.7a), *then*

$$\begin{aligned}f(x) &= f_1(x) + A(x) + a + b,\\ g(x) &= g_1(x) + A(x) + a,\\ h(x) &= h_1(x) + A(x) + b,\end{aligned} \tag{13.7b}$$

where A is additive and $a,\ b$ are constants.

Proof. It is easy to verify that ϕ given by (13.7a) is a solution of (13.7). Conversely, suppose (13.7) holds. Then (13.7) can be rewritten as

$$\alpha(z, x+y) + \alpha(x, y) = \alpha(x, y+z) + \alpha(y, z), \tag{13.8}$$

where

$$\alpha(x, y) = \phi(x, y) - \phi(0, y) - \phi(x, 0) \quad \text{for } x, y, z \in G. \tag{13.8a}$$

Now (13.8a) yields $\alpha(x, 0) = -\phi(0, 0) = \phi(0, y) = \text{constant}$. Then $x = 0$ in (13.8) gives $\alpha(z, y) = \alpha(y, z)$; that is, α is symmetric. Indeed, (13.8) is (13.2). From Result 13.1, there is a $k : G \to H$ such that

$$\alpha(x, y) = k(x+y) - k(x) - k(y),$$

which with (13.8a) gives (13.7a).

If (f, g, h) and (f_1, g_1, h_1) are solutions of (13.7a), then

$$f(x+y) - f_1(x+y) = g(x) - g_1(x) + h(y) - h_1(y),$$

which is Pexider (PA). Equation (13.7b) now follows.

When (13.7) holds, the set of solutions of (13.7a) is given by (13.7b). This proves the result. □

A generalization of the cocycle equation (13.2) is the equation

$$\phi_1(p, q) + \phi_2(p+q, r) = \phi_3(p, r) + \phi_4(p+r, q) \tag{13.2a$_1$}$$

(which arises in the determination of all branding measures of inset information on open domains) by reducing it to two other similar equations,

$$k(p, q) + k(p+q, r) = h(p, r) + k(p+r, q)$$

and

$$f(p, q) + g(p+q, r) + f(p, r) + g(p+r, q) = 0,$$

where ϕ_i, f, g, h, $k : D_2 \to \mathbb{R}$, $D_n = \{(p_1, \ldots, p_n) : p_i \in J, \text{ for all } i, \sum_{i=1}^{n} p_i \in I_0\}$, $I_0 =]0, 1[^m$, and $(p, q, r) \in D_3$. The general solution of (13.2a) is given by (Ebanks [237])

$$\begin{aligned}
\phi_1(p, q) &= \alpha_1(p) + \beta_1(q) - \alpha_2(p+q) + \psi(p, q),\\
\phi_2(p, q) &= \alpha_2(p) + \beta_2(q) + \gamma(p+q) - \psi(p, q),\\
\phi_3(p, q) &= \alpha_1(p) + \beta_2(q) - \alpha_3(p+q) - \psi(p, q),\\
\phi_4(p, q) &= \alpha_3(p) + \beta_1(q) + \gamma(p+q) + \psi(p, q),
\end{aligned}$$

where $\alpha_i, \beta_j, \gamma : J \to \mathbb{R}$ $(i = 1, 2, 3;\ j = 1, 2)$ are completely arbitrary and $\psi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is an arbitrary skew-symmetric and biadditive map.

The most general "cocycle" equation is

$$F_1(x+y,z)+F_2(y+z,x)+F_3(z+x,y)+F_4(x,y)+F_5(y,z)+F_6(z,x)=0 \tag{13.9}$$

with six unknown functions $F_i : G \times G \to X$ $(i = 1, \ldots, 6)$. Here G is an arbitrary Abelian group and X is a vector space over the rational field. The general solution of (13.9) is presented in the following result.

Result 13.7. (Ebanks and Ng [243]). *The general solution* $F_i : G^2 \to X$ $(i = 1, \ldots, 6)$ *of* (13.9) *is given by*

$$F_1(x,y) = A_1(x,y) + f_3(x) - (f_5 + f_7)(y) + f_{10}(x+y) - B_2(x+y,y), \tag{13.9a}$$

$$F_2(x,y) = B_1(x,y) + f_6(x) - (f_1 + f_8)(y) + f_{11}(x+y) - B_2(x+y,y), \tag{13.9b}$$

$$F_3(x,y) = C_1(x,y) + f_9(x) - (f_2 + f_4)(y) - (f_{10} + f_{11})(x+y) + B_2(x+y,x), \tag{13.9c}$$

$$F_4(x,y) = -B_1(y,x) - C_1(x,y) + f_1(x) + f_2(y) - f_3(x+y), \tag{13.9d}$$

$$F_5(x,y) = -C_1(y,x) - A_1(x,y) + f_4(x) + f_5(y) - f_6(x+y), \tag{13.9e}$$

$$F_6(x,y) = -A_1(y,x) - B_1(x,y) + f_7(x) + f_8(y) - f_9(x+y), \tag{13.9f}$$

for arbitrary maps $A_1, B_1, C_1 : G^2 \to X$ *additive in the first variable, arbitrary map* $B_2 : G^2 \to X$ *additive in the second variable, and arbitrary maps* $f_i : G \to X$ $(i = 1, \ldots, 11)$.

13.1.1 Differences that Depend on the Product

In lattice theory, one considers functions f that satisfy the equation

$$f(a \vee b) + f(a \wedge b) = f(a) + f(b)$$

for all pairs a, b. If one applies the substitutions

$$a \vee b = a + b - ab \quad \text{and} \quad a \wedge b = ab$$

for some transforming lattice concepts to Boolean rings, one has the functional equation

$$f(x + y - xy) + f(xy) = f(x) + f(y), \tag{13.10}$$

known as the Hosszú functional equation, which is connected to the additive function (Kuczma [558]) treated by many others [125, 195, 739, 782, 573, 240].

Hosszú determined the differentiable solution as

$$f(x) = ax + b, \tag{13.11}$$

where $f : \mathbb{R} \to \mathbb{R}$. In Daróczy [195, 197], the integrable solution of (13.10) on I_0 is obtained as (13.1). In Swiatak [782], the following results are proved.

Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies (13.10). Then:

(i) If $\phi(x) = f(x) - f(0)$ is odd, (13.10) can be reduced to Jensen's equation (J); that is, (13.10) $\Longleftrightarrow$ (J).
(ii) If f is continuous at 0 or 1 or at points a and $a + \frac{1}{a} - 1$, then f has the form (13.1).
(iii) If f is integrable on $]0, \varepsilon[$, then f is given by (13.1).
(iv) In [125], it is stated that any solution $f : \mathbb{R} \to \mathbb{R}$ of (13.10) is (J); that is, $f(x) = A(x) + b$, where A is additive (no assumptions made). The same result holds for $x, y \in \mathbb{C}$ also.
(v) In [197], the same result is obtained by connecting to (13.2) as follows. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a solution of (13.10). Define

$$\phi(x, y) = f(x) + f(y) - f(xy), \quad \text{for } x, y \in \mathbb{R},$$

and obtain

$$\phi(x, y) + \phi(xy, z) = \phi(y, z) + \phi(x, yz), \tag{13.2a}$$

which is cocyclic and, since $\phi(x, y) = f(x + y - xy)$, with $z = \frac{1}{y}$, becomes

$$f\left(xy + \frac{1}{y} - x\right) + f(x + y - xy) = f(1) + f\left(y + \frac{1}{y} - 1\right);$$

that is,

$$f(u + 1) + f(v + 1) = f(1) + f(u + v + 1)$$

by the substitutions $xy + \frac{1}{y} - x = u + 1$, $x + y - xy = v + 1$. Thus $A(x) = f(x+1) - f(1)$ satisfies (A) for all $(x, y) \in S = \{(x, y) : x+y > 0 \text{ or } x = 0 = y \text{ or } x + y + 4 \leq 0\}$. To prove that A is additive on $\mathbb{R}$, let $u, v \in \mathbb{R}$ and $(u, v) \notin S$. Then there exists an x such that (x, u), $(x+u, v)$, $(x, u+v)$ are in S owing to the fact that, for fixed $(u, v) \notin S$, the inequalities $x + u > 0$, $x + u + v > 0$ can be made to hold for x. But then $A(x+u) = A(x) + A(u)$, $A(x+u+v) = A(x+u) + A(v)$, and $A(x+u+v) = A(x) + A(u+v)$. It follows from these equations that $A(u+v) = A(u) + A(v)$ for $(u, v) \notin S$ also. Thus A is additive on $\mathbb{R}$ and the result follows. □

Let G be an Abelian group and F be a field. Let $\mathcal{B}(\mathbb{R}, G)$ be all functions $f : \mathbb{R} \to G$ satisfying (13.10). Then $\mathcal{B}(\mathbb{R}, G)$ is an Abelian group.

Let F be a field with more than five elements. Then $f \in \mathcal{B}(F, G)$ if and only if f is affine ($f(x + y) + f(0) = f(x) + f(y)$); that is, f satisfies (J) (see Davison [208, 209]). The structures of $\mathcal{B}(\mathbb{R}, \mathbb{R})$, $\mathcal{B}(\mathbb{C}, \mathbb{C})$, $\mathcal{B}(F, G)$, and $\mathcal{B}(\mathbb{R}, G)$ are known.

Let $\phi_i : Z \to Z$ $(i = 1, 2, 3, 4)$ be defined by $\phi_1(n) = 1$, $\phi_2(n) = n$, $\phi_3(n) = \left[-\frac{n}{2}\right]$, and $\phi_4(n) = \left[-\frac{n}{3}\right]$, and left $\phi_5 : Z \to Z_2$ be defined by

$$\phi_5(n) = \begin{cases} 1, & n \equiv 2 \bmod 4, \\ 0, & n \not\equiv 2 \bmod 4. \end{cases}$$

Result 13.8. [208, 209]. *$f \in \mathcal{B}(Z, G)$ if and only if there exist c_1 to c_5 with $2c_5 = 0$ such that*

$$f(x) = \sum_{i=1}^{5} c_i \phi_i(x).$$

Let F be a field and G be an Abelian group. In [227] was characterized all real-valued functions f such that $f(xy) - f(x)f(y)$ depends on the sum $x + y$ for all $x, y \in \mathbb{R}$. Our aim now is to characterize all functions $f : F \to G$ for which the Cauchy difference (CD) depends only on the product xy for all $x, y \in F$. Further, we apply this characterization to determine all solutions of a generalized Hosszú equation

$$f(x + y + \alpha xy) + g(xy) = h(x) + k(y), \quad x, y \in F, \tag{13.10a}$$

where $\alpha \in F$ is a parameter chosen a priori. Notice that when $\alpha = -1$ and $f = g = h = k$, the functional equation (13.10a) reduces to (13.10). If we assume $\alpha = 0$, $f = -h = -k$, and $g = \alpha$, then (13.10a) reduces to

$$f(x + y) - f(x) - f(y) = \alpha(xy) \quad \text{for } x, y \in F. \tag{13.12}$$

If $\alpha = 1$, $h = k = -g = f$, (13.10a) becomes the Pompeiu functional equation (Pompeiu [671, 672]) that characterizes field endomorphisms (see the next section).

First we consider (13.12). Observe that the right-hand side of (13.12) is a Cauchy difference, so it satisfies the cocycle equation. That is, defining $\phi : F \times F \to G$ by

$$\phi(x, y) := f(x + y) - f(x) - f(y), \quad x, y \in F, \tag{13.1}$$

we have by this definition the cocycle equation (13.2). By (13.1), (13.2), and (13.12), α must satisfy

$$\alpha(xy) + \alpha((x + y)z) = \alpha(x(y + z)) + \alpha(yz), \quad x, y, z \in F.$$

Setting $y = -x$, we obtain

$$\alpha(-x^2) + \alpha(0) = \alpha(-x^2 + xz) + \alpha(-xz), \quad x, y \in F.$$

If $x \neq 0$, then from the equation above with $z = -x^{-1}t$, we get

$$\alpha(-x^2) + \alpha(0) = \alpha(-x^2 - t) + \alpha(t), \quad x \in F^*, \ t \in F.$$

Next, defining $A : F \to G$ by

$$A(x) := \alpha(x) - \alpha(0), \quad x \in F, \tag{13.13}$$

we have

$$A(-x^2 - t) = A(-x^2) - A(t), \quad x \in F^*, \ t \in F. \tag{13.14}$$

Next, we solve the functional equation (13.14). Afterwards, we can easily obtain the general solution of (13.12).

Result 13.9. (Ebanks, Kannappan, and Sahoo [240]). *If F is neither Z_2 nor Z_3 nor an infinite field of characteristic* 2, *then every solution of* (13.14) *is additive.*

The sharpness of this result is illustrated in the following examples.

Example 1. Let $F = Z_2$, $G = \mathbb{R}$, and $A : Z_2 \to \mathbb{R}$ be defined by $A(0) = 0$ and $A(1)$ arbitrary but nonzero. Then A satisfies (13.14) but is not additive. In fact, it can be easily verified that the general solution of (13.14) on Z_2 is any map A that satisfies $A(0) = 0$.

Example 2. The general solution of (13.14) on Z_3 is $A(0) = 0$, $A(1) = \lambda$, and $A(2) = 2\lambda$ for arbitrary $\lambda \in G$. If $3\lambda \neq 0$ in G, then A is not additive.

Example 3. Let $F = Z_2(x)$, the field of rational functions in x over Z_2. Then F^2 is a subfield of F since $s^2+t^2 = (s+t)^2$ and $s^2t^2 = (st)^2$. If we define $A : Z_2(x) \to Z_2$ by

$$A(s) = \begin{cases} 0, & \text{if } s \in F^2, \\ 1, & \text{if } s \in F\backslash F^2, \end{cases}$$

then A satisfies (13.14) but is not additive. Indeed, in this situation, equation (13.14) reduces to

$$A(u^2 + t) = A(t), \quad u \in F^*,\ t \in F.$$

If $t \in F^2$, then $u^2 + t \in F^2$ and $A(u^2 + t) = 0 = A(t)$; if $t \in F\backslash F^2$, then $u^2 + t \in F\backslash F^2$, so $A(u^2 + t) = 1 = A(t)$. This shows that A satisfies (13.14). But A is not additive because $x, x^3, x + x^3 \in F\backslash F^2$, and $A(x) = A(x^3) = 0 \neq 1 = A(x + x^3)$.

Theorem 13.10. [240]. *Suppose F is a field and G is a uniquely* 2*-divisible Abelian group. If $F \neq Z_2$ or Z_3, then the maps α, $f : K \to G$ satisfy* (13.12) *if and only if they have the form*

$$\alpha(x) = 2A_1(x) + c, \qquad x \in F, \tag{13.15}$$

$$f(x) = A_1(x^2) + A_2(x) - c, \quad x \in F, \tag{13.15a}$$

for arbitrary additive maps $A_1, A_2 : F \to G$ and arbitrary constant $c \in G$.

Proof. By (13.14) and Result 13.9, the map $A : F \to G$ defined by (13.13) satisfies (13.14) and is additive. Call this map $2A_1$. Then (13.11) gives us

$$\alpha(x) = 2A_1(x) + c, \quad x \in F,$$

where $c = \phi(0)$. This is (13.15); substituting this into (13.12), we obtain

$$f(x + y) - f(x) - f(y) = 2A_1(xy) + c, \quad x, y \in F.$$

On the other hand, the equation above can be rewritten as

$$\begin{aligned} f(x + y) - f(x) - f(y) &= 2A_1(xy) + c \\ &= A_1(2xy) + c \\ &= A_1((x + y)^2 - x^2 - y^2) + c \\ &= \{A_1((x + y)^2) - c\} - \{A_1(x^2) - c\} - \{A_1(y^2) - c\}. \end{aligned}$$

From this we see that the map $A_2 : F \to G$ defined by

$$A_2(x) := f(x) - A_1(x^2) + c, \quad x \in F,$$

is additive. This provides (13.15a). The converse is easily verified and completes the proof of the theorem. □

Remarks. Theorem 13.10 is sharp; it is false for $F = Z_2$ or Z_3. The following statements are easily verified.

The general solution $\alpha,\ f : Z_2 \to \mathbb{R}$ of (13.12) is given by

$$\alpha(x) = dx + c, \quad f(x) = -\frac{1}{2}dx - c, \quad x \in Z_2,$$

for arbitrary real constants $c,\ d$. This will not be of the form (13.15), (13.15a) unless $d = 0$.

The general solution $\alpha,\ f : Z_3 \to \mathbb{R}$ of (13.12) is given by

$$\alpha(x) = dx + c, \quad x \in Z_3,$$
$$f(x) = \begin{cases} -c & \text{if } x = 0, \\ -d - c & \text{if } x \in Z_3 \backslash \{0\}, \end{cases}$$

for arbitrary $c, d \in \mathbb{R}$. Again, this will not be of the form (13.15), (13.15a) unless $d = 0$.

The next result is a rephrasing of Theorem 13.10 and characterizes functions whose Cauchy difference depends on the product of arguments.

Corollary 13.10a. *Suppose F is a field and G is a uniquely 2-divisible Abelian group. Let $f : F \to G$ be a function. If $F \neq Z_2$ or Z_3, then the Cauchy difference $f(x+y) - f(x) - f(y)$ depends only on the product xy for all $x, y \in F$ if and only if f is of the form*

$$f(x) = A_1(x^2) + A_2(x) + c, \tag{13.15a}$$

where $A_1, A_2 : F \to G$ are additive maps and c is an arbitrary but fixed element in G (see also Ecsedi [247]).

This result is extended in the following corollary.

Corollary 13.10b. [240]. *Maps $f, g, h, k : F \to G$ satisfy the functional equation (the Pexider difference depends on the product xy)*

$$f(x + y) - g(x) - h(y) = k(xy) \tag{13.12a}$$

for all $x, y \in F$, where F is a field different from Z_2 or Z_3, and G is a uniquely 2-divisible Abelian group, if and only if $f,\ g,\ h$, and k are given by

$$\begin{aligned} f(x) &= A_1(x^2) + A_2(x) + a, \\ g(x) &= A_1(x^2) + A_2(x) - b_1, \\ h(x) &= A_1(x^2) + A_2(x) - b_2, \\ k(x) &= 2A_1(x) + b_1 + b_2 + a, \end{aligned} \tag{13.15b}$$

where $A_1, A_2 : F \to G$ are additive functions and $a, b_1, b_2 \in G$ are constants (*see also Kannappan and Sahoo* [509]).

Proof. Letting $x = 0$ and $y = 0$ in (13.12a) separately, we obtain

$$h(y) = f(y) - g(0) - k(0) \quad \text{and} \quad g(x) = f(x) - h(0) - k(0), \tag{13.16}$$

respectively. Note that g and h differ from f by a constant. Inserting (13.16) into (13.12a), we get (13.12), where

$$\alpha(x) := k(x) - g(0) - h(0) - 2k(0).$$

By Theorem 13.10 and (13.16), we obtain the asserted solution with $b_1 := h(0) + k(0) - a$ and $b_2 := g(0) + k(0) - a$. □

Theorem 13.11. [240]. *Suppose that K is a ring with unity and G is a multiplicative group. Let $f : K \to G$ be a function for which the Cauchy difference $f(xy)f(y)^{-1}f(x)^{-1}$ depends only on the sum $x + y$ for all $x, y \in K$. Then f is constant.*

The same conclusion holds if we suppose that $f(xy)f(x)^{-1}f(y)^{-1}$ depends only on $x + y$.

Proof. First, suppose that

$$f(xy)f(y)^{-1}f(x)^{-1} = h(x + y), \quad x, y \in K, \tag{13.17}$$

for some map $h : K \to G$. Putting $y = 0$ here, we get

$$f(x)^{-1} = h(x), \quad x \in K,$$

and with this (13.17) becomes

$$f(xy)f(y)^{-1}f(x)^{-1} = f(x + y)^{-1}, \quad x, y \in K.$$

The last equation can be written as

$$f(x + y)f(xy) = f(x)f(y), \quad x, y \in K.$$

Putting $x = 1$, we obtain $f(1 + y)f(y) = f(1)f(y)$; that is, $f(1 + y) = f(1)$ for all $y \in K$. Hence f is constant.

The proof is completely analogous if

$$f(xy)f(x)^{-1}f(y)^{-1} = h(x + y)$$

holds instead of (13.17). □

Solution of (13.10a)

Theorem 13.12. [240]. *Suppose that F is a field with at least five elements and that $f, g, h, k : F \to G$, where G is an Abelian group, satisfy the functional equation* (13.10a) *for some fixed $\alpha = F^*$. Then, and only then, we have*

$$\begin{aligned} f(x) &= -A(\alpha x) + a, \\ g(x) &= A(\alpha^2 x) + a + b_1 + b_2, \\ h(x) &= -A(\alpha x) + a + b_1, \\ k(x) &= -A(\alpha x) + a + b_2, \end{aligned} \tag{13.18}$$

where $A : F \to G$ is an additive function and $a, b_1, b_2 \in G$ are arbitrary constants (*see also* [509]).

Proof. Letting $y = 0$ and $x = 0$ in (13.10a) separately, we see that

$$h(x) = f(x) + g(0) - k(0) \quad \text{and} \quad k(y) = f(y) + g(0) - h(0), \tag{13.19}$$

respectively. Note that h and k differ from f by a constant. Using (13.19) in (13.10a) yields

$$f(x + y + \alpha xy) + g(xy) = f(x) + f(y) + b_1 + b_2, \tag{13.20}$$

where

$$b_1 = g(0) - k(0), \quad b_2 = g(0) - h(0).$$

Substitution of $y = -\frac{1}{\alpha}$ (recall $\alpha \neq 0$) in (13.20) gives

$$g(x) = f(-\alpha x) + b_1 + b_2. \tag{13.21}$$

Hence, by (13.21), (13.20) becomes

$$f(x + y + \alpha xy) + f(-\alpha xy) = f(x) + f(y) \tag{13.22}$$

for all $x, y \in F$. Since $\alpha \neq 0$, replacing x by $-\frac{x}{\alpha}$ and y by $-\frac{y}{\alpha}$ into (13.22) and defining

$$\psi(x) := f\left(-\frac{x}{\alpha}\right), \tag{13.23}$$

we obtain

$$\psi(x + y - xy) + \psi(xy) = \psi(x) + \psi(y) \tag{13.10}$$

for all $x, y \in F$. Using Result (iii) from above, we obtain $\psi(x) = A(x) + u$, where A is additive. From this and (13.19), (13.21), and (13.23), we obtain the form of f, g, h, and k as asserted in (13.18). This completes the proof of the theorem.

□

13.1.2 Pompeiu Functional Equation and Its Generalizations

Let $G = \mathbb{R} - \{1\}$. Then $G(o)$ is an Abelian group, where the group operation o is defined as

$$x \circ y = x + y + xy, \quad x, y \in G.$$

The difference $f(x \circ y) - f(x) - f(y)$ is a function of x, y, say $\alpha(x, y)$. If $\alpha(x, y) = f(x) f(y)$, we obtain the functional equation

$$f(x + y + xy) = f(x) + f(y) + f(x) f(y) \quad \text{for } x, y \in \mathbb{R}, \tag{13.24}$$

known as the Pompeiu functional equation [671, 672, 515, 646], which characterizes field endomorphisms in fields with more than four elements.

To see this, add 1 to both sides of (13.24) and write $F(x) = 1 + f(x)$. Then (13.24) reduces to $F(x + y + xy) = F(x)F(y)$. Now, replacing x by $x - 1$ and y by $y - 1$, we obtain $M(xy) = M(x)M(y)$, where $M(x) = F(x - 1)$. Thus, M is multiplicative and $f(x) = F(x) - 1 = M(x + 1) - 1$, which solves (13.24).

In a special case, f is an automorphism of the field $\mathbb{R}$. Suppose M is also additive. Then M is a ring homomorphism of $\mathbb{R}$. If M is a nontrivial homomorphism, then $f(x) = M(x) = x$; that is, f is an automorphism of the field $\mathbb{R}$.

Note that the solution of (13.24) is obtained without assuming any regularity condition on f.

We determine the general solution $f, p, q, g, h : \mathbb{R}^* \to \mathbb{R}$ of the functional equations

$$f(x + y + xy) = p(x) + q(y) + g(x)h(y), \quad \text{for all } x, y \in \mathbb{R}^*, \tag{13.24a}$$

and

$$f(ax + by + cxy) = f(x) + f(y) + f(x) f(y), \tag{13.24b}$$

which are generalizations of the Pompeiu functional equation (13.24). We present a method that is simple and direct to determine the general solutions of (13.24a) and (13.24b) without any regularity assumptions.

The following two lemmas are instrumental for establishing the solutions of (13.24a) and (13.24b).

Lemma 13.13. *Let $g, h : \mathbb{R}^* \to \mathbb{R}$ satisfy the functional equation*

$$g(xy) = g(y) + g(x)h(y) \tag{13.25}$$

for all $x, y \in \mathbb{R}^*$. Then they are given by

$$\begin{aligned} &g(x) = 0, && h(y) = \textit{arbitrary}, \\ &g(x) = L(x), && h(y) = 1, \\ &g(x) = \alpha[M(x) - 1], && h(y) = M(y), \end{aligned} \tag{13.25a}$$

where $M : \mathbb{R}^* \to \mathbb{R}$ is a multiplicative map not identically one, $L : \mathbb{R}^* \to \mathbb{R}$ is a logarithmic function not identically zero, and α is an arbitrary nonzero constant.

Proof. If $g = 0$, then h is arbitrary and we obtain the trivial solution. We assume hereafter that $g \neq 0$. Using (13.25) a few times, we have

$$\begin{aligned} g(xyz) &= g(xy \cdot z) \\ &= g(z) + g(xy)h(z) \\ &= g(z) + [g(y) + g(x)h(y)]h(z) \\ &= g(x \cdot yz) \\ &= g(y)h(z) + g(z) + g(x)h(yz); \end{aligned}$$

that is, since $g \neq 0$,

$$h(yz) = h(y)h(z), \quad y, z \in \mathbb{R}^*,$$

or

$$h(x) = M(x), \quad x \in \mathbb{R}^*,$$

where M is multiplicative.

If $h = M = 1$, then (13.25) shows that $g = L$ is logarithmic. Suppose $h = M \not\equiv 1$. Then interchanging x with y in (13.25) and comparing the resulting equation to (13.25), we get

$$g(y)[h(x) - 1] = g(x)[h(y) - 1].$$

Since $h(x) \neq 1$, the equation above gives

$$g(x) = \alpha[h(x) - 1] = \alpha(M(x) - 1),$$

where α is a nonzero constant, since $g \neq 0$. This proves the lemma.

Lemma 13.13a. (Kannappan and Sahoo [514]). *The general solutions* $f, g, h : \mathbb{R}^* \to \mathbb{R}$ *of the functional equation*

$$f(xy) = f(x) + f(y) + \alpha g(x) + \beta h(y) + g(x)h(y) \quad \text{for all } x, y \in \mathbb{R}^*, \tag{13.26}$$

where α *and* β *are a priori chosen constants, are given by*

$$f(x) = L(x) + \alpha\beta, \quad g(x) = \text{arbitrary}, \quad h(y) = -\alpha; \tag{13.26a}$$

$$f(x) = L(x) + \alpha\beta, \quad g(x) = -\beta, \quad h(y) = \text{arbitrary}; \tag{13.26b}$$

$$\begin{cases} f(x) = L_0(x) + \frac{1}{2}cL_1^2(x) + \alpha\beta, \\ g(x) = cL_1(x) - \beta, \\ h(y) = L_1(y) - \alpha; \end{cases} \tag{13.26c}$$

$$\begin{cases} f(x) = L(x) + \gamma\,\delta[M(x) - 1] + \alpha\beta \\ g(x) = \gamma\,[M(x) - 1] - \beta \\ h(y) = \delta[M(y) - 1] - \alpha, \end{cases} \tag{13.26d}$$

where $M : \mathbb{R}^* \to \mathbb{R}$ *is a multiplicative map not identically one,* $L_0, L_1, L : \mathbb{R}^* \to \mathbb{R}$ *are logarithmic functions with* L_1 *not identically zero, and* c, δ, γ *are arbitrary nonzero constants.*

13.1.3 Solution of the Functional Equation (13.24a)

Theorem 13.14. *The functions $f, p, q, g, h : \mathbb{R}^* \to \mathbb{R}$ satisfy the functional equation*

$$f(x+y+xy) = p(x) + q(y) + g(x)h(y) \tag{13.24a}$$

for all $x, y \in \mathbb{R}^*$ if and only if

$$\begin{cases} f(x) = L(x+1) + \alpha\beta + a + b, \\ p(x) = L(x+1) + b, \\ q(y) = L(y+1) + \alpha\beta + a + \beta h(y), \\ g(x) = -\beta, \\ h(y) = \textit{arbitrary}; \end{cases} \tag{13.27a}$$

$$\begin{cases} f(x) = L(x+1) + \alpha\beta + a + b, \\ p(x) = L(x+1) + \alpha\beta + b + \alpha g(x), \\ q(y) = L(y+1) + a, \\ g(x) = \textit{arbitrary}, \\ h(y) = -\alpha; \end{cases} \tag{13.27b}$$

$$\begin{cases} f(x) = L(x+1) + \gamma\,\delta[M(x+1) - 1] + \alpha\beta + a + b, \\ p(x) = L(x+1) + (\delta + \alpha)\gamma\,[M(x+1) - 1] + b, \\ q(y) = L(y+1) + (\gamma + \beta)\delta[M(y+1) - 1] + a, \\ g(x) = \gamma\,[M(x+1) - 1] - \beta, \\ h(y) = \delta[M(y+1) - 1] - \alpha; \end{cases} \tag{13.27c}$$

$$\begin{cases} f(x) = L_0(x+1) + \frac{1}{2}cL_1^2(x+1) + \alpha\beta + a + b, \\ p(x) = L_0(x+1) + \frac{1}{2}cL_1^2(x+1) + \alpha cL(x+1) + b, \\ q(y) = L_0(y+1) + \frac{1}{2}cL_1^2(y+1) + \beta L_1(y+1) + a, \\ g(x) = cL(x+1) - \beta, \\ h(y) = L_1(y+1) - \alpha, \end{cases} \tag{13.27d}$$

where $M : \mathbb{R}^* \to \mathbb{R}$ is a multiplicative map not identically zero, $L_0, L_1, L : \mathbb{R}^* \to \mathbb{R}$ are logarithmic maps with L_1 not identically zero, and $\alpha,\ \beta,\ \gamma,\ \delta,\ a,\ b,\ c$ are arbitrary real constants.

Proof. First, we substitute $y = 0$ and then we put $x = 0$ in (13.24a) to obtain

$$p(x) = f(x) - a + \alpha g(x) \tag{13.28}$$

and

$$q(y) = f(y) - b + \beta h(y), \tag{13.29}$$

where $a := q(0),\ b := p(0),\ \alpha := -h(0),\ \beta := -g(0)$. Letting (13.28) and (13.29) into (13.24a), we have

$$f(x+y+xy) = f(x) + f(y) - a - b + \alpha g(x) + \beta h(y) + g(x)h(y)$$

for $x, y \in \mathbb{R}^*$, which by replacing x by $u-1$ and y by $v-1$ and then defining

$$F(u) := f(u-1) - a - b, \quad G(u) := g(u-1), \quad H(u) := h(u-1)$$

for all $u \in \mathbb{R}^*$ yields

$$F(uv) = F(u) + F(v) + \alpha G(u) + \beta H(v) + G(u)H(v) \tag{13.26}$$

for all $u, v \in \mathbb{R}^*$. The general solution can now be obtained from Lemma 13.13a. The first two solutions of Lemma 13.13a (see (13.26a) and (13.26b)) together with (13.28) and (13.29) yield the solutions (13.27a) and (13.27b). The next two solutions (that is, solutions (13.26c) and (13.26d)) yield together with (13.28) and (13.29) the asserted solutions (13.27c) and (13.27d). This completes the proof of the theorem. □

13.1.4 Solution of the Functional Equation (13.24b)

The only constant solutions of (13.24b) are $f \equiv 0$ and $f \equiv -1$, so we look for nonconstant solutions of the functional equation.

Result 13.14a. (Kannappan and Sahoo [515]). *The function $f : \mathbb{R} \to \mathbb{R}$ is a solution of* (13.24b) *if and only if f is given by*

$$f(x) = \begin{cases} M(cx) - 1 & \text{if } a = 0 = b,\ c \neq 0, \\ E(x) - 1 & \text{if } a = 1 = b,\ c = 0, \\ M(cx+1) - 1 & \text{if } a = 1 = b,\ c \neq 0, \\ k & \text{otherwise,} \end{cases}$$

where $M : \mathbb{R} \to \mathbb{R}$ is multiplicative, $E : \mathbb{R} \to \mathbb{R}$ is exponential, and k is a constant satisfying $k(k+1) = 0$.

13.2 Quadratic Differences

Let G and H be Abelian groups and H be divisible. For a map $f : G \to H$, the difference

$$f(x+y) + f(x-y) - 2f(x) - 2f(y), \quad \text{for } x, y \in G, \tag{13.30}$$

is called the *quadratic difference*.

If we denote the quadratic difference (13.30) by F,

$$F(x,y) = f(x+y) + f(x-y) - 2f(x) - 2f(y), \tag{13.30a}$$

where $F : G \times G \to H$, then F satisfies the functional equation [765],

(13.31) $F(x+y,z)+F(x-y,z)-2F(y,z) = F(x,y+z)+F(x,y-z)-2F(x,y).$

It is known that this equation is not sufficient to guarantee (13.30a), even when $G = H = R$ (the additive real group). On the other hand, some sets of sufficient conditions for (13.30a) are known. For instance, (13.31) and boundedness of (13.31) and two times continuous differentiability are sufficient when $F : \mathbb{R}^2 \to \mathbb{R}$.

Some sets of conditions that are both necessary and sufficient for $F : G \to H$ to have the representation (13.30a) are given in [244]. Let G be an Abelian group and H be a *uniquely* divisible Abelian group.

The following result on Cauchy differences is used to prove Result 13.16.

Result 13.15. (Ebanks and Ng [244]). *A map $K : G^3 \to H$ is symmetric and satisfies the condition that*

(13.32) $(x,y) \mapsto K(x,y,w)$ *is a* (CD) *Cauchy difference for each $w \in G$*

if and only if K itself is a Cauchy difference of order 2; *i.e., there exists a map $g : G \to H$ such that*

(13.33) $K(x,y,z) = g(x+y+z)-g(x+y)-g(x+z)-g(y+z)+g(x)+g(y)+g(z)$

for all $x, y, z \in G$.

For a given $F : G \times G \to H$, let F_o and F_e be the canonical odd and even parts of F. That is, define $F_o, F_e : G \times G \to H$ by

$$F_o(x,y) = \frac{1}{2}[F(x,y) - F(-x,-y)],$$
$$F_e(x,y) = \frac{1}{2}[F(x,y) + F(-x,-y)].$$

Furthermore, we decompose F_e into its symmetric and skew-symmetric parts F_{e+} and F_{e-},

$$F_{e+}(x,y) = \frac{1}{2}[F_e(x,y) + F_e(y,x)],$$
$$F_{e-}(x,y) = \frac{1}{2}[F_e(x,y) - F_e(y,x)].$$

Result 13.16. [244]. *In order for $F : G \times G \to H$ to have quadratic decomposition* (13.30a) *in terms of an arbitrary $f : G \to H$, it is necessary and sufficient that*

(a) F satisfy (13.31),
(b) the map K_{e+} defined by means of F_{e+} satisfy (13.32), *and*
(c) F satisfy

(13.34) $F(x+y,x-y)+2F(x,y) = F(x,x)+F(y,y)+F(0,0)$ *for $x, y \in G$.*

13.2.1 Differences in a Prescribed Set

First we consider the Cauchy difference and then the Pexider difference.

Let X be a real linear topological space (always assumed to be Hausdorff) and $f : X \to \mathbb{R}$ such that (CD) $\in Z$; that is,

$$f(x+y) - f(x) - f(y) \in Z \quad \text{for } x, y \in X. \tag{13.35}$$

The question is under what condition f has the form

$$f = A + k, \tag{13.36}$$

where $A : X \to \mathbb{R}$ is a continuous linear functional and $k : X \to \mathbb{R}$ takes integer values only.

An example in Godini [321] shows that (13.35) is not generally true, where $f : \mathbb{R} \to \mathbb{R}$ must be of the form (13.36), where A is additive, and $k : \mathbb{R} \to Z$. However, such a representation is possible under some regularity condition imposed on f.

Result 13.17. (Baron [98]). *If the Cauchy difference* (CD) (*where* $f : \mathbb{R} \to \mathbb{R}$) *as a function of two variables is Lebesgue measurable and takes integer values only, then there exists an additive* $A : \mathbb{R} \to \mathbb{R}$ *and a Lebesgue measurable* $k : \mathbb{R} \to \mathbb{R}$ *such that* f *has the form* (13.36).

Result 13.18. (Baron and Volkmann [102]). *If a Lebesgue measurable function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies* (13.35), *then there exists a real constant* c *and a Lebesgue measurable* $k : \mathbb{R} \to Z$ *such that* $f(x) = cx + k(x)$ *for* $x \in \mathbb{R}$.

Result 13.19. [102]. *Let* X *be a real linear topological space. If* $f : X \to \mathbb{R}$ *satisfies* (13.35) *for all* $x, y \in X$ *and there exists a set* $P \subseteq X$ *such that* $0 \in \operatorname{int}(P - P)$ *and* $f(x) \in Z + \left(-\frac{1}{6}, \frac{1}{6}\right)$ *for every* $x \in X$, *then there exists a continuous linear functional* $A : X \to \mathbb{R}$ *and a* $k : X \to Z$ *such that* f *has the form* (13.36).

Result 13.20. [99, Theorem 4]. *Let* X *and* Y *be topological vector spaces and* P *be a countable, discrete subgroup of* Y.

Suppose $f : X \to Y$ *satisfies*

$$f(x+y) - f(x) - f(y) \in P \quad \text{for } x, y \in E. \tag{13.35a}$$

Then, whether X *is a Baire space and* f *is Baire measurable or* X *is a separable* F*-space and* f *is Christensen measurable, there always exists an additive function* $A : X \to Y$ *such that*

$$f(x) - A(x) \in P, \quad \text{for } x \in X, \tag{13.36a}$$

of a local basis with the additional assumption of a local function $A : X \to Y$ *such that* (13.36a) *holds.*

Result 13.21. (Baron and Kannappan [100]). *Assume that* X *is a real topological vector space and* $\mathcal{M}$ *is a* σ*-algebra of its subsets such that there exists a* σ*-ideal* $\mathcal{T}$ *of subsets of* X *fulfilling the following three conditions:*

1. $X \notin \mathcal{T}$.
2. *If* $P \in \mathcal{M} \cap \mathcal{T}$, *then* $nP \in \mathcal{T}$ *for every positive integer* n.
3. *If* $P \in \mathcal{M} \backslash \mathcal{T}$, *then* $0 \in \operatorname{int}(P - P)$.

Let Y be a locally convex real topological vector space and P be a countable discrete subgroup of Y. If a function $f : X \to Y$ fulfills (13.35a) *and $f^{-1}(W) \in \mathcal{M}$ for every open subset W of Y, then there exists a continuous linear operator $A : X \to Y$ such that* (13.36a) *holds.*

Motivated by the results above, the following results are proved in [137]. Let X be a real linear topological space, $G(+)$ a topological group, P a normal discrete subgroup of G, and $\mathcal{M}$ a family of subsets of X.

Definition. A topological group $G(+)$ is σ-bounded if and only if for each neighbourhood V of 0 there exists a sequence $\{x_n\}$ in G such that $G = U\{V + x_n : n \in Z_+^*\}$.

Result 13.22. (Brzdek [137]). *Let $f : X \to G$ be continuous at 0 and satisfy* (13.35a). *Then there exists a continuous additive $A : X \to G$ such that* (13.36a) *holds.*

Result 13.23. [137]. *Suppose that G is a σ-bounded topological group and there is a σ-ideal, $\mathcal{T}$, of subsets of X such that $0 \in \operatorname{int}(C - C)$ for each $C \in \mathcal{M} \backslash \mathcal{T}$. If a function $f : X \to G$ fulfills* (13.35a) *and there is a set $H \subset X$, $H \notin \mathcal{T}$, such that the function $f|_A$ is $\mathcal{M}$-measurable, then there exists a continuous additive function $A : X \to G$ such that* (13.36a) *holds.*

Result 13.24. *Assume X is a topological vector space, G is a group, P is a normal subgroup of G, and $f : X \to G$ satisfies* (13.35a). *If $U \subset X$ is a balanced neighbourhood of origin and C is a symmetric subset of F (i.e., $-C = C$) such that*

$$P \cap (C + C + C) = \{0\} \quad \text{and} \quad f(U) \subset P + C,$$

then there exists an additive function $A : E \to F$ such that $A(U) \subset C$ and $f(x) - A(x) \in P$ for every $x \in X$.

Let X be a real topological vector space, Y be a real Banach space, P be a subgroup of Y, and C be a symmetric (i.e., $-C = C$) subset of Y. Consider for $f : X \to Y$ the congruence (difference)

$$f(x + y) - f(x) - f(y) \in P + C \quad \text{for all } x, y \in X. \tag{13.35b}$$

Theorem 13.25. (Baron, Simon, and Volkmann [101]). *Assume X is a real topological vector space, Y is a real Banach space, P is a discrete subgroup of Y, and C is a symmetric, convex, and compact subset of Y such that $P \cap (6C) = \{0\}$. If $f : X \to Y$ satisfies* (13.35b) *and is continuous at at least one point, then there is a continuous linear function $A : E \to F$ such that*

$$f(x) - A(x) \in P + C \quad \text{for } x \in X.$$

13.3 Pexider Difference

Now we consider the *Pexider difference*

$$f(x+y)-g(x)-h(y), \tag{13.37}$$

where $f, g, h : G \to H$, G is a groupoid with identity 0, and H is a group, and obtain representation of f, g, h when (13.37) takes values in a fixed subgroup of H.

We start with the following useful lemma that is in tune with the extensions considered in Chapter 1.

Lemma 13.26. [99]. *Suppose X is a topological vector space (real or complex), U is a balanced neighbourhood of the origin, and H is an Abelian group. If $A_0 : U \to H$ is additive (i.e.,*

$$A_0(x+y) = A_0(x) + A_0(y)$$

for all $x, y \in U$ with $x+y \in U$), then there exists a unique additive function $A : X \to H$ such that A is an extension of A_0.

Proof. Using induction, it is easy to prove that

$$A_0(nx) = nA_0(x) \tag{13.38}$$

for every $x \in U$ with $nx \in U$. (Recall that U being balanced implies $\alpha U \subset U$ for $|\alpha| \le 1$.) For x, y in U and integers m, n with $mx = ny$, we have, using (13.38),

$$mA_0(x) = mnA_0\left(\frac{x}{n}\right) = mnA_0\left(\frac{y}{m}\right) = nA_0(y). \tag{13.39}$$

Now we introduce a function $A : X \to H$ by

$$A(x) = nA_0\left(\frac{x}{n}\right), \quad x \in X,$$

where n is any integer such that $x/n \in U$. This function A is well defined because of (13.39). It is easy to verify that A indeed is additive on X, is equal to A_0 on U, and is unique. This completes the proof of the lemma.

Now we prove the following theorem regarding the Pexider difference (13.37).

Theorem 13.27. [99]. *Suppose G is a groupoid with identity 0 and H is a group. Let P be a subgroup of H. A triple (f, g, h) of functions mapping G into H fulfills the condition*

$$f(x+y)-g(x)-h(y) \in P \tag{13.37a}$$

for all $x, y \in G$ if, and only if, there are functions $\alpha, \beta : G \to K$, $\phi : G \to H$ and constants $a, b \in H$ such that

$$\phi(x+y)-\phi(x)-\phi(y) \in P, \tag{13.40}$$

for all $x, y \in G$, and

$$f(x) = \alpha(x)+\phi(x)+a, \quad g(x) = b-a+\phi(x)+a, \quad h(x) = \beta(x)+\phi(x)+a-b,$$

for every $x \in G$.

Proof. Assume that (13.37a) holds for all $x, y \in G$. Then, setting in (13.37a) first $y = 0$ and then $x = 0$, we have

$$\mu(x) = f(x) - g(x) - h(0) \in P, \quad \text{for } x \in G, \tag{13.41}$$

and

$$\nu(y) = f(y) - g(0) - h(y) \in P, \quad \text{for every } y \in G. \tag{13.41a}$$

From the two relations above, we get

$$g(x) = -h(0) - \mu(x) + f(x), \quad h(x) = -\nu(x) + f(x) - g(0), \quad \text{for } x \in G. \tag{13.42}$$

Putting (13.42) into (13.37a) and using (13.41a), we obtain

$$f(x+y) - f(x) + \mu(x) + h(0) + g(0) - f(y) \in P \quad \text{for all } x, y \in G. \tag{13.43}$$

Let $x = 0 = y$ in (13.37a) to get $f(0) - g(0) - h(0) \in P$. Define $\alpha, \beta, \phi : G \to H$ by

$$\alpha(x) = \mu(x) + h(0) + g(0) - f(0), \quad \beta(x) = -\nu(x) + k(x), \tag{13.44}$$

$$\phi(x) = -k(x) + f(x) - f(0). \tag{13.45}$$

Indeed, $\alpha(x)$ and $\beta(x)$ belong to P, and

$$\begin{aligned}\phi(x+y) - \phi(x) - \phi(y) &= -\alpha(x+y) + f(x+y) - f(x) + \alpha(x) + f(0) - f(y) + \alpha(y) \\ &= -\alpha(x+y) + [f(x+y) - f(x) + \mu(x) + h(0) + g(0) - f(y)] + \alpha(y)\end{aligned}$$

also belongs to P for all $x, y \in G$ because of (13.43). Thus (13.40) holds for all $x, y \in G$.

Finally, putting $a = f(0)$, $b = g(0)$, from (13.45), (13.42), and (13.44), we obtain the desired forms of f, g, and h. The converse is easy to check.

Result 13.28. [99]. *Let X be a topological vector space, F be a topological Abelian group, and P be a discrete subgroup of F. Suppose $\phi : E \to F$ satisfies* (13.40) *for all $x, y \in X$ and is continuous at the origin. Then there exists a continuous additive function $A : X \to F$ such that*

$$\phi(x) - A(x) \in P \quad \textit{for every } x \in X; \tag{13.46}$$

i.e.,

$$\phi(x) = A(x) + \lambda(x) \quad \textit{for every } x \in X,$$

where $\lambda : X \to K$ is continuous at the origin.

Proof. Since P is discrete (that is, every element of P has a neighbourhood contained in F containing only that element), there exists a neighbourhood $V \subseteq F$ of zero such that $P \cap V = \{0\}$. Further, there exists a neighbourhood $W \subset F$ of zero such that

$$-W = W, \quad W + W + W \subset V$$

(Rudin [715]), and a balanced neighbourhood $U \subset E$ of zero such that

$$\phi(U) \subset \phi(0) + W$$

[715]. Then, for $x, y \in U$ with $x + y \in U$,

$$[\phi(x+y) - \phi(0)] - [\phi(x) - \phi(0)] - [\phi(y) - \phi(0)] \in W - W - W \subset V.$$

On the other hand, from (13.40), $\phi(0) \in P$ and

$$[\phi(x+y) - \phi(0)] - [\phi(x) - \phi(0)] - [\phi(y) - \phi(0)] \in P.$$

Consequently,

$$\phi(x+y) - \phi(0) = [\phi(x) - \phi(0)] + [\phi(y) - \phi(0)]$$

for all $x, y \in U$ with $x+y \in U$. Therefore, by Lemma 13.26, there exists an additive function $A : X \to F$ such that

$$\phi(x) - \phi(0) = A(x) \quad \text{for every } x \in U. \tag{13.47}$$

In particular, A is continuous everywhere. Further, if $x \in E$, then $\frac{x}{n} \in U$ for some positive integer n, and using (13.47) and (13.40), we get

$$\begin{aligned} \phi(x) - A(x) &= \phi\left(n\frac{x}{n}\right) - n\phi\left(\frac{x}{n}\right) + n\phi\left(\frac{x}{n}\right) - nA\left(\frac{x}{n}\right) \\ &= \left[\phi\left(n\frac{x}{n}\right) - n\phi\left(\frac{x}{n}\right)\right] + n\phi(0) \in P. \end{aligned}$$

This proves the result. □

In the next result, we assume a measurability condition on ϕ instead of the continuity at the origin.

Result 13.29. [99]. *Let X and F be topological vector spaces and P be a countable discrete subgroup of F. Suppose $\phi : X \to F$ satisfies* (13.38) *for all $x, y \in X$. Then, whether X is a Baire space and ϕ is Baire measurable or X is a separable F-space and ϕ is Christensen measurable, there always exists an additive function $A : X \to F$ such that* (13.46) *holds.*

13.3.1 Some Generalizations

We now give the general solutions of the functional equation

$$f(q,u) - g(pq, uv) - h(q - pq, uw) = A(p,q,v,w) \tag{13.48}$$
$$(p, q \in I,\ u, v, w \in]0,1[^s \text{ with } v + w \in]0,1[^s)$$

in which $f, g, h : I \times]0,1[^s \to \mathbb{R}$ are Lebesgue measurable functions in their first arguments, $A(p, \cdot, v, w)$ is an affine map of I into $\mathbb{R}$, and s is a nonzero cardinal number. This functional equation is motivated from a functional equation that arises from the study of measures of information.

13.3.2 Measurable Solutions of the Functional Equation (13.48a)

$$F_1(q,u) - F_2(pq,uv) - F_3(q-pq,uw) = A(p,q,v,w). \tag{13.48a}$$

We shall extend the following result. The measurable solutions of the functional equation

$$\begin{aligned} &F_1(q) - F_2(pq) - F_3(q-pq) - A(p,q),\\ &q \to A(p,q) \text{ affine}, \quad \text{for } p,q \in I, \end{aligned}$$

are given by

$$\begin{aligned} F_i(p) &= ap\log p + b_i p + c_i, \quad i = 1,2,3,\\ A(p,q) &= a[-p\log p - (1-p)\log(1-p)]q + (b_3 - b_2)pq\\ &\quad + (b_1 - b_3)q + c_1 - c_2 - c_3, \quad \text{for } p,q \in I, \end{aligned}$$

where $a,\ b_i,\ c_i$ are constants.

Result 13.30. (Kannappan and Sahoo [514]). *Let $F_i : I\times\ ]0,1[^s \to \mathbb{R}$ be functions that are Lebesgue measurable in their first arguments ($i = 1,2,3$). Then the solutions of the functional equation* (13.48a) *with $q \to A(p,q,v,w)$ affine, for $p,q \in I,\ u,v,w \in\ ]0,1[^s$ with $v+w \in\ ]0,1[^s$, are given by*

$$\begin{aligned} F_1(p,u) &= ap\log p + [L(u) + b_1]p + (L_2 + L_3)(u) + c_1,\\ F_2(p,u) &= ap\log p + [L(u) + b_2]p + L_2(u) + c_2,\\ F_3(p,u) &= ap\log p + [L(u) + b_3]p + L_3(u) + c_3,\\ A(p,q,v,w) &= a[-p\log p - (1-p)\log(1-p)]q + [b_1 - b_2 p - b_3(1-p)\\ &\quad - pL(y) - (1-p)L(w)]q - L_2(v) - L_3(w) + c_1 - c_2 - c_3,\\ &\quad (p,q \in I,\ u,v,w \in\ ]0,1[^s \text{ with } v+w \in\ ]0,1[^s), \end{aligned}$$

where $L,\ L_2,\ L_3$ are logarithmic (i.e., satisfy (L)*) and $a,\ b_i,\ c_i$ are constants.*

Let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a function on the plane $\mathbb{R}^2$. The *second difference* of f is

$$f(x_1+y_1, x_2+y_2) - f(x_1+y_1, x_2-y_2) - f(x_1-y_1, x_2+y_2) + f(x_1-y_1, x_2-y_2),$$

where y_1 and y_2 represent the increments. The second difference of f is said to admit *product form* if there exist functions $g,h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} f(x_1+y_1, x_2+y_2) - f(x_1+y_1, x_2-y_2) - f(x_1-y_1, x_2+y_2)\\ + f(x_1-y_1, x_2-y_2) = g(x_1,x_2)h(y_1,y_2) \end{aligned} \tag{13.49}$$

for all $x_i, y_i \in \mathbb{R}$ $(i = 1,2)$. A special case of (13.49) is the equation

$$\begin{aligned} f(x_1+y_1, x_2+y_2) - f(x_1+y_1, x_2-y_2)\\ - f(x_1-y_1, x_2+y_2) + f(x_1-y_1, x_2-y_2) = 0, \end{aligned} \tag{13.50}$$

for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$. The functional equation above reduces to

$$f(x_1+t,x_2+t) - f(x_1+t,x_2) - f(x_1,x_2+t) + f(x_1,x_2) = 0, \quad \text{for all } x_1,x_2,t \in \mathbb{R}, \tag{13.51}$$

if one first substitutes $y_1 = y_2 = \frac{t}{2}$ and replaces x_1 by $x_1 + \frac{t}{2}$ and x_2 by $x_2 + \frac{t}{2}$. The functional equation (13.51) was investigated by many authors (see [628]).

We study the class of functions whose second difference admits product form. In particular, we determine the general solution of the functional equation (13.49) for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$. Since our proof extends to structures more general than $\mathbb{R}$, we in fact consider the domain of $f,\ g,\ h$ to be a product of arbitrary groups and the range to be a quadratically closed field of characteristic different from 2. Here, for $f, g, h : G_1 \times G_2 \to F$, we consider the functional equation

$$f(x_1y_1,x_2y_2) - f(x_1y_1,x_2y_2^{-1}) - f(x_1y_1^{-1},x_2y_2) + f(x_1y_1^{-1},x_2y_2^{-1}) = g(x_1,x_2)h(y_1,y_2) \tag{13.49a}$$

with f satisfying the conditions

$$\text{(KC)} \qquad f(x_1y_1z_1,x_2) = f(x_1z_1y_1,x_2) \quad \text{and} \quad f(x_1,x_2y_2z_2) = f(x_1,x_2z_2y_2)$$

for all $x_i, y_i, z_i \in G_i$ $(i = 1, 2)$. Since we do not assume the groups to be Abelian, the condition (KC) plays an important role for solving (13.49a). The general solution of (13.49a) with (KC) is determined by deriving two related functional equations,

$$g(x_1y_1,x_2y_2) + g(x_1y_1,x_2y_2^{-1}) + g(x_1y_1^{-1},x_2y_2) + g(x_1y_1^{-1},x_2y_2^{-1}) = g(x_1,x_2)p(y_1,y_2)$$

and

$$h(x_1y_1,x_2y_2) - h(x_1y_1,x_2y_2^{-1}) - h(x_1y_1^{-1},x_2y_2) + h(x_1y_1^{-1},x_2y_2^{-1}) = p(x_1,x_2)h(y_1,y_2).$$

We determine the general solution of (13.49a) in the following result.

Result 13.31. (Kannappan and Ng [493]). *Let $G_1,\ G_2$ be arbitrary groups and let F be a quadratically closed field of characteristic different from* 2. *The complete list of functions $f, g, h : G_1 \times G_2 \to F$ satisfying the functional equation* (13.49a) *with* (KC) *is*

$$\begin{cases} f(x_1,x_2) = B(x_1,x_2), \\ g(x_1,x_2) = 0 \qquad \textit{and } h \textit{ arbitrary, or} \\ h(x_1,x_2) = 0 \qquad \textit{and } g \textit{ arbitrary,} \end{cases}$$

$$\begin{cases} f(x_1,x_2) = B(x_1,x_2) + \alpha\{[\alpha_1 E_1(x_1) - \alpha_2 E_1(x_1)^{-1}]E_2(x_2) \\ \qquad\qquad - [\beta_1 E_1(x_1) - \beta_2 E_1(x_1)^{-1}]E_2(x_2)^{-1}\}, \\ g(x_1,x_2) = [\alpha_1 E_1(x_1) + \alpha_2 E_1(x_1)^{-1}]E_2(x_2) + [\beta_1 E_1(x_1) \\ \qquad\qquad + \beta_2 E_1(x_1)^{-1}]E_2(x_2)^{-1}, \\ h(y_1,y_2) = \alpha[E_1(y_1 - E_1(y_1)^{-1}][E_1(y_2) - E_1(y_2)^{-1}], \\ \qquad\qquad \text{with } E_i(x_i) \not\equiv E_i(x_i)^{-i} \quad (i = 1,2); \end{cases}$$

$$\begin{cases} f(x_1,x_2) = B(x_1,x_2) + E_{1o}(x_1)\{[\alpha_1 A(x_1) + a_1]A(x_1)E_2(x_2) \\ \qquad\qquad + [\alpha_2 A(x_1) + a_2]A(x_1)E_2(x_2)^{-1}\}, \\ g(x_1,x_2) = E_{1o}(x_1)\{[4\alpha_1 A(x_1) + 2a_1]E_2(x_2) + [4\alpha_2 A(x_1) + 2a_2]E_2(x_2)^{-1}\}, \\ h(y_1,y_2) = E_{1o}(y_1)A(y_1)[E_2(y_2) - E_2(y_2)^{-1}], \\ \qquad\qquad \text{with } E_2(x_2) \not\equiv E_2(x_2)^{-1}; \end{cases}$$

$$\begin{cases} f(x_1,x_2) = B(x_1,x_2) + \{E_1(x_1)[\alpha_1 A(x_2) + a_1]A(x_2) \\ \qquad\qquad + E_1(x_1)^{-1}[\alpha_2 A(x_2) + a_2]A(x_2)\}E_{2o}(x_2), \\ g(x_1,x_2) = \{E_1(x_1)[4\alpha_1 A(x_2) + 2a_1] + E_1(x_1)^{-1}[4\alpha_2 A(x_2) + 2a_2]\}E_{2o}(x_2), \\ h(y_1,y_2) = [E_1(y_1) - E_1(y_1)^{-1}]A(y_2)E_{2o}(y_2), \\ \qquad\qquad \text{with } E_1(x_1) \not\equiv E_1(x_1)^{-1}; \end{cases}$$

$$\begin{cases} f(x_1,x_2) = B(x_1,x_2) + E_{1o}(x_1)E_{2o}(x_2)\big[\alpha_3 A_1(x_1)^2 A_2(x_2)^2 \\ \qquad\qquad + \alpha_1 A_1(x_1)^2 A_2(x_2) + \alpha_2 A_1(x_1)A_2(x_2)^2 + \dfrac{a}{4}A_1(x_1)A_2(x_2)\big], \\ g(x_1,x_2) = E_{1o}(x_1)E_{2o}(x_2)[16\alpha_3 A_1(x_1)A_2(x_2) + 8\alpha_1 A_1(x_1) + 8\alpha_2 A_2(x_2) + a], \\ h(y_1,y_2) = E_{1o}(y_1)E_{2o}(y_2)A_1(y_1)A_2(y_2); \end{cases}$$

$$\begin{cases} f(x_1,x_2) = B(x_1,x_2) + \dfrac{a}{4}E_{1o}(x_1)E_{2o}(x_2)B(x_1,x_2), \\ g(x_1,x_2) = aE_{1o}(x_1)E_{2o}(x_2), \\ h(y_1,y_2) = E_{1o}(y_1)E_{2o}(y_2)B(y_1,y_2), \end{cases}$$

where α, α_i $(i = 1,2,3)$, β_j, a_j $(j = 1,2)$ *are arbitrary constants, the maps* A, A_1, A_2 *are additive, the map* B *is biadditive,* E_i, E_{io} *are exponential maps with*

$$E_{io}(x_i) \equiv E_i(x_i) \equiv E_i(x_i)^{-1},$$

and β *is an arbitrary solution of*

$$\beta(x_1y_1, x_2y_2) - \beta(x_1y_1, x_2y_2^{-1}) - \beta(x_1y_1^{-1}, x_2y_2) + \beta(x_1y_1^{-1}, x_2y_2^{-1}) = 0$$

with β *satisfying* (KC).

14

Characterization of Special Functions

This chapter is devoted to functional equations connected to a few special functions such as the gamma function, beta function, Riemann's zeta function, Riemann's function, Cantor-Lebesgue singular functions, Minkowski's function, and Dirichlet's distribution.

14.1 Gamma Function

Euler used the gamma function as an interpolation function for the factorials $n! = 1 \cdot 2 \cdots n$ [261]. Legendre [604] introduced the notation Γ such that $\Gamma(n) = (n-1)!$ for $n \in Z_+^*$. As a matter of fact, it was D. Bernoulli [121] who gave the first presentation of an interpolating function of the factorials in the form of an infinite product, later known as the gamma function. Euler gave representations by integrals and formulated interesting theorems on the properties of this function. The gamma function appears in physical problems and applications. The gamma function is especially useful to develop other functions that have physical applications. The function

$$\Gamma(n) = (n-1)!, \quad \text{for } n \in Z_+^*, \tag{14.1}$$

is defined for positive integers n by recurrence $\Gamma(1) = 1,\ \Gamma(n+1) = n\Gamma(n)$.

It is natural to expect to extend this to positive reals as solutions of the equation

$$f(x+1) = xf(x) \quad \text{for } x > 0. \tag{14.2}$$

Some of the well-known representations of the gamma function Γ are (Gronau [324])

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$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)}, \quad x \in]0,\infty[, \tag{14.3}$$

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \qquad x > 0, \tag{14.4}$$

$$\Gamma(x) = \int_0^1 (-\log y)^{x-1}dy, \qquad x > 0, \tag{14.5}$$

$$\Gamma(x) = \int_{-\infty}^\infty e^{sx}e^{-e^s}ds, \qquad x > 0, \tag{14.6}$$

and Γ satisfies, in addition to (14.2) to (14.5),

$$n^{x-\frac{1}{2}} \prod_{k=0}^{n-1} f\left(\frac{x+k}{n}\right) = (2\pi)^{\frac{1}{2}(n-1)} f(x), \quad x > 0,\ n \in Z_+^*, \tag{14.7}$$

$$f(x)f(1-x) = \frac{\pi}{\sin \pi x}, \qquad x \in]0,1[, \tag{14.8}$$

$$\Gamma(x) = \frac{1}{x}\prod_{n=1}^\infty \left\{\left(1+\frac{1}{n}\right)^x \left(1+\frac{x}{n}\right)^{-1}\right\}, \quad x > 0 \text{ (due to Euler)}. \tag{14.9}$$

The integrals (14.4), (14.5), and (14.6) are equivalent. The substitutions $y = e^{-t}$, $s = \log t$ separately in (14.4) result in (14.5) and (14.6), respectively ($y = e^{e^s}$ reduces (14.5) to (14.6); $t = e^s$ transforms (14.6) to (14.4)). The functional equation (14.2) is the most well known among them since it is the simplest, most elegant, and most remarkable formula. It is also known as the *gamma functional equation*.

The logarithmically convex function satisfying (14.2) with $f(1) = 1$ is called *Euler's gamma function.*

Remark. $\Gamma(x)$ is not the only solution of (14.2). For example,

$$f(x) = \Gamma(x)e^{p(x)} \quad \text{or} \quad \Gamma(x)p(x), \quad x > 0,$$

where p is an arbitrary periodic (analytic) function of period 1 is also a solution of (14.2) (see [554]). Indeed,

$$f(x+1) = \Gamma(x+1)e^{p(x+1)} = x\Gamma(x)e^{p(x)} = xf(x).$$

So, some additional condition is necessary on f to characterize Γ through (14.2).

Notation. For $n, \nu \in Z_+$, define $(\nu)_n = \nu(\nu+1)\cdots(\nu+n-1)$, $(\nu)_0 = 1$.

Theorem 14.1. (Anastassiadis [76], Bohr and Mollerup [126]). $\Gamma :]0,\infty[\to]0,\infty[$ *logarithmically convex is the unique solution of equation* (14.2) *with* $\Gamma(1) = 1$.

Proof. Let f satisfy (14.2) with $f(1) = 1$ and be logarithmically convex for $x > 0$. From (14.2) there follows, for $n \geq 2$, $0 < x \leq 1$,

$$f(x+n) = (x)_n f(x), \quad f(n) = (n-1)!. \tag{14.10}$$

Since f is logarithmically convex, $e^{ax}f(x)$ is also convex for $x > 0$ and a a constant.

For

$$n-1<n<n+x\le n+1 \quad (n \text{ in } Z_+^*\ge 2,\ 0<x\le 1),$$

one has from convexity

$$\frac{e^{a(n-1)}f(n-1)-e^{an}f(n)}{-1}\le\frac{e^{a(n+x)}f(n+x)-e^{an}f(n)}{x}$$
$$\le\frac{e^{a(n+1)}f(n+1)-e^{an}f(n)}{1}$$

and from (14.10)

$$\frac{x(n-1)!+(n-1)!-xe^{-a}(n-2)!}{e^{ax}(x)_n}\le f(x)\le\frac{xe^{a}n!-x(n-1)!+(n-1)!}{e^{ax}(x)_n}.$$

Replacing n by $(n+1)$ in this inequality, one gets

$$\frac{xn!+n!-xe^{-a}(n-1)!}{e^{ax}(x)_{n+1}}\le f(x)\le\frac{xe^{a}n!-x(n-1)!+(n-1)!}{e^{ax}(x)_n}.$$

Taking $a=-\log n$ in the inequality above, we have

$$\frac{n!n^x}{(x)_{n+1}}\le f(x)\le\frac{n!n^x}{(x)_{n+1}}\,\frac{x+n}{n}$$

or

$$f(x)=\lim_{n\to\infty}\frac{n!n^x}{(x)_{n+1}}=\Gamma(x)\quad(x\in(0,1]). \tag{14.3}$$

We have $f(x)=\Gamma(x)$ for $x\in(0,1]$. For the other values of x, (14.3) results from the functional equation (14.2).

Proof from Kuczma [554]. Theorem 5.11 in [554] shows that there exists a unique function fulfilling the hypothesis. Since $\log x$ is completely monotonic, the function $-\log\Gamma(x)$ is also completely monotonic (cf. Theorem 5.12 and the corollary in [554]) and thus $\Gamma(x)$ is analytic in $(0,\infty)$. Further (by Formula 5.27 in [554]), we obtain

$$\Gamma(x)=\prod_{n=0}^{\infty}\frac{(n+1)(n+2)^{x-1}}{(n+x)(n+1)^{x-1}}$$
$$=\lim_{n\to\infty}\prod_{k=0}^{n-2}\frac{(k+1)(k+2)^{x-1}}{(k+x)(k+1)^{x-1}}$$
$$=\lim_{n\to\infty}\frac{(n-1)!n^{x-1}}{x(x+1)\cdots(x+n-2)},$$

whence, since $\lim_{n\to\infty}\frac{n^2}{(x+n-1)(x+n)}=1$,

$$\Gamma(x)=\lim_{n\to\infty}\frac{n!n^x}{x(x+1)\cdots(x+n)}. \tag{14.3}$$

Theorem 14.2. *If a function* $f(x)$ *satisfies equation* (14.2) *and, moreover,*

$$\lim_{x\to\infty}\left(\frac{e}{x}\right)^x\sqrt{\frac{x}{2\pi}}\,f(x)=1,$$

then

$$f(x)=\Gamma(x).$$

Proof. Taking (in Theorem 5.8 in [554]) $Q(x)=\left(\frac{e}{x}\right)^2\sqrt{\frac{x}{2\pi}}$, we obtain

$$\begin{aligned} f(x) &= \lim_{n\to\infty}\frac{\sqrt{2\pi}\,e^{-(x+n)}(x+n)^{x+n}}{\sqrt{x+n}\,x(x+1)\cdots(x+n-1)} \\ &= \lim_{n\to\infty}\frac{\sqrt{2\pi}\,e^{-(x+n)(x+n)^{x+n+1}}}{n!n^x\sqrt{x+n}}\cdot\frac{n!n^x}{x(x+1)\cdots(x+n)}. \end{aligned} \tag{14.11}$$

Further,

$$\frac{\sqrt{2\pi}\,e^{-(x+n)}(x+n)^{x+n+1}}{n!n^x\sqrt{x+n}}=\frac{\sqrt{2\pi}\,e^{-n}n^{n+1}}{n!\sqrt{x+n}}e^{-x}\left(\frac{x+n}{n}\right)^{x+n+1}. \tag{14.12}$$

In view of Stirling's formula

$$n!=n^ne^{-n}\sqrt{2\pi n}e^{\theta/12n},\quad 0<\theta\le 1,$$

expression (14.12) tends to 1 as $n\to\infty$. Thus $f(x)=\Gamma(x)$ results by (14.11) and (14.3).

Result 14.3. *If a function* $\phi(x)$ *satisfies* (14.2) *with* $\phi(1)=1$ *and, moreover,* $(e/x)^x\phi(x)$ *is monotonic, then* $\phi(x)=\Gamma(x)$.

14.1.1 Further Properties of the Gamma Function

Set

$$\psi(x)=2^x\Gamma\left(\frac{1}{2}x\right)\Gamma\left(\frac{1}{2}(x+1)\right).$$

The function $\psi(x)$ is logarithmically convex in $(0,\infty)$; moreover,

$$\psi(x+1)=2^{x+1}\Gamma\left(\frac{1}{2}(x+1)\right)\Gamma\left(\frac{1}{2}x+1\right)=x\,\psi(x).$$

Thus $\psi(x)$ may differ from $\Gamma(x)$ only by a multiplicative constant,

$$2^x\Gamma\left(\frac{1}{2}x\right)\Gamma\left(\frac{1}{2}(x+1)\right)=d\Gamma(x), \tag{14.13}$$

where $d=\Gamma\left(\frac{1}{2}\right)$ (Legendre's formula).

Theorem 14.4. (Artin [80]). *If a positive function* $\phi \in C^1[(0,\infty)]$ *satisfies equation* (14.2) *and*

$$2^x \phi\left(\frac{x}{2}\right)\phi\left(\frac{x+1}{2}\right) = d\phi(x), \quad d = 2\Gamma\left(\frac{1}{2}\right), \tag{14.14}$$

then $\phi(x) = \Gamma(x)$.

Proof. The function $\alpha(x) = \log\phi(x) - \log\Gamma(x)$ satisfies the equation

$$\alpha\left(\frac{x}{2}\right) + \alpha\left(\frac{x+1}{2}\right) = \alpha(x) \quad \text{for } x > 0. \tag{14.15}$$

Indeed,

$$\begin{aligned}
\alpha\left(\frac{x}{2}\right) + \alpha\left(\frac{x+1}{2}\right) &= \log\phi\left(\frac{x}{2}\right) + \log\phi\left(\frac{x+1}{2}\right) \\
&\quad - \log\Gamma\left(\frac{x}{2}\right) - \log\Gamma\left(\frac{x+1}{2}\right) \\
\text{by (14.14) and (14.13)} \quad &= \log d\phi(x) - \log 2^x - (\log d\Gamma(x) - \log 2x) \\
&= \log\phi(x) - \log\Gamma(x) = \alpha(x).
\end{aligned}$$

From [80, p. 38], it follows that $\alpha(x) = 0$. Thus $\phi(x) = \Gamma(x)$. □

14.1.2 Definitions of the Function $\Gamma(x)$

For $\nu, n \in Z_+^*$, it is obvious that

$$(\nu-1)! = \frac{n!}{(\nu)_{n+1}} n^\nu \left[\frac{n+1}{n}\,\frac{n+2}{n}\cdots\frac{n+\nu}{n}\right].$$

Since

$$\lim_{n\to\infty}\left[\frac{n+1}{n}\,\frac{n+2}{n}\cdots\frac{n+\nu}{n}\right] = 1,$$

one can have

$$(\nu-1)! = \lim_{n\to\infty}\frac{n!n^\nu}{(\nu)_{n+1}}.$$

Now treat ν as a variable (positive) and define

$$\Gamma_n(x) = \frac{n!n^x}{(x)_{n+1}},$$

where x is a variable greater than zero. Then

$$\Gamma_n(x) = \frac{n!n^x}{(x)_{n+1}} = e^{x\left(\log n - 1 - \frac{1}{2} - \cdots - \frac{1}{n}\right)}\frac{1}{x}\prod_{\nu=1}^{n}\frac{e^{\frac{x}{\nu}}}{1+\frac{x}{\nu}}. \tag{14.16}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right) = \gamma = \int_0^\infty\left(\frac{1}{1-e^{-t}}-\frac{1}{t}\right)e^{-t}dt$$

is known as *Euler's constant.*

Since the infinite product

$$\prod_{n=1}^{\infty} \frac{e^{\frac{x}{n}}}{1+\frac{x}{n}}$$

converges uniformly and the series

$$\sum_{n=1}^{\infty} \left\{ \log\left(1+\frac{x}{n}\right) - \frac{x}{n} \right\}$$

converges absolutely and uniformly, one can define the function

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x \cdot n!}{(x)_{n+1}} \quad \text{for } x > 0. \tag{14.3}$$

It can be shown that (14.14) converges in the interval $(-k, -k+1)$, $k \in Z_+^*$, so Γ can be defined by (14.3) for all values of real x except for the values $0, -1, -2, \ldots$. It follows from (14.14) that

$$\Gamma(x) = e^{-\gamma x} \cdot \frac{1}{x} \prod_{n=1}^{\infty} \frac{e^{\frac{x}{n}}}{1+\frac{x}{n}} \quad \text{(Weierstrass formula)},$$

from which follows Euler's formula (14.9).

Remark. Replace x by $x+1$ in (14.14) to get

$$\Gamma_n(x+1) = \frac{n^{x+1} \cdot n!}{(x+1)_{n+1}} = x\Gamma_n(x) \cdot \frac{n}{x+n+1}$$

so that $\Gamma(x+1) = x\Gamma(x)$.

Let us now take up (14.8). Set $\alpha(x) = \Gamma(x)\Gamma(1-x)\sin \pi x$. This function is defined everywhere except at $x = -n$, $n = 0, 1, 2, \ldots$, and is periodic with period 1, $\alpha(x+1) = \alpha(x)$, so that it can be extended onto the whole real axis. Moreover, $\alpha \in C^1$ in $\bigcup_{-\infty}^{\infty} (n-1, n)$. Since

$$\alpha(x) = x\Gamma(x)\Gamma(1-x)\frac{\sin \pi x}{x} = \Gamma(x+1)\Gamma(1-x)\frac{\sin \pi x}{x}, \tag{14.17}$$

$\alpha(x)$ is of class C^1 in $(-\infty, \infty)$. By (14.13),

$$\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(1-\frac{x}{2}\right) = d2^{x-1}\Gamma(1-x),$$

whence

$$\begin{aligned}\alpha\left(\frac{x}{2}\right)\alpha\left(\frac{x+1}{2}\right) &= \Gamma\left(\frac{x}{2}\right)\Gamma\left(1-\frac{x}{2}\right)\sin\frac{\pi x}{2}\Gamma\left(\frac{x+1}{2}\right)\Gamma\left(\frac{1-x}{2}\right)\cos\frac{\pi x}{2} \\ &= \frac{d}{2^x}\Gamma(x)d2^{x-1}\Gamma(1-x)\frac{1}{2}\sin \pi x = \frac{d^2}{4}\alpha(x).\end{aligned}$$

Consequently, the function $\zeta(x) = \log \frac{4}{d^2}\alpha(x)$ satisfies (14.15), whence it follows [80, p. 38] that $\alpha(x) \equiv d^2/4$. Setting $x = 0$ in (14.17), we obtain $\alpha(0) = \pi$, whence $\alpha(x) \equiv \pi$ and $d = 2\sqrt{\pi}$.

The equality $\alpha(x) \equiv \pi$ also immediately gives the relation

$$\Gamma(x)\Gamma(1-x) = \frac{x}{\sin \pi x} \tag{14.8}$$

discovered by Euler. Equation (14.8) is sometimes referred to as *Euler's functional equation*. Now (14.13) holds.

More generally, one can prove the relation (due to Gauss and known as *Gauss's multiplication formula*)

$$p^x \prod_{j=0}^{p-1} \Gamma\left(\frac{x+j}{p}\right) = d_p \Gamma(x),$$

$p = 2, 3, 4, \ldots$, where $d_p = p^{1/2}(2\pi)^{(p-1)/2}$. Whereas each of the equations

$$p^x \prod_{j=0}^{p-1} \phi\left(\frac{x+j}{p}\right) = d_p \phi(x), \quad p = 2, 3, 4, \ldots, \tag{14.18}$$

has continuous solutions satisfying (14.2) but different from $\Gamma(x)$, it may be proved that $\phi(x) = \Gamma(x)$ is the only continuous solution of equation (14.2) fulfilling *all* the equations (14.8) [80].

The *Stirling formula* for $\Gamma(x)$ is

$$\Gamma(x) = \sqrt{2\pi}\, x^{x-1/2} e^{-x+\theta(x)/12x},$$

where $\theta(x)$ fulfills $0 < \theta(x) < 1$.

Now we treat the complex versions of (14.8) and (14.18).

Theorem 14.5. (Haruki [356]). *Let $S = \{0, -1, -2, \ldots\}$ denote the set of all non-positive integers and $K = \mathbb{C}\backslash S$ its complement. If*

(i) *f is a meromorphic function of a complex variable z on $\mathbb{C}$,*
(ii) *f is analytic and vanishes nowhere on D,*
(iii) *and f satisfies the functional equation*

$$n^{z-\frac{1}{2}} \prod_{k=0}^{n-1} f\left(\frac{z+k}{n}\right) = (2\pi)^{\frac{n-1}{2}} f(z) \tag{14.7a}$$

(*Gauss multiplication formula*)

on K, where n is an arbitrarily fixed positive integer greater than 1 and $n^{z-\frac{1}{2}}$ denotes the principal value $\exp\left(\left(z-\frac{1}{2}\right)\operatorname{Log} n\right)$,

then the only function satisfying (i), (ii), *and* (iii) *is*

$$f(z) = \exp\left(a\left(z - \frac{1}{2}\right) + \frac{2m\pi i}{n-1}\right)\Gamma(z),$$

where a *is an arbitrary complex constant and* m *is an arbitrary integer.*

Equation (14.7a) *characterizes* $\Gamma(x)$ *by a multiplicative factor. Here* $n^{z-\frac{1}{2}}$ *denotes* $\exp\left(\left(z - \frac{1}{2}\right)\operatorname{Log} n\right)$, *where* Log *is the principal value of the logarithm.*

Remarks. As is well known, the function $f(z) = \Gamma(z)$ satisfies conditions (i), (ii), and (iii) of Theorem 14.5. When $n = 2$, (14.7a) becomes

(14.14a) $2^{z-1}\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right) = \sqrt{\pi}\,\Gamma(z)$ (Legendre's duplication formula).

Proof of the theorem. Replace z by $z + 1$ in (14.7a) to obtain

$$n^{z+\frac{1}{2}}\prod_{k=1}^{n} f\left(\frac{z+k}{n}\right) = (2\pi)^{\frac{n-1}{2}} f(z+1). \tag{14.19}$$

Observing that, by condition (ii), $f(z)$ does not vanish anywhere on K and dividing (14.19) by (14.7a) side by side yields on K

$$\frac{f\left(\frac{z}{n}+1\right)}{\frac{z}{n} f\left(\frac{z}{n}\right)} = \frac{f(z+1)}{z f(z)}. \tag{14.20}$$

Set

$$F(z) = \frac{f(z+1)}{z f(z)} \tag{14.21}$$

so that, by conditions (i) and (ii), F is meromorphic for $|z| < +\infty$ and analytic on K. Furthermore, by (14.20), we get on K

$$F\left(\frac{z}{n}\right) = F(z). \tag{14.22}$$

Let z_0 be an arbitrarily fixed point belonging to K, and we set $b = F(z_0)$. Then, by (14.22) and iteration, we have

$$F\left(\frac{z_0}{n^k}\right) = b \quad (k = 1, 2, 3, \dots). \tag{14.23}$$

Furthermore, by (14.21) and condition (ii), we obtain $b \neq 0$.

Since F is meromorphic for $|z| < +\infty$, the point $z = 0$ is a pole or a removable singular point of F. We claim that the point $z = 0$ is a removable singular point.

Assume the contrary. Then the point $z = 0$ is a pole of F. If we set $z_k = \frac{1}{n^k}$ $(k = 1, 2, 3, \dots)$, then we obtain $\lim_{k\to+\infty} z_k = 0$. Hence we have $\lim_{k\to+\infty} F(z_k) = \infty$.

On the other hand, by (14.23), we get $\lim_{k\to+\infty} F(z_k) = b$, which is a contradiction.

Therefore, the point $z = 0$ is a removable singular point. Hence, by Riemann's theorem on removable singular points (see [67, pp. 124–125]), F is analytic on the point $z = 0$. If we apply the Identity Theorem with $f(z) = F(z)$ and $z_0 = 0$, then we obtain $F(z) \equiv b$ on $K \cup \{0\}$.

Thus, by (14.21), we get

$$f(z+1) = bzf(z). \tag{14.24}$$

Now we prove that the point $z = 0$ is a simple pole of f. By (14.24) and by the fact that f is analytic at the point $z = 1$, we obtain

$$\lim_{z\to 0}(zf(z)) = \lim_{z\to 0}\frac{f(z+1)}{b} = \frac{f(1)}{b}. \tag{14.25}$$

By condition (ii), $\frac{f(1)}{b}$ is a nonzero complex constant. But then (14.25) shows that 0 is a simple pole of f.

We claim that the point $z = -k$ ($k \in Z_+^*$) is also a simple pole of f. Repeated application of (14.24) yields on K

$$f(z+k) = b^k(z+k-1)(z+k-2)(z+k-3)\cdots(z+1)zf(z)$$

or

$$f(z) = \frac{f(z+k)}{b^k(z+k-1)(z+k-2)(z+k-3)\cdots(z+1)z}. \tag{14.26}$$

From (14.26) and (14.25) there results

$$\begin{aligned}\lim_{z\to -k}((z+k)f(z)) &= (-1)^k\frac{1}{b^k k!}\lim_{z\to -k}((z+k)f(z+k))\\ &= (-1)^k\frac{1}{b^k k!}\lim_{z\to 0}(zf(z))\\ &= (-1)^k\frac{f(1)}{b^{k+1}k!} \neq 0.\end{aligned}$$

Thus, the point $z = -k$ is a simple pole of f. Let

$$g(z) = \frac{f(z)}{\Gamma(z)}. \tag{14.27}$$

Since f and Γ are analytic and $\Gamma(z) \neq 0$ on K by (14.27), g is analytic on K. The set of all poles of f and Γ is the same: $S = \{0, -1, -2, \ldots, -k, \ldots\}$. Furthermore, each pole of f and Γ is a simple pole. Therefore, by Riemann's theorem on removable singular points, g is analytic at each point belonging to S, so g is an entire function of z.

Substituting $f(z) = \Gamma(z)g(z)$ into (14.19) yields

$$\prod_{k=0}^{n-1} g\left(\frac{z+k}{n}\right) = g(z) \tag{14.28}$$

for all complex z.

Now $g(z)$ is never zero on K by condition (ii). Furthermore, since each of $\frac{1}{z+k}$ $(k = 0, 1, 2, \dots)$ cancelled in the numerator and the denominator, the entire function $g(z)$ is never zero. Hence there exists an entire function ϕ of z such that

$$g(z) = \exp(\phi(z)) \tag{14.29}$$

holds for all complex z.

By (14.28) and (14.29), we have, for all $z \in \mathbb{C}$,

$$\exp\left(\sum_{k=0}^{n-1} \phi\left(\frac{z+k}{n}\right) - \phi(z)\right) = 1. \tag{14.30}$$

Differentiating (14.30) twice with respect to z gives

$$\phi''(z) = \frac{1}{n^2} \sum_{k=0}^{n-1} \phi''\left(\frac{z+k}{n}\right).$$

Set $m = \mathrm{Max}(|\phi''(z)|)$ for $|z| \le 1$. Since $\left|\frac{z+k}{n}\right| \le \frac{1}{n}(|z| + k)$, $\frac{1}{n}(1+k) \le \frac{1}{n}(1 + (n-1)) = 1$ $(k = 0, 1, 2, \dots, n-1)$ for $|z| \le 1$, we obtain

$$\left|\phi''\left(\frac{z+k}{n}\right)\right| \le m \quad (k = 0, 1, 2, \dots, n-1).$$

Thus we obtain

$$\begin{aligned} |\phi''(z)| &= \frac{1}{n^2}\left|\sum_{k=0}^{n-1} \phi''\left(\frac{z+k}{n}\right)\right| \\ &\le \frac{1}{n^2} \sum_{k=0}^{n-1} \left|\phi''\left(\frac{z+k}{n}\right)\right| \\ &\le \frac{1}{n^2} \cdot nm = \frac{m}{n}. \end{aligned}$$

Hence we obtain $(0 \le)\ m \le \frac{m}{n}$ $(n > 1)$ and so $m = 0$. Therefore we have $\phi''(z) = 0$ for $|z| \le 1$. By the Identity Theorem, we get for all complex z

$$\phi(z) = az + b, \tag{14.31}$$

where a, b are complex constants.

Substituting (14.31) back into (14.30) and using $\sum_{k=0}^{n-1} k = \frac{1}{2}n(n-1)(n > 1)$ yields

$$\exp\left((n-1)\left(\frac{a}{2}+b\right)\right) = 1;$$

that is, $b = -\frac{a}{2} + \frac{2m\pi i}{n-1}$, where m is an integer.

Hence,

$$\phi(z) = a\left(z - \frac{1}{2}\right) + \frac{2m\pi i}{n-1}. \tag{14.32}$$

From (14.32), (14.29), and (14.27), we get on K the result sought,

$$f(z) = \exp\left(a\left(z-\frac{1}{2}\right) + \frac{2m\pi i}{n-1}\right)\Gamma(z).$$

This proves the theorem. □

Remark. Since the left-hand side of the functional equation (14.28) is a product, we may think of taking the logarithms on both sides. However, since the complex logarithmic function is multiple valued, we need (14.29).

Corollary 14.5a. *Under the additional conditions that* $f(1) = 1$ *and* $f'(1) = \Gamma'(1) = -\gamma$, *where* γ *denotes Euler's constant, the only function satisfying* (14.7a) *is* $bf(z) = \Gamma(z)$.

Corollary 14.5b. *Under the additional conditions that* $f\left(\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ *and* $f'\left(\frac{1}{2}\right) = \Gamma'\left(\frac{1}{2}\right) = -\sqrt{\pi}(\gamma + 2\ \mathrm{Log}\ 2)$, *where* γ *denotes Euler's constant, the function satisfying* (i), (ii), *and* (iii) *is* $f(z) = \Gamma(z)$.

Euler's gamma function satisfies on K

$$\Gamma(z)\Gamma(1-z) = \frac{z}{\sin(\pi z)}. \tag{14.8a}$$

The formula above is called Euler's functional equation. As an application of Corollary 14.5b, we shall prove (14.8a) later.

Proof of Euler's functional equation (14.8a). Let

$$\psi(z) = \frac{\pi}{\Gamma(1-z)\sin(\pi z)}$$

and

$$p(z) = \Gamma(1-z)\sin(\pi z).$$

We shall show first that p is an entire function of z. Since it is obvious that p is analytic on the complement of Z_+^*, we shall prove that p is analytic at $z = m$ with $p(m) \neq 0$, where m is an arbitrary positive integer.

By using the fact that the residue of Γ at $z = -k \in S$ is $(-1)^k \frac{1}{k!}$, we obtain

$$\begin{aligned}
\lim_{z\to m} p(z) &= \lim_{z\to m} (\Gamma(-z+1)\sin(\pi z)) \\
&= \lim_{z\to m} ((-z\Gamma(-z))\sin(\pi z)) \\
&= -m \lim_{z\to m} \left(\frac{\sin(\pi z)}{-z+m}\right) \lim_{z\to m} ((-z+m)\Gamma(-z)) \\
&= (-m)(-1)^{m-1}\pi \lim_{z\to -m} ((z+m)\Gamma(z)) \\
&= (-m)(-1)^{m-1}\pi(-1)^m \frac{1}{m!} \\
&= \frac{\pi}{(m-1)!} \neq 0.
\end{aligned}$$

Hence, by Riemann's theorem on removable singularity, p is analytic at $z = m$ with $p(m) \neq 0$. Thus p is an entire function, and the set of all zeros of p is S. Hence, ψ is meromorphic in $|z| < +\infty$ and is analytic on K. Furthermore, ψ never vanishes on K, so ψ satisfies conditions (i) and (ii).

Next we shall prove that $\psi(z)$ satisfies condition (iii) with $n = 2$. By Legendre's duplication formula, we obtain on K

$$\begin{aligned}
2^{z-\frac{1}{2}}\psi\left(\frac{z}{2}\right)\psi\left(\frac{z+1}{2}\right) &= 2^{z-\frac{1}{2}} \frac{\pi^2}{\Gamma\left(\frac{2-z}{2}\right)\Gamma\left(\frac{1-z}{2}\right)\sin\left(\frac{\pi z}{2}\right)\sin\left(\frac{\pi(z+1)}{2}\right)} \\
&= \frac{\sqrt{2}\pi^2}{2^{(1-z)-1}\Gamma\left(\frac{1-z}{2}\right)\Gamma\left(\frac{(1-z)+1}{2}\right)\left(2\sin\left(\frac{\pi z}{2}\right)\cos\left(\frac{\pi z}{2}\right)\right)} \\
&= \frac{\sqrt{2}\pi^2}{\sqrt{\pi}\Gamma(1-z)\sin(\pi z)} \\
&= \sqrt{2\pi}\frac{\pi}{\Gamma(1-z)\sin(\pi z)} \\
&= (2\pi)^{\frac{2-1}{2}}\psi(z).
\end{aligned}$$

Hence $\psi(z)$ satisfies the functional equation

$$2^{z-\frac{1}{2}}\psi\left(\frac{z}{2}\right)\psi\left(\frac{z+1}{2}\right) = (2\pi)^{\frac{2-1}{2}}\psi(z) \tag{14.14a}$$

on K.

Furthermore, by $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\sin\left(\frac{\pi}{2}\right) = 1$, we get $\psi\left(\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)$ and $\psi'\left(\frac{1}{2}\right) = \Gamma'\left(\frac{1}{2}\right)$. So, by Corollary 14.5b, we have $\psi(z) = \Gamma(z)$ on K and (14.8a) holds. □

14.2 Beta Function

Definition. The function $\beta : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}$ (a real-valued function defined in the first quadrant), monotonic in the first variable x for a fixed second variable y (enough to assume the monotonicity for large x) and fulfilling the conditions

$$\beta(x+1, y) = \frac{x}{x+y}\beta(x, y), \quad x, y \in (0, \infty), \tag{14.33}$$

$$\beta(1, y) = \frac{1}{y}, \tag{14.33a}$$

is called *Euler's beta function.*

Theorem 14.6. (Kuczma [554, 549]). *We have*

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \textit{for } x, y > 0. \tag{14.34}$$

Proof. Keeping y fixed, we have (by virtue of Theorem 5.4 in [554])

$$\begin{aligned}
\beta(x, y) &= \frac{1}{y}\prod_{n=0}^{\infty}\frac{1+n}{1+n+y}\cdot\frac{x+n+y}{x+n} \\
&= \lim_{n\to\infty}\frac{n!(x+y)(x+y+1)\cdots(x+y+n-1)}{y(y+1)\cdots(y+n)x(x+1)\cdots(x+n-1)} \\
&= \lim_{n\to\infty}\frac{x+n}{x+y+n}\cdot\frac{n!n^x}{x(x+1)\cdots(x+n)}\cdot\frac{n!n^y}{y(y+1)\cdots(y+n)} \\
&\quad\cdot\frac{(x+y)(x+y+1)\cdots(x+y+n)}{n!n^{x+y}},
\end{aligned}$$

and the theorem follows in view of (14.3). □

Theorem 14.6a. [76]. *The function* $f : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}$ *satisfying* (14.33) *and* (14.33a) *and logarithmically convex or decreasing for large* x *is the beta function* β *given by* (14.34).

Proof. One can, by using (14.33), write

$$f(x+n; y) = \frac{(x)_n}{(x+y)_n}f(x, y), \quad \text{for } n \in Z_+^*,\ x \in]0, 1[, \tag{14.35}$$

and

$$f(n+1, y) = \frac{n!}{(y)_{n+1}}. \tag{14.35a}$$

Since $f(x, y)$ is supposed to be decreasing for large x,

$$f(n, y) \geq f(x+n, y) \geq f(n+1, y) \quad \text{for } x \in]0, 1].$$

Because of (14.35) and (14.35a), we have

$$\frac{(n-1)!}{(y)_n} \geq \frac{(x)_n}{(x+y)_n} f(x,y) \geq \frac{n!}{(y)_{n+1}}$$

or

$$\frac{(n-1)!(x+y)_n}{(y)_n(x)_n} \geq f(x,y) \geq \frac{n!(x+y)_n}{(x)_n(y)_{n+1}}.$$

This inequality is valid for all n. Replacing n by $(n+1)$, one finds

$$\frac{n!(x+y)_{n+1}}{(y)_{n+1}(x)_{n+1}} \geq f(x,y) \geq \frac{n!(x+y)_{n+1}}{(y)_{n+1}(x)_{n+1}} \frac{x+n}{x+y+n},$$

and since $\frac{x+n}{x+y+n} \to 1$ as $n \to \infty$, we have

$$\begin{aligned} f(x,y) &= \lim_{n\to\infty} \frac{n!(x+y)_{n+1}}{(y)_{n+1}(x)_{n+1}} \\ &= \frac{\lim\limits_{n\to\infty} \frac{n!n^x}{(x)_{n+1}} \lim\limits_{n\to\infty} \frac{n!n^y}{(y)_{n+1}}}{\lim\limits_{n\to\infty} \frac{n!n^{x+y}}{(x+y)_{n+1}}} \\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \beta(x,y). \end{aligned}$$

□

Remark. β given by (14.34) is symmetric. Differentiating partially with respect to x, we obtain

$$\beta_x(x,y) = \frac{\Gamma(x+y)\Gamma'(x) - \Gamma(x)\Gamma'(x+y)}{\Gamma^2(x+y)}\Gamma(y).$$

Since $\frac{\Gamma'(x)}{\Gamma(x)}$ is a decreasing function for $x > 0$, we have

$$\frac{\Gamma'(x)}{\Gamma(x)} - \frac{\Gamma'(x+y)}{\Gamma(x+y)} < 0 \quad (y > 0),$$

and since $\Gamma(t) > 0$ for $t > 0$, $\beta_x(x,y) < 0$.

In the same way, $\beta(x,y)$ is decreasing for $y > 0$. $x = 1$ in (14.34) yields (14.33a).

14.2.1 Integral Representation of β

One can represent β by an integral

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad \text{for } x,y > 0. \tag{14.36}$$

Legendre called this integral *Euler's integral of the first kind* or the *beta function.*

Theorem 14.7. [76]. *β given by* (14.36) *is indeed* (14.34).

Proof. Replace x by $x+1$ in (14.36) to have

$$\beta(x+1,y)=\int_0^1 (1-t)^{x+y-1}\cdot\left(\frac{t}{1-t}\right)^x dt.$$

Set

$$k=\int_\epsilon^{1-n}(1-t)^{x+y-1}\left(\frac{t}{1-t}\right)^x dt,$$

which by integration by parts gives

$$k=-\frac{(1-t)^{x+y}}{x+y}\cdot\left(\frac{t}{1-t}\right)^x\Bigg|_\epsilon^{1-\eta}+\int_\epsilon^{1-\eta}\frac{(1-t)^{x+y}}{x+y}\cdot x\left(\frac{1}{1-t}-1\right)^{x-1}\cdot\frac{1}{(1-t)^2}dt.$$

By allowing $\epsilon,\eta\to 0$, we obtain

$$\beta(x+1,y)=\frac{x}{x+y}\int_0^1(1-t)^{y-1}\cdot t^{x-1}dt=\frac{x}{x+y}\beta(x,y),$$

which is (14.33).

Set $x=1$ in (14.36) to have

$$\beta(1,y)=\int_0^1(1-t)^{y-1}d=\frac{1}{y},$$

which is (14.33a).

The integral is a decreasing function of x for $x>0$ since $t\in\,]0,1[$. Hence, by Theorem 14.6 or 14.6a, β is given by (14.34). This completes the proof.

Now I insert the nice, interesting writeup of gamma and beta functions from Dragomir, Agarwal, and Barnett [228] (see also [77]).

Introduction

In the eighteenth century, L. Euler (1707–1783) [261] concerned himself with the problem of interpolating between the numbers

$$n!=\int_0^\infty e^{-t}t^n dt,\quad n=0,1,2,\ldots, \tag{14.4}$$

with noninteger values of n. This problem led Euler, in 1729, to the now famous *gamma function*, a generalization of the factorial function that gives meaning to $x!$, where x is any positive number.

The notation $\Gamma(x)$ is not due to Euler, however, but was introduced in 1809 by A. Legendre (1752–1833) [604], who was also responsible for the *duplication formula* (14.3) for the gamma function.

Nearly 150 years after Euler discovered it, the theory concerning the gamma function was greatly expanded by means of the theory of entire functions developed by K. Weierstrass (1815–1897).

The gamma function has several equivalent definitions, most of which are due to Euler. To begin with, we define

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)(x+2)\cdots(x+n)}. \tag{14.3}$$

If x is not zero or a negative integer, it can be shown that the limit (14.3) exists. It is apparent, however, that $\Gamma(x)$ cannot be defined at $x = 0, -1, -2, \ldots$ since the limit becomes infinite for any of these values.

By setting $x = 1$ in (14.3), we see that

$$\Gamma(1) = 1.$$

Other values of Γ are not so easily obtained, but the substitution of $x + 1$ for x in (14.3) leads to the *recurrence formula*

$$\Gamma(x+1) = x\Gamma(x). \tag{14.2}$$

Equation (14.2) is the basic functional relation for the gamma function; it is in the form of a difference equation.

A direct connection between the gamma function and factorials can be obtained from (14.2) as

$$\Gamma(n+1) = n!, \quad n = 0, 1, 2, \ldots. \tag{14.1}$$

Integral Representation

The gamma function rarely appears in the form (14.3) in applications. Instead, it most often arises in the evaluation of certain integrals; for example, Euler was able to show that

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \quad x > 0. \tag{14.4}$$

This *integral representation* of $\Gamma(x)$ is the most common way in which the gamma function is now defined. Lastly, we note that (14.4) is an improper integral due to the infinite limit of integration and because the factor t^{x-1} becomes infinite if $t = 0$ for values of x in the interval $0 < x < 1$. Nonetheless, the integral (14.4) is *uniformly convergent* for all $a \le x \le b$, where $0 < a \le b < \infty$.

A consequence of the uniform convergence of the defining integral for $\Gamma(x)$ is that we may differentiate the function under the integral sign to obtain [77, p. 54]

$$\Gamma'(x) = \int_0^\infty e^{-t} t^{x-1} \log t \, dt, \quad x > 0, \tag{14.4a}$$

and

$$\Gamma''(x) = \int_0^\infty e^{-t} t^{x-1} (\log t)^2 dt, \quad x > 0.$$

The integrand in (14.4a) is positive over the entire interval of integration, and thus it follows that $\Gamma''(x) > 0$; i.e., Γ is convex on $(0, \infty)$.

In addition to (14.4), there are a variety of other integral representations of $\Gamma(x)$, most of which can be derived from that one by simple changes of variable (Andrews [77, p. 57]),

$$\Gamma(x) = \int_0^1 \left(\log \frac{1}{u} \right)^{x-1} du, \quad x > 0, \tag{14.5}$$

and

$$\frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)} = \int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta \, d\theta, \quad x, y > 0.$$

By setting $x = y = \frac{1}{2}$ in the integral above, we deduce the special value

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

14.2.2 Other Special formulas

A formula involving gamma functions that is somewhat comparable to the double-angle formulas for trigonometric functions is the *Legendre duplication formula*

$$2^{2x-1}\Gamma(x)\Gamma\left(x + \frac{1}{2}\right) = \sqrt{\pi}\Gamma(2x), \quad x > 0. \tag{14.13}$$

An especially important case of (14.13) occurs when $x = n$ $(n = 0, 1, 2, \ldots)$ [77, p. 55],

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{2^{2n} n!}\sqrt{\pi}, \quad n = 0, 1, 2, \ldots.$$

Although it was originally found by Schlömlich in 1844, thirty-two years before Weierstrass's famous work on entire functions, Weierstrass is usually credited with the *infinite product* definition of the gamma function,

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{\frac{-x}{n}}, \tag{14.9a}$$

where γ is the *Euler-Mascheroni constant* defined by

$$\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \log n \right] = 0.577215 \ldots.$$

An important identity involving the gamma function and sine function can now be derived by using (14.9a). We obtain the identity

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad (x \text{ noninteger}). \tag{14.8}$$

The following properties of the gamma function also hold [77, p. 63–65]:

$$\Gamma(x) = s^x \int_0^\infty e^{-st} t^{x-1} dt, \qquad x, s > 0;$$

$$\Gamma(x) \int_{-\infty}^\infty \exp(xt - e^t) dt, \qquad x > 0;$$

$$\Gamma(x) = \int_1^\infty e^{-t} t^{x-1} dt + \sum_{n=1}^\infty \frac{(-1)^n}{n!(x+n)}, \qquad x > 0; \tag{14.4}$$

$$\Gamma(x) = (\log b)^x \int_0^\infty t^{x-1} b^{-t} dt, \qquad x > 0,\ b > 1;$$

$$\Gamma(x) = \Gamma'(x+1) - x\Gamma'(x), \qquad x > 0;$$

$$\Gamma(x) = \int_0^\infty e^{-t}(t-x)t^{x-1} \log t\, dt, \qquad x > 0;$$

$$\Gamma\left(\frac{1}{2} - n\right) = \frac{(-1)^n 2^{2n-1}(n-1)!\sqrt{\pi}}{(2n-1)!}, \qquad n = 0, 1, 2, \ldots;$$

$$\Gamma\left(\frac{1}{2} + n\right)\Gamma\left(\frac{1}{2} - n\right) = (-1)^n \pi, \qquad n = 0, 1, 2, \ldots;$$

$$\Gamma(3x) = \frac{1}{2\pi} 3^{3x-\frac{1}{2}} \Gamma(x)\Gamma\left(x + \frac{1}{3}\right)\Gamma\left(x + \frac{2}{3}\right), \quad x > 0;$$

$$[\Gamma'(x)]^2 \le \Gamma(x)\Gamma''(x), \qquad x > 0.$$

Beta Function

A useful function of two variables is the *beta function* [77, p. 66], where

$$\beta(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x > 0,\ y > 0. \tag{14.36}$$

The utility of the beta function is often overshadowed by that of the gamma function, perhaps partly because it can be evaluated in terms of the gamma function. However, since it occurs so frequently in practice, a special designation for it is widely accepted.

It is obvious that the beta mapping has the symmetry property

$$\beta(x, y) = \beta(y, x),$$

and the following connection between the beta and gamma functions holds:

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x > 0,\ y > 0. \tag{14.34}$$

The following properties of the beta mapping also hold (see, for example, [77, p. 68–70]):

$$
\begin{aligned}
&\beta(x+1,y)+\beta(x,y+1)=\beta(x,y), && x,y>0;\\
&\beta(x,y+1)=\frac{y}{x}\beta(x+1,y)=\frac{y}{x+y}\beta(x,y), && x,y>0;\\
&\beta(x,x)=2^{1-2x}\beta\left(x,\frac{1}{2}\right), && x>0;\\
&\beta(x,y)\beta(x+y,z)\beta(x+y+z,w)\\
&\qquad=\frac{\Gamma(x)\Gamma(y)\Gamma(z)\Gamma(w)}{\Gamma(x+y+z+w)}, && x,y,z,w>0;\\
&\beta\left(\frac{1+p}{2},\frac{1-p}{2}\right)=\pi\sec\left(\frac{p\pi}{2}\right), && 0<p<1;\\
&\beta(x,y)=\frac{1}{2}\int_0^1\frac{t^{x-1}+t^{y-1}}{(t+1)^{x+y}}dt\\
&\qquad=p^x(1+p)^{x+y}\int_0^1\frac{t^{x-1}(1-t)^{y-1}}{(t+p)^{x+y}}dt && \text{for } x,y,p>0.
\end{aligned}
\tag{14.33}
$$

14.3 Riemann's Zeta Function

The formula

$$\zeta(x)=\sum_{n=1}^{\infty}n^{-x} \tag{14.37}$$

defines the function $\zeta(x)$ for $\operatorname{Re} x>1$. This function can be continued onto the whole complex plane except at $x=1$, where it has a simple pole; thus it is a meromorphic function. $\zeta(x)$, known as *Riemann's zeta function*, satisfies the functional equation

$$\zeta(1-x)=2(2\pi)^{-x}\cos\frac{\pi x}{2}\Gamma(x)\zeta(x), \tag{14.38}$$

where $\Gamma(x)$ is Euler's gamma function. In view of relations (14.16) and (14.8), with x replaced by $(1-x)/2$, (14.38) is equivalent to

$$\zeta(x)\Gamma\left(\frac{x}{2}\right)\pi^{-x/2}=\zeta(1-x)\Gamma\left(\frac{1-x}{2}\right)\pi^{(x-1)/2}.$$

The function $\zeta(x)$ plays an important role in number theory.

Remark. The first satisfactory proof of (14.38) was given by B. Riemann in 1859, but the equation itself (called the *Riemann equation*) together with a motivation was already given by Euler in 1768.

Equation (14.38) is to a great extent characteristic for function (14.37). Namely, we have the following.

Result 14.8. (Hamburger [338], Kuczma [554], Hecke [362]). *Let $\phi(x)$ be analytic in every finite domain except at most a finite number of poles, and suppose that there exist positive constants r and μ such that*

$$|\phi(x)| \leq e^{|x|^{\mu}} \quad \text{for } |x| \geq r.$$

Further, let $\phi(x)$ be expansible for $\operatorname{Re} x > 1$ *in an absolutely convergent Dirichlet series*

$$\phi(x) = \sum_{n=1}^{\infty} a_n n^{-x},$$

and let the function $\psi(x)$ defined by

$$\psi(1-x) = 2(2\pi)^{-x} \cos\left(\frac{1}{2}\pi x\right) \Gamma(x)\phi(x)$$

be expansible in a Dirichlet series

$$\psi(x) = \sum_{n=1}^{\infty} b_n n^{-x}$$

absolutely convergent on a vertical line $\operatorname{Re} x = u_0 > 0$. *Then $\phi(x) = \psi(x) = c\zeta(x)$, where c is a constant.*

14.3.1 The Theta Function

Given functions $\chi, \phi : \mathbb{R}^n \to \mathbb{C}$, define the subspaces $V(\chi, \phi)$ and $W(\chi, \phi)$ of the complex vector space $(\mathbb{R}^n)^{\mathbb{C}}$ of all complex-valued functions defined on $\mathbb{R}^n$. The space $V(\chi, \phi)$ is the span of all functions $u \mapsto \chi(u+v)\phi(u-v)$ with fixed $v \in \mathbb{R}^n$, and $W(\chi, \phi)$ is the span of all functions $v \mapsto \chi(u+v)\phi(u-v)$ with fixed $u \in \mathbb{R}^n$. Note that $V_{(\chi,\phi)} = V_{(\phi,\chi)}$. If $\chi = \phi = \theta$, we simply write $V(\theta)$ for $V(\theta, \theta)$ and $W(\theta)$ for $W(\theta, \theta)$. Identifying the space $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, we can also use this notation for functions χ and ϕ defined on $\mathbb{C}^n$.

Definition. Let $u = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{C}^n$. An analytic trivial theta function is a function $\psi : \mathbb{C}^n \to \mathbb{C}$ that can be written as

$$\psi(u) = c \exp(\beta(u) + \alpha(u)) \quad \text{for } u \in \mathbb{C}^n,$$

where $c \in \mathbb{C}^*$ is a constant, $\alpha : \mathbb{C}^n \to \mathbb{C}$, given by

$$\alpha(u) = \sum_{\nu=1}^{n} a_\nu \xi_\nu, \quad a_\nu \in \mathbb{C}, \text{ for } \nu \in \{1, \ldots, n\},$$

is a $\mathbb{C}$-linear mapping, and $\beta : \mathbb{C}^n \to \mathbb{C}$, given by

$$\beta(u) = \sum_{\nu,\mu=1}^{n} b_{\nu\mu} \xi_\nu \xi_\mu, \quad b_{\nu\mu} \in \mathbb{C}, \text{ for } \nu, \mu \in \{1, \ldots, n\},$$

is a complex-valued quadratic form where $u = (\xi_1, \xi_n, \ldots, \xi_n)$. Alternatively, the analytic trivial theta functions are exactly the functions of type $u \mapsto \exp(P(u))$, where $P : \mathbb{C}^n \to \mathbb{C}$ is an n-variable polynomial with complex coefficients of degree at most 2.

Equations of the Type (14.39)

$$\chi(u+v)\phi(u-v) = f_1(u)g_1(v) + f_2(u)g_2(v), \tag{14.39}$$

where $\chi, \phi, f_1, f_2, g_1, g_2$ are complex-valued functions on $\mathbb{R}^n$, $n \in Z_+^*$ arbitrary, and χ and ϕ are continuous is well known in the theory of theta functions of one complex variable. There they are called addition formulas for theta functions. Apart from degeneracy and some obvious modifications, theta functions are the only continuous functions satisfying an addition formula of type (14.39).

Result 14.9. (Bernoulli [121]). *Suppose $\chi, \phi : \mathbb{R}^n \to \mathbb{C}$ are continuous functions for which there exist functions $f_1, f_2, g_1, g_2 : \mathbb{R}^n \to \mathbb{C}$ so that* (14.39) *is valid for all $u, v \in \mathbb{R}^n$. If $\chi \not\equiv 0$ and $\phi \not\equiv 0$, then we have the following possibilities for the pair χ, ϕ:*

$$\begin{aligned}
&\text{(a1)} && \chi(u) = c_1 \exp(\beta(u) + \alpha_1(u))(\zeta_1 + \alpha(u)),\\
&&& \phi(u) = c_2 \exp(\beta(u) + \alpha_2(u)),\\
&\text{(a2)} && \chi(u) = c_1 \exp(\beta(u) + \alpha_1(u)) \sin(\zeta_1 + \alpha(u)),\\
&&& \phi(u) = c_2 \exp(\beta(u) + \alpha_2(u)),\\
&\text{(b1)} && \chi(u) = c_1 \exp(\beta(u) + \alpha_1(u))(\zeta_1 + \alpha(u)),\\
&&& \phi(u) = c_2 \exp(\beta(u) + \alpha_2(u))(\zeta_2 + \alpha(u)),\\
&\text{(b2)} && \chi(u) = c_1 \exp(\beta(u) + \alpha_1(u)) \sin(\zeta_1 + \alpha(u)),\\
&&& \phi(u) = c_2 \exp(\beta(u) + \alpha_2(u)) \sin(\zeta_2 + \alpha(u)),\\
&\text{(b3)} && \chi(u) = c_1 \exp(\beta(u) + \alpha_1(u))\sigma(\zeta_1 + \alpha(u)),\\
&&& \phi(u) = c_2 \exp(\beta(u) + \alpha_2(u))\sigma(\zeta_2 + \alpha(u)).
\end{aligned}$$

Here $c_1, c_2 \in \mathbb{C}^*$, $\zeta_1, \zeta_2 \in \mathbb{C}$ are constants, $\alpha_1, \alpha_2, \alpha : \mathbb{R}^n \to \mathbb{C}$ are $\mathbb{R}$-linear mappings, $\beta : \mathbb{R}^n \to \mathbb{C}$ is a complex-valued quadratic form, and σ is a Weierstrass σ-function.

In cases (a1) *and* (a2), *the roles of χ and ϕ may be reversed. Conversely, each of the listed function pairs satisfies the functional equation* (14.39).

14.4 Singular Functions

A nonconstant $\varphi : [0, 1] \to [0, 1]$ is called (strictly) singular if it is continuous and (strictly) increasing with $\varphi'(x) = 0$ a.e. Singular functions are not absolutely continuous. We consider now three singular functions associated with Cantor, Minkowski, and de Rham [412].

14.4.1 Cantor-Lebesgue Singular Function

Cantor's ternary set W is obtained by removing from $[0,1]$ the open intervals $I_{1,1} = \left(\frac{1}{3}, \frac{2}{3}\right)$, $I_{2,1} = \left(\frac{1}{9}, \frac{2}{9}\right)$, $I_{2,2} = \left(\frac{7}{9}, \frac{8}{9}\right)$, $I_{3,1} = \left(\frac{1}{27}, \frac{2}{27}\right), \ldots$ and

$$W = \left\{ \sum_{n=1}^{\infty} \frac{2x_n}{3^n};\ x_n \in \{0,1\} \right\}.$$

The Cantor function $C : [0,1] \to [0,1]$ is defined by

$$C\left(\sum_{n=1}^{\infty} \frac{2x_n}{3^n} \right) := \sum_{n=1}^{\infty} \frac{x_n}{2^n} \quad \text{and} \quad C(x) := \frac{2k-1}{2^n}, \quad \text{for } x \in I_{n,k}.$$

C was introduced in 1884 by G. Cantor [141], who had already stated its singularity. The function C later appears in H. Lebesgue's classic book on integration [602] and is referred to by some authors as Lebesgue's singular function.

C satisfies

$$\int_0^1 C'(x)dx = 0, \quad \int_0^1 C(x)dx = \frac{1}{2},$$
$$\int_0^1 \sqrt{1 + (C'(x))^2}dx = 1, \quad \text{arclength (graph } C) = 2.$$

Sierpinski [739] showed that C satisfies the functional equations

$$f\left(\frac{x}{3}\right) = \frac{1}{2} f(x), \quad f\left(\frac{x+1}{3}\right) = \frac{1}{2},$$
$$f\left(\frac{x+2}{3}\right) = \frac{1}{2} f(x) + \frac{1}{2}, \quad x \in [0,1], \tag{14.40}$$

and proved the following characterization: *Assume that* $f : [0,1] \to \mathbb{R}$ *is bounded and satisfies* (14.40)*. Then* $f = C$. (See also [554].)

14.4.2 Minkowski's Function

H. Minkowski introduced the function $K : [0,1] \to [0,1]$ in his *Geometrie der Zahlen* [636]. The aim was to get a geometrical picture of Lagrange's criterion for real quadratic irrationals. This function was in fact a bit mysterious, and even now some of its properties are hard to obtain. K is defined as follows. $K(0) := 0$, $K(1) := 1$. If $\frac{p}{q}$ and $\frac{p'}{q'}$ are two adjacent Farey fractions ($pq' - qp' = 1$), then

$$K\left(\frac{p+p'}{q+q'}\right) := \frac{1}{2}\left[K\left(\frac{p}{q}\right) + K\left(\frac{p'}{q'}\right) \right].$$

For example,

$$K\left(\frac{1}{2}\right) = \frac{1}{2}\left[K(0) + K(1)\right] = \frac{1}{2},$$
$$K\left(\frac{1}{4}\right) = \frac{1}{2}\left[K(0) + K\left(\frac{1}{3}\right)\right] = \frac{1}{8},$$
$$K\left(\frac{3}{4}\right) = \frac{1}{2}\left[K\left(\frac{2}{3}\right) + K(1)\right] = \frac{7}{8}.$$

The remaining values are obtained by continuous extension.

K is a strictly singular function. The connection with Lagrange's criterion "x is a quadratic irrational iff its simple continued fraction is periodic" is as follows. Let $x \in (0, 1)$ be given by a simple continued fraction $[0, a_1, a_2, \dots]$, $a_\nu \in X$. Then

$$K(x) = \sum_{n=1}^{\infty} (-1)^{n-1} 2^{1-(a_1+a_2+\cdots+a_n)}.$$

Thus we have the equivalent statements

(i) $[0, a_1, a_2, \dots]$ finite, (ii) x rational, (iii) $K(x)$ dyadic rational

and

(i) $[0, a_1, a_2, \dots]$ periodic, (ii) x quadratic irrational, (iii) $K(x)$ nondyadic rational.

A first hint on functional equations for K can be found in the 1956 paper of G. de Rham [702]. There he considers the system

$$f\left(\frac{x}{2}\right) = \frac{f(x)}{1 + f(x)}, \quad f\left(\frac{x+1}{2}\right) = f(x) + 1, \quad x \in [0, 1), \tag{14.41}$$

which has a unique positive solution m; m is continuous and strictly increasing with $\lim_{x\to 1} m(x) = \infty$.

K satisfies

$$2K\left(\frac{x}{x+1}\right) = K(x), \quad 2K\left(\frac{1}{2-x}\right) = K(x) + 1, \quad x \in [0, 1]. \tag{14.42}$$

The most comprehensive facts on functional equations and characterizations for K has been given by R. Girgensohn in [313]. He showed that *K satisfies the functional equations*

$$f\left(\frac{x}{x+1}\right) = \frac{1}{2} f(x), \qquad x \in [0, 1], \tag{14.43}$$
$$f(x) + f(1 - x) = 1, \qquad x \in [0, 1], \tag{14.44}$$
$$f\left(\frac{1}{1+x}\right) = 1 - \frac{1}{2} f(x), \quad x \in [0, 1], \tag{14.45}$$
$$f\left(\frac{1}{2-x}\right) = \frac{1}{2} + \frac{1}{2} f(x), \quad x \in [0, 1], \tag{14.46}$$

and proved the following characterizations:

Assume that $f : [0,1] \to \mathbb{R}$ *is bounded and satisfies* (14.43) *and* (14.45). *Then* $f = K$.

Assume that $f : [0,1] \to \mathbb{R}$ *is bounded and satisfies* (14.43) *and* (14.46). *Then* $f = K$.

Moreover, he showed that (14.43) and (14.45) together imply (14.44) and (14.46) but that there are unbounded solutions of (14.43) and (14.46) that do not satisfy (14.44).

In proving the functional equations and the characterizations, continued fractions are essentially used.

14.4.3 De Rham's Function

G. de Rham proved in [702] the following theorem: *Fix* $\alpha \in (0,1)$. *Then there is exactly one bounded* $R_\alpha : [0,1] \to \mathbb{R}$ *that satisfies the functional equations*

$$f\left(\frac{x}{2}\right) = \alpha f(x), \quad f\left(\frac{x+1}{2}\right) = (1-\alpha) f(x) + \alpha, \quad x \in [0,1]. \tag{14.47}$$

R_α *is strictly singular for* $\alpha \neq \frac{1}{2}$, $R_{1/2}(x) = x$. R_α *is given by*

$$R_\alpha(0) = 0, \quad R_\alpha\left(\sum_{n=0}^{\infty} 2^{-\gamma_n}\right) = \sum_{n=0}^{\infty} \left(\frac{1-\alpha}{\alpha}\right)^n \alpha^{\gamma_n},$$
$$\gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n \in \mathbb{N}.$$

The main property of R_α, its strict singularity, was proved by de Rham essentially using the defining functional equations, not its representation given above. He further provided a probabilistic interpretation of his system (14.47). De Rham was aware of the fact that R_α had already been investigated by E. Cesàro in [148], etc., but without the use of functional equations.

It is amazing that the strictly singular R_α has applications in real life. Let $f(x)$ be the probability of eventual success under bold play for a gambler with initial fortune x and α the probability of winning a turn. Then f satisfies (14.47), and hence $f = R_\alpha$.

Furthermore, R_α was used by B. Bernstein, T. Erber, and M. Karamolengos [126] to describe certain aspects of the behaviour of plastics.

L. Takács in [800] introduced the functions $T_\rho : [0,1] \to \mathbb{R}$, $\rho \in (0,\infty)$, given by

$$T_\rho(0) := 0, \quad T_\rho\left(\sum_{k=0}^{\infty} 2^{-\nu_k}\right) := \sum_{k=0}^{\infty} \rho^k (1+\rho)^{-\nu_k},$$
$$\nu_0 < \nu_1 < \nu_2 < \cdots < \nu_k \in \mathbb{N}.$$

He proved that T_ρ is strictly singular for $\rho \neq 1$ using the representation given above. It is easy to check that T_ρ satisfies the functional equations

$$
\begin{aligned}
f\left(\frac{x}{2}\right) &= \left(\frac{1}{1+\rho}\right) f(x), \\
f\left(\frac{x+1}{2}\right) &= \frac{\rho}{1+\rho} f(x) + \frac{1}{1+\rho}, \quad x \in [0,1],
\end{aligned}
\tag{14.48}
$$

and is its unique bounded solution. In fact, $T_\rho = R_{1/(1+\rho)}$ for $\rho \in (0,\infty)$. We add the observation that $\tilde{R}_\alpha = R_\alpha - \alpha$ satisfies the single replicativity equation $f(\frac{x}{2}) + f(\frac{x+1}{2}) = f(x)$ on $[0,1]$.

15

Miscellaneous Equations

This chapter is devoted to solutions by the methods of determinants and means and revisits logarithmic, trigonometric, and polynomial (mean value theorem) functions and inner product spaces.

We start off with a method known as the method of determinants, due to E. Vincze, and solve many of the equations solved before.

15.1 A General Method: Method of Determinants

E. Vincze [818] has introduced a method of solving functional equations applicable to a rather large class of equations. It is based essentially on the notion of linear dependence and its expression in terms of identities involving determinants.

Consider the determinant

$$\begin{vmatrix} F_1(z_1) & F_2(z_1) & \cdots & F_n(z_1) \\ F_1(z_2) & F_2(z_2) & \cdots & F_n(z_2) \\ \cdots & \cdots & \cdots & \cdots \\ F_1(z_n) & F_2(z_n) & \cdots & F_n(z_n) \end{vmatrix}$$

consisting of complex or real functions defined in some arbitrary subset S of $\mathbb{R}$ or $\mathbb{C}$. We can abbreviate this by

$$\Delta[F_1(z_1), F_2(z_2), \ldots, F_n(z_n)] = \Delta[F_1, F_2, \ldots, F_n].$$

Some Properties

Suppose that the $n(k+1)$ functions $F_i : S \to \mathbb{C}$ ($i = 1$ to $n(k+1)$) satisfy the identity

$$\sum_{\nu=1}^{k} \Delta[F_{\nu n+1}(z_1), F_{\nu n+2}(z_2), \ldots, F_{\nu n+n}(z_n)] = 0 \tag{15.1}$$

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for all $z_1, z_2, \ldots, z_n$ in S. Then

$$\sum_{\nu=1}^{k} \Delta[F_{\nu n+1}(z_1), F_{\nu n+2}(z_2), \ldots, F_{\nu n+n}(z_n), F_0(z_{n+1})] = 0 \tag{15.1a}$$

also holds for all values of $z_1, z_2, \ldots, z_n, z_{n+1}$ in S for an arbitrary F_0. Vincze calls the passage from the first equation to the second "extension or expansion by $F_0(z)$". It is the basic tool in his discussion (see also [12]).

Let us further note that $F_1, F_2, \ldots, F_n$ from S to $\mathbb{C}$ are linearly independent in $S \subseteq \mathbb{C}$; that is, there exists c_1 to c_n not all zero such that

$$\sum_{i=1}^{n} c_i f_i(z) = 0$$

if and only if

$$\Delta[F_1(z_1), F_2(z_2), \ldots, F_n(z_n)] = 0 \tag{15.2}$$

for all $z_1, z_2, \ldots, z_n$ in S [12, 818].

Remarks. If the functions $F_1, \ldots, F_n$ are linearly dependent, one of the following holds:

(i) $F_n(z) = 0$ for all $z \in \mathbb{C}$.
(ii) $F_k(z) = b_{k+1}F_{k+1}(z) + \cdots + b_nF_n(z)$, where the b_k $(k = 2, \ldots, n)$ denote complex constants.

We note that in equation (15.1) we can introduce a parameter w, and it follows from

$$\Delta[F_1(z_1 + w), F_2(z_2), \ldots, F_n(z_n)] = 0$$

that

$$b_1(w)F_1(z + w) + \cdots + b_n(w)F_n(z) = 0$$

for all $z, w \in \mathbb{C}$.

From this it follows as above that at least one of the following equations holds:

(i) $F_n(z) = 0$ for all $z \in \mathbb{C}$.
(ii) $F_k(z) = b_{k+1}F_{k+1}(z) + \cdots + b_nF_n(z)$ $(k = 2, \ldots, n)$.
(iii) $F_1(z + w) = b_2(w)F_2(z) + \cdots + b_n(w)F_n(z)$, where b_k $(k = 3, \ldots, n)$ denote complex constants and the $b_k(w)$ $(k = 2, \ldots, n)$ are complex functions.

These remarks will be exploited in what follows.

Vincze illustrates his method with a number of examples. We will illustrate a couple of them. The first is one of the simplest.

Example 1. It is desired to find functions F, G, H, K, N satisfying the equation

$$F(z_1 \circ z_2) = G(z_1) + H(z_2) + K(z_1)N(z_2) \tag{15.3}$$

for $z_1, z_2, z_1 \circ z_2$ belonging to a set S (see Chapter 1, (1.94a,b)) that is an Abelian semigroup with respect to the operation "$\circ$". In particular, it is assumed that there is a point $a \in S$ such that the equation $a \circ z = c$ has at least one solution for every given c.

Since the left side of (15.3) is symmetric in z_1 and z_2,

$$G(z_1) + H(z_2) + K(z_1)N(z_2) = G(z_2) + H(z_1) + K(z_2)N(z_1),$$

or in determinant notation

$$\Delta(G, 1) + \Delta(1, H) + \Delta(K, N) = 0. \tag{15.4}$$

Now we "extend by 1" to obtain

$$\Delta(G, 1, 1) + \Delta(1, H, 1) + \Delta(K, N, 1) = 0$$

and then

$$\Delta(K, N, 1) = 0;$$

that is, the functions $K,\ N,\ 1$ are linearly dependent. Then, by the remark, there are two possibilities, either

$$(1)\ N(z) \equiv \beta_1$$

or

$$(2)\ K(z) = \beta_1 + \beta_2 N(z).$$

We shall discuss case (1). Substituting in (15.4), we obtain

$$\begin{aligned}\Delta(G, 1) + \Delta(1, H) + \Delta(K, \beta_1) &= \Delta(G, 1) - \Delta(H, 1) + \Delta(\beta_1 K, 1)\\ &= \Delta(G - H + \beta_1 K, 1) = 0.\end{aligned}$$

Thus there is a constant β_2 such that

$$G(z) - H(z) + \beta_1 K(z) = 2\beta_2.$$

Combining, we get from (15.3)

$$F(z_1 \circ z_2) = [G(z_1) + \beta_1 K(z_1) - \beta_2] + [G(z_2) + \beta_1 K(z_2) - \beta_2],$$

which is the Pexider equation (PA), (PL) considered in Chapter 1. Thus the final solution in case (1) is given by

$$\begin{cases} F(z) = A(z) + 2\beta_3, \quad G(z) = A(z) - \beta_1 K(z) + \beta_2 + \beta_3, \\ H(z) = A(z) + \beta_3 - \beta_2, \quad K(z) \text{ arbitrary}, \quad N(z) = \beta_1, \end{cases} \tag{15.4a}$$

where $A(u \circ v) = A(u) + A(v)$.

Case (2) is more complicated and also involves the solution of another Cauchy equation,

$$E(z_1 \circ z_2) = E(z_1)E(z_2).$$

Aside from this, no new principle is involved. We briefly indicate how the proof goes.

Substituting (2) in (15.4) yields

$$\Delta(G, 1) + \Delta(1, H) + \Delta(\beta_1 + \beta_2 N, N) = \Delta(G - H - \beta_1 N, 1) = 0;$$

that is,

$$G(z) = H(z) + \beta_1 N + \beta_3,$$

which in (15.3) gives

$$F(z_1 \circ z_2) = G(z_1) + G(z_2) + \beta_2 N(z_1)N(z_2) - \beta_3.$$

Using associativity of $\circ$, we get

$$\Delta[G(z_1 \circ w), 1] + \Delta[1, G(z_2)] + \Delta[N(z_1 \circ w), \beta_2 N(z_2)] = 0.$$

Expanding by 1 results in

$$\Delta[N(z_1 \circ w), \beta_2 N(z_2), 1] = 0.$$

Now apply the remark a couple of times. The remaining solutions of (15.3) are [818]

(15.4b)
$$\begin{cases} F(z) = \alpha_1 E(z) + A(z) + \alpha_2, \\ G(z) = \alpha_3 E(z) + A(z) + \alpha_4, \\ H(z) = \alpha_5 E(z) + A(z) + \alpha_6, \\ K(z) = \alpha_7 E(z) + \alpha_8, \\ N(z) = \alpha_9 E(z) + \alpha_{10}; \end{cases}$$

(15.4c)
$$\begin{cases} F(z) = \alpha_1 A(z)^2 + \alpha_2 A(z) + A_1(z) + \alpha_3, \\ G(z) = \alpha_1 A(z)^2 + A_1(z) + \alpha_4, \\ H(z) = \alpha_1 A(z)^2 + \alpha_5 A(z) + A_1(z) + \alpha_6, \\ K(z) = 2\alpha_1 A(z) + \alpha_7, \\ N(z) = A(z) + \alpha_8, \end{cases}$$

where the constants α_i satisfy certain conditions, and A and E are as given above.

Vincze has also applied his method to generalized sine and cosine equations where addition is replaced by an associative operation $\circ$. The method appears to be quite powerful.

Example 2. We consider now the "trigonometric" functional equation

(15.5)
$$C(z + w) = C(z)C(w) - S(z)S(w),$$

where $C, S : \mathbb{C} \to \mathbb{C}$ (see Chapter 3, (3.72)). We shall determine the most general complex solutions of this equation by using the method of determinants in the following theorem.

Theorem 15.1. [818]. *Suppose $C, S : \mathbb{C} \to \mathbb{C}$ satisfy equation* (15.5)*. Then C and S are given by*

$$\text{(c1)}\quad C(z) = \frac{1}{2\beta}[(\alpha+\beta)E_2(z) - (\alpha-\beta)E_1(z)], \quad S(z) = \frac{1}{2\beta}(E_1(z) - E_2(z)),$$

with $\alpha^2 - \beta^2 = 1,\ \beta \neq 0,$

$$\text{(c2)}\qquad C(z) = E(z)(1 \pm A(z)), \quad S(z) = E(z)A(z),$$

where A is additive, E_1 and E_2 are exponential (E)*, and α and β are complex constants. There are no other solutions.*

Proof. We consider solutions c_1 and c_2. It is clear that with the mentioned restrictions on the constants α and β, these functions actually do satisfy equation (15.5). Therefore, we need only show that all solutions are either of the form (c1) or (c2). The idea is first to obtain a formula for $S(z+w)$ similar to C in (15.5).

By the associativity of addition $+$, we have

$$\begin{aligned} C(z+\zeta+w) &= C(z+\zeta)C(w) - S(z+\zeta)S(w) \\ &= C(\zeta+w)C(z) - S(\zeta+w)S(z), \end{aligned}$$

from which we obtain

$$\text{(15.6)}\qquad \Delta[C(z+\zeta), C(w)] - \Delta[S(z+\zeta), S(w)] = 0,$$

which by "enlarging" by $C(u)$ we obtain

$$\Delta[S(z+\zeta), S(w), C(u)] = 0.$$

Hence we have by the remark one of the following cases:

(i) $C(u) \equiv 0$.
(ii) $S(z) = \beta_1 C(z)$ (β_1 constant).
(iii) $S(z+\zeta) = \alpha_1(\zeta)S(z) + \alpha_2(\zeta)C(z)$.

(i) If we set $C(z) = 0$ in (15.5), then $S(z_1)S(z_2) = 0$, and hence $S(z) \equiv 0$. This solution is contained in both c_1 and c_2.
(ii) In this case, we obtain from (15.5) the Pexider functional equation (PL)

$$C(z+w) = (1-\beta_1^2)C(z) \cdot C(w).$$

Thus, with $1 - \beta_1^2 \neq 0$,

$$C(z) = \frac{1}{1-\beta_1^2}E_1(z),$$

where E_1 is exponential, and then from (ii),

$$S(z) = \frac{\beta_1}{1-\beta_1^2} E_1(z),$$

which gives (c1).

We should stress that if C and S are going to be linearly dependent, either (i) or (ii) must then hold. We therefore assume in what follows that they are *linearly independent*.

(iii) We notice the symmetry of the left side of (iii) and see that

$$\Delta(\alpha_1, S) + \Delta(\alpha_2, C) = 0. \tag{15.6a}$$

"Enlarging" by C yields

$$\Delta(\alpha_1, S, C) = 0.$$

But this equality can only hold (see remark) if

$$\alpha_1(z) = \beta_1 S(z) + \beta_2 C(z) \quad (\beta_1, \beta_2 \text{ are constants}) \tag{15.7}$$

since we supposed the functions $S,\ C$ to be linearly independent.

If we use (15.7) in (15.6a), we obtain

$$\begin{aligned} \Delta(\beta_1 S + \beta_2 C, S) + \Delta(\alpha_2, C) &= \Delta(\beta_2 C, S) + \Delta(\alpha_2, C) \\ &= \Delta(-\beta_2 S, C) + \Delta(\alpha_2, C) \\ &= \Delta(\alpha_2 - \beta_2 S, C) = 0. \end{aligned}$$

Since $C \not\equiv 0$, we have

$$\alpha_2(z) - \beta_2 S(z) = \beta_3 C(z) \quad (\beta_3 \text{ constant}). \tag{15.8}$$

Making use of (15.7) and (15.8), we obtain from (iii)

$$\begin{aligned} S(z+w) &= \alpha_1(w)S(z) + \alpha_2(w)C(z) \\ &= [\beta_2 S(w) + \beta_2 C(z)]S(z) + [\beta_2 S(w) + \beta_3 C(w)]C(z) \\ &= \beta_1 S(z)S(w) + \beta_2 S(z)C(w) + \beta_2 S(w)C(z) + \beta_3 C(z)C(w), \end{aligned} \tag{15.9}$$

a formula similar to (15.5) for S.

In order to obtain further restrictions on the constants $\beta_1,\ \beta_2,\ \beta_3$, we set (15.5) and (15.9) into (15.6):

$$\begin{aligned} &\Delta[C(z_1)C(\zeta) - S(z)S(\zeta), C(w)] - \Delta[\beta_1 S(z)S(\zeta) \\ &\qquad + \beta_2 S(z)C(\zeta) + \beta_2 S(\zeta)C(z) + \beta_3 C(z)C(\zeta), S(w)] \\ &\quad = C(\zeta)\Delta[C(z), C(w)] - S(\zeta)\Delta[S(z)C(w)] \\ &\qquad - \beta_1 S(\zeta)\Delta[S(z), S(w)] - \beta_2 C(\zeta)\Delta[S(z), S(w)] \\ &\qquad - \beta_2 S(\zeta)\Delta[C(z), S(w)] - \beta_3 C(\zeta)\Delta[C(z), S(w)] \\ &\quad = [(1-\beta_2)S(\zeta) - \beta_3 C(\zeta)]\Delta[C(z), S(w)] = 0. \end{aligned}$$

Because C and S were assumed linearly independent, we have

$$\Delta[C(z), S(w)] \not\equiv 0$$

and hence

$$(1-\beta_2)S(\zeta) - \beta_3 C(\zeta) \equiv 0,$$

from which it follows since S and C are independent that $\beta_2 = 1,\ \beta_3 = 0$. Now equation (15.9) can be written as

$$S(z+w) = \beta_1 S(z)S(w) + S(z)C(w) + S(w)C(z). \tag{15.10}$$

Now we determine γ so that

$$C(z+w)+\gamma\, S(z+w) = (C(z)+\gamma\, S(z))(C(w)+\gamma\, S(w)), \quad z, w \in \mathbb{C}. \tag{15.11}$$

Utilize (15.5) and (15.10) in (15.11) to have

$$\begin{aligned} C(z)C(w) - S(z)S(w) + \gamma\beta_1 S(z)S(w) + \gamma\, S(z)C(w) + \gamma\, S(w)C(z) \\ = C(z)C(w) + \gamma\, C(z)S(w) + \gamma\, S(z)C(w) + \gamma^2 S(z)S(w); \end{aligned}$$

that is, since $S \neq 0,\ \gamma^2 - \gamma\beta_1 + 1 = 0$ or

$$\gamma = \frac{\beta_1 \pm \sqrt{\beta_1^2 - 4}}{2}.$$

Case i. Roots are distinct, say $r_1,\ r_2$. Let

$$E_i(z) = C(z) + r_i S(z), \quad i = 1, 2.$$

Then the E_i's are exponential. Putting r equal to $r_1,\ r_2$ in (15.11) and subtracting one from the other, we get

$$(r_1 - r_2)S(z+w) = E_1(z)E_1(y) - E_2(z)E_2(y)$$

or

$$S(z) = \frac{1}{r_1 - r_2}(E_1(z) - E_2(z)), \quad r_1 \neq r_2.$$

Use

$$C(z+w) + r_1 S(z+w) = E_1(z)E_1(w)$$

and an expression for S to get

$$C(z) = \frac{1}{r_1 - r_2}[r_1 E_2(z) - r_2 E_1(z)],$$

which gives (c1).

Case ii. Roots are equal. Then $\beta_1 = \pm z$ and $r = \pm 1$. Now (15.11) becomes $E(z+w) = E(z)E(w)$; that is, E is exponential, where $E(z) = C(z) \pm S(z)$. From (15.9), we have

$$S(z+w) = S(z)E(w) + S(w)E(z);$$

that is,

$$\frac{S(z+w)}{E(z+w)} = \frac{S(z)}{E(z)} + \frac{S(w)}{E(w)}$$

and

$$S(z) = E(z)A(z),$$

where $A(z) = \frac{S(z)}{E(z)}$ and is additive. Finally,

$$C(z) = E(z) \mp S(z) = E(z)(1 \mp A(z)),$$

which gives (c2).

Some of the other equations solved by this method include

$$\begin{aligned} S(z+w) &= S(z)C(w) + S(w)C(z), \\ C(z+w) &= C(z)C(w) + S(z)S(w), \\ S(z+w) &= G(z)H(w) + K(z)N(w), \end{aligned}$$

where $S, C, G, H, K, N : \mathbb{C} \to \mathbb{C}$.

Vincze also proved the following result (see also Chapter 1).

Result. (Vincze [819]). *Let $S(*)$ be an Abelian semigroup. Then the functional equation*

$$f(x * y)^n = [f(x) + f(y)]^n$$

is equivalent to $f(x * y) = f(x) + f(y)$, where $f : S \to \mathbb{C}$.

15.2 Means

Now we devote this section to various means and the functional equations connected with them. We start off with the *weighted arithmetic mean* (w.a.m.),

$$\text{w.a.m.} = \frac{ax + by}{a+b} = (1-p)x + py,$$

for $x, y \in \mathbb{R},\ a, b \in \mathbb{R},\ 0 \le p \le 1,\ p = \frac{b}{a+b}$.

Note that $a = 1 = b$ yields the *arithmetic mean*

$$\text{A.M.} = \frac{x+y}{z}.$$

Theorem 15.2. (See also [12, 44].) *The only solution of the* transitivity equation

$$F(x+t, y+t) = F(x, y) + t, \quad x, y, t \in \mathbb{R}, \tag{15.12}$$

and the homogeneity equation

$$F(xu, yu) = F(x, y)u, \quad x, y, u \in \mathbb{R}, \tag{15.13}$$

is the w.a.m. where $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Proof. Set $y = 0,\ x = 1$ in (15.13) to get

$$F(u, 0) = F(1, 0)u = (1 - p)u, \quad p \text{ a constant.} \tag{15.14}$$

Put $t = -y$ in (15.12) and use (15.14) to have

$$\begin{aligned} F(x, y) &= F(x - y, 0) + y \\ &= (1 - p)(x - y) + y \\ &= (1 - p)x + py, \end{aligned}$$

which is w.a.m. This proves the theorem. □

Remark. Note that the pair of equations is solved without assuming any regularity condition on F.

A considerable amount of literature (see [623]) about the concept of the mean (or average) and the properties of several means (such as the median, the arithmetic mean, the geometric mean, the power mean, the harmonic mean, etc.) was produced in the nineteenth century and often treated the significance and the interpretation of these specific aggregation functions.

Cauchy [147] considered in 1821 the mean of n independent variables $x_1, \ldots, x_n$ as a function $M(x_1, \ldots, x_n)$, which should be internal to the range of the x_i's values:

$$\min\{x_1, \ldots, x_n\} \le M(x_1, \ldots, x_n) \le \max\{x_1, \ldots, x_n\}. \tag{15.15}$$

The concept of the mean as an *average* is usually ascribed to Chisini [153]. The solution of Chisini's equation corresponds to the arithmetic mean, the geometric mean, the quadratic mean, the harmonic mean, the exponential mean, and the anti-harmonic mean, which is defined as

$$M(x_1, \ldots, x_n) = \frac{\sum_i x_i^2}{\sum_i x_i}. \tag{15.16}$$

Unfortunately, Chisini's definition is so general that it does not even imply that the "mean" (provided there exists a real and unique solution to the equation above) fulfills Cauchy's internality property (15.15).

In 1930, Kolmogoroff [543] and Nagumo [642] considered that the mean should be more than just a Cauchy mean or an average in the sense of Chisini. They defined a *mean value* to be an infinite sequence of continuous, symmetric, and strictly increasing (in each variable) real functions

$$M_1(x_1) = x_1,\ M_2(x_1, x_2), \ldots, M_n(x_1, \ldots, x_n), \ldots$$

satisfying the *reflexive* law, $M_n(x, \ldots, x) = x$ for all n and all x, and a certain kind of associative law,

$$\begin{aligned} &M_k(x_1, \ldots, x_k) = x \\ &\qquad \Rightarrow M_n(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \\ &\qquad = M_n(x, \ldots, x, x_{k+1}, \ldots, x_n), \end{aligned} \tag{15.17}$$

for every natural integer $k \leq n$. They proved, independently of each other, that these conditions are necessary and sufficient for the quasiarithmeticity of the mean; that is, for the existence of a continuous strictly monotonic function f such that M_n may be written in the form

$$M_n(x_1, \ldots, x_n) = f^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right], \quad n \in Z_+^*. \tag{15.18}$$

The *quasiarithmetic means* (see also Chapter 17) comprise most of the algebraic means of common use and allow one to specify f in relation to operational conditioning:

Examples of quasiarithmetic means

$f(x)$	$M_n(x_1, \ldots, x_n)$	name
x	$\frac{1}{n}\sum x_i$	arithmetic
x^2	$\sqrt{\frac{1}{n}\sum x_i^2}$	quadratic
$\log x$	$\sqrt[n]{\prod x_i}$	geometric
x^{-1}	$\frac{1}{\frac{1}{n}\sum \frac{1}{x_i}}$	harmonic
$x^\alpha\ (\alpha \in \mathbb{R}_0)$	$\left(\frac{1}{n}\sum x_i^\alpha\right)^{\frac{1}{\alpha}}$	power
$e^{\alpha x}\ (\alpha \in \mathbb{R}_0)$	$\frac{1}{\alpha}\log\left[\frac{1}{n}\sum e^{\alpha x_i}\right]$	exponential

The properties above defining a mean value seem to be natural enough. For instance, one can readily see that, for increasing means, the reflexivity property is equivalent to Cauchy's internality (15.15), and both are accepted by all statisticians as requisites for means.

Associativity of means (15.17) was introduced first in 1926 by Bemporad [116, p. 87] in a characterization of the arithmetic mean. Under reflexivity, this condition seems more natural, for it becomes equivalent to

$$\begin{gathered} M_k(x_1,\dots,x_k) = M_k(x'_1,\dots,x'_k) \\ \Downarrow \\ M_n(x_1,\dots,x_k,x_{k+1},\dots,x_n) = M_n(x'_1,\dots,x'_k,x_{k+1},\dots,x_n), \end{gathered}$$

which says that the mean does not change when altering some values without modifying their partial mean. This property is called "decomposability" in order not to confuse it with the classical associativity property. Some means, however, do not belong to this family: The antiharmonic mean (15.16) is not increasing in each variable, and the median is not associative in the sense of (15.16). Kolmogoroff and Nagumo proved that the quasiarithmetic means are exactly the decomposable sequences of continuous, symmetric, strictly increasing in each variable, and reflexive functions.

Observe, however, that this concept has been defined for symmetric means. When symmetry is not assumed, it is necessary to rewrite the decomposability property in such a way that the first variables are not privileged. Marichal et al. [624] proposed the general form

$$\begin{aligned} M_k(x_{i_1},\dots,x_{i_k}) = x \Rightarrow M_n &\left(\sum_{i\in K} x_i e_i + \sum_{i\notin K} x_i e_i\right) \\ &= M_n\left(\sum_{i\in K} x e_i + \sum_{i\notin K} x_i e_i\right) \end{aligned} \tag{15.19}$$

for every subset $K = \{i_1,\dots,i_k\} \subseteq \{1,\dots,n\}$ with $i_1 < \cdots < i_k$ (e_i represents the unit vectors of $\mathbb{R}^n$). Of course, under symmetry, this definition coincides with (15.17).

We now show that symmetry is not necessary in the Kolmogoroff-Nagumo characterization, provided that decomposability is considered in its general form. Thus we show that any decomposable sequence $(M_n)_{n\in\mathbb{N}_0}$ of continuous, strictly increasing, and reflexive functions is a quasiarithmetic mean value (15.18).

The characterization of the quasiarithmetic mean was done by using some of its simple properties.

We first show that, for any decomposable sequence $(M_n)_{n\in\mathbb{N}_0}$ of reflexive functions, the two-place function M_2 fulfills the *bisymmetry functional equation*

$$M_2(M_2(x_1,x_2),M_2(x_3,x_4)) = M_2(M_2(x_1,x_3),M_2(x_2,x_4)). \tag{15.20}$$

The bisymmetry property (also called mediality) is very easy to handle and has been investigated from the algebraic point of view by using it mostly in structures without the property of associativity—in a certain respect, it has been used as a substitute for associativity and also for symmetry (see Chapter 15).

Proposition 15.3. (Marichal [623]). *For any decomposable sequence* $(M_n)_{n\in\mathbb{N}_0}$ *of reflexive functions, we have for all* $k, n \in Z_+^*$ $(n \geq 2)$:

(1) $M_{k,n}(x_{11}, \ldots, x_{1k}; \ldots; x_{n1}, \ldots, x_{nk}) = M_n(M_k(x_{11}, \ldots, x_{1k}); \ldots; M_k(x_{n1}, \ldots, x_{nk}))$.
(2) *The function* M_2 *fulfills the bisymmetry functional equation* (15.20).

Proof. We have successively

$$\begin{aligned}
&M_2(M_2(x_1, x_2), M_2(x_3, x_4)) \\
&\quad = M_4(x_1, x_2, x_3, x_4) && \text{(by (1))} \\
&\quad = M_4(M_2(x_1, x_3), M_2(x_2, x_4), M_2(x_1, x_3), M_2(x_2, x_4)) && \text{(decomposability)} \\
&\quad = M_2(M_2(M_2(x_1, x_3), M_2(x_2, x_4)), \\
&\qquad M_2(M_2(x_1, x_3), M_2(x_2, x_4))) && \text{(by (1))} \\
&\quad = M_2(M_2(x_1, x_3), M_2(x_2, x_4)), && \text{(reflexivity)}
\end{aligned}$$

which proves the result. □

Now, consider a decomposable sequence $(M_n)_{n\in Z_+^*}$ of continuous, strictly increasing, and reflexive functions. Since M_2 fulfills the bisymmetry equation, it must have a particular form. Actually, it has been proved by Aczél [12] that the general continuous, strictly increasing in both variables, reflexive, and symmetric real solution of the bisymmetry equation (15.20) is given by the *quasilinear mean*.

Result 15.4. [12]. *Let* I' *be any real interval, finite or infinite. A two-place function* $M : I' \times I' \to \mathbb{R}$ *is continuous, strictly increasing in each variable, reflexive, and symmetric and fulfills the bisymmetric equation* (15.20) *if and only if there exists a continuous strictly monotonic function* $f : I' \to \mathbb{R}$ *and a real number* $\alpha \in]0, 1[$ *such that*

$$M(x, y) = f^{-1}[\alpha f(x) + (1 - \alpha) f(y)] \quad \text{for all } x, y \in I'.$$

Result 15.4a. [623]. *Let* I' *be any* (*finite or infinite*) *real interval and* $\{M_n\}_{n\in\mathbb{N}_0}$ *be a decomposable sequence of continuous, symmetric, strictly increasing, and reflexive functions* $M_n : (I')^n \to \mathbb{R}$. *Then and only then there exists a continuous strictly monotonic function* $f : I' \to \mathbb{R}$ *such that, for all* $n \in \mathbb{Z}_+^*$,

$$M_n(x_1, \ldots, x_n) = f^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right] \quad \text{for } x_1, \ldots, x_n \in I'. \tag{15.18}$$

As already mentioned, the symmetry property is not essential in the previous result.

Result 15.4b. [623]. *Let* I' *be any* (*finite or infinite*) *real interval and* $\{M_n\}$ *be a decomposable sequence of continuous, strictly increasing, and reflexive functions* $M_n : (I')^n \to \mathbb{R}$. *Then and only then there exists a continuous strictly monotonic function* $f : I' \to \mathbb{R}$ *such that, for all* $n \in \mathbb{Z}_+^*$,

$$M_n(x_1, \ldots, x_n) = f^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right] \quad \text{for } x_1, \ldots, x_n \in I'. \tag{15.18}$$

Definitions. A function $M : \mathbb{R}^2 \to \mathbb{R}$ is said to be a *quasilinear function* provided there exists a continuous and strictly monotonic function $f : \mathbb{R} \to \mathbb{R}$ and constants $a,\ b,\ c$ such that

$$M(x, y) = f^{-1}(af(x) + bf(y) + c), \quad a, b \neq 0,\ c, x, y \in \mathbb{R}. \tag{15.21}$$

When $a + b = 1$ and $c = 0$,

$$M(x, y) = f^{-1}((1 - a) f(x) + af(y)), \quad x, y \in \mathbb{R}, \tag{15.22}$$

is called a *quasilinear mean*. When $a = \frac{1}{2}$,

$$M(x, y) = f^{-1}\left(\frac{f(x) + f(y)}{2}\right), \quad \text{for } x, y \in \mathbb{R}, \tag{15.18a}$$

is called a *quasiarithmetic mean.*

Result 15.4c. (Hosszú [378]). *The most general cancellative, continuous solutions of the right and left autodistributive laws* $(xy)\cdot z = (xz)\cdot(yz)$ *and* $z\cdot(xy) = (zx)\cdot(zy)$ *are the quasilinear means* (15.22). *Let* $M : \mathbb{R}^2 \to \mathbb{R}$ *be increasing in its first variable, have partial derivatives of first order, and satisfy* (*right*) *autodistributivity. Then* M *is a quasilinear mean* (15.22).

15.2.1 Characterizations—Arithmetic and Exponential Means

Theorem 15.5. [44]. *The only solutions* $M : \mathbb{R}^2 \to \mathbb{R}$ *of the transitivity* (15.12) *and quasiarithmetic mean* (15.18a) *are*

$$A.M. = \frac{x + y}{2}, \quad \textit{exponential mean } (E.M.) = \frac{1}{c}\log\left(\frac{e^x + e^y}{2}\right),$$

where $c\ (\neq 0)$ *is a constant.*

Proof. Using (15.18a) in (15.12), we get

$$f^{-1}\left(\frac{f(x + t) + f(y + t)}{2}\right) = f^{-1}\left(\frac{f(x) + f(y)}{2}\right) + t$$

for $x, y, t \in \mathbb{R}$.

Let $k_t(x) = f(x + t)$ for a fixed t. Note that f is continuous and strictly monotonic. Then the equation above can be written as

$$f^{-1}\left(\frac{k_t(x) + k_t(y)}{2}\right) = f^{-1}\left(\frac{f(x) + f(y)}{2}\right) + t.$$

Replace x by $f^{-1}(x)$ and y by $f^{-1}(y)$ to obtain

$$\frac{(k_t f^{-1})(x) + k_t f^{-1}(y)}{2} = f\left[f^{-1}\left(\frac{x+y}{2}\right) + t\right] = k_t\left(f^{-1}\left(\frac{x+y}{2}\right)\right)$$

by the definition of k; that is,

$$(k_t f^{-1})\left(\frac{x+y}{2}\right) = \frac{(k_t f^{-1})(x) + (k_t f^{-1})(y)}{2},$$

which is Jensen (J). So there exist an additive $A : \mathbb{R} \to \mathbb{R}$ (continuous since f is) and a constant $b \in \mathbb{R}$ such that

$$(k_t f^{-1})(x) = A(x) + b, \quad \text{for } x \in \mathbb{R},$$

or

$$f(x+t) = k_t(x) = A(f(x)) + b = cf(x) + b = c(t) f(x) + b(t),$$

where c and b are functions of t, which is a special case of (1.94a). By Theorem 1.50, we have either $f(x) = \alpha x + \beta$ or $f(x) = \alpha e^{\gamma x} + \beta$, $\alpha, \beta, \gamma \in \mathbb{R}$. The first f in (15.18a) yields A.M. and the latter f in (15.18a) yields E.M. This proves the theorem. □

15.2.2 Geometric Mean and the Root Mean Power

Theorem 15.6. (See [44], [250], and Chapter 17, Section 17.1.2). *Let $M : \mathbb{R} \to \mathbb{R}$ satisfy the homogeneity* (15.13) *and the quasiarithmetic mean* (15.18a)*. Then the only solutions are either*

$$M(x, y) = (xy)^{\frac{1}{2}}$$

(*geometric mean, G.M.*) *or*

$$M(x, y) = \left(\frac{x^p + y^p}{2}\right)^{\frac{1}{p}}$$

(*the root mean power, R.M.P.*), *where* $p \in \mathbb{R}$.

Proof. The proof runs similar to that in Theorem 15.5. From (15.13) and (15.18a) there results

$$f^{-1}\left(\frac{f(x) + f(y)}{2}\right) u = f^{-1}\left(\frac{f(xu) + f(yu)}{2}\right).$$

Replace x by $f^{-1}(x)$ and y by $f^{-1}(y)$ to get

$$f^{-1}\left(\frac{x+y}{2}\right) u = f^{-1}\left(\frac{f(f^{-1}(x)u) + f(f^{-1}(y)u)}{2}\right).$$

For fixed u, set $f_u(x) = f^{-1}(x)u$, so that we have

$$f\left(f_u\left(\frac{x+y}{2}\right)\right) = \frac{f(f_u(x)) + f(f_u(y))}{2},$$

which is Jensen in ff_u so that

$$(ff_u)(x) = A(x) + b$$

and

$$f(f^{-1}(x)u) = A(x) + b$$

or

$$f(xu) = A(f(x)) + b = c(u)f(x) + b(u),$$

where c and b are functions of u, which is a special case of (1.94b) (also (1.94a)). By Result 1.61, we obtain either

$$f(x) = \alpha \log x + \beta \quad \text{or} \quad f(x) = \alpha x^p + \beta, \quad \alpha, \beta, p \in \mathbb{R}.$$

From the first form, we get the G.M. $(xy)^{\frac{1}{2}}$, and from the second form, we get the root mean power $\left(\frac{x^p+y^p}{2}\right)^{\frac{1}{p}}$. This completes the proof of this theorem. □

Remark. From the last two theorems, it follows that if

$$f^{-1}\left(\frac{f(x) + f(y)}{2}\right) = g^{-1}\left(\frac{g(x) + g(y)}{2}\right)$$

holds, where $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, strictly monotonic functions, then there exist constants α, β such that

$$f(x) = \alpha g(x) + \beta \quad \text{for all } x \in \mathbb{R}.$$

Definition. Let $M : I' \times I' \to \mathbb{R}$ be represented by

$$M(x, y) = \frac{p(x)x + p(y)y}{p(x) + p(y)} \quad \text{for } x, y \in I', \tag{15.23}$$

where $p : I' \to \mathbb{R}_+^*$ is an arbitrary function and I' is an interval in $\mathbb{R}$.

Equation (15.23) is a generalization of the quasiarithmetic mean (15.18a) and is called an *arithmetic mean with weight function.*

Theorem 15.7. (Páles and Volkmann [661]). *A function $M : I' \times I' \to \mathbb{R}$ is an arithmetic mean with weight function if and only if it has the Cauchy mean value property* (15.15) *and satisfies*

$$\begin{aligned}(x - M(x,z))(y - M(x,y))(z - M(z,y)) \\ = -(x - M(x,y))(y - M(z,y))(z - M(x,z)) \\ \text{for all } x, y, z \in I'.\end{aligned} \tag{15.24}$$

Proof. "If part": (15.23) obviously implies the mean value property (15.15) of M as well as (15.24) in the case where at least two of the $x,\ y,\ z$ coincide. If $x,\ y,\ z$ are three different elements from I', (15.23) yields

$$\frac{p(x)}{p(y)} = \frac{y - M(x,y)}{M(x,y) - x}, \quad \frac{p(z)}{p(x)} = \frac{M(x,z) - x}{z - M(x,z)}, \quad \frac{p(y)}{p(z)} = \frac{M(z,y) - z}{y - M(z,y)},$$

and (15.24) then follows from

$$\frac{p(x)}{p(y)} : \frac{p(y)}{p(z)} : \frac{p(z)}{p(x)} = 1.$$

"Only if part": Now we suppose that (15.15) and (15.24) are fulfilled. Let $x,\ y,\ z$ be three different elements of I'. Because of (15.15), we observe that all three differences $x - M(x,z)$, etc., occurring in (15.24) are different from zero. We interchange y and z in (15.24) and multiply the resulting equation by (15.24). After cancellation, we obtain

$$(y - M(y,z))(z - M(z,y)) = (y - M(z,y))(z - M(y,z)),$$

which implies $M(z,y) = M(y,z)$. So M is a symmetric function. With $x = y = z$, we obtain from (15.23)

$$M(x,x) = x, \quad x \in I'. \tag{15.25}$$

Now we fix an element z in I' and we define $p : I' \to \mathbb{R}_+^*$ by

$$p(x) = \begin{cases} \frac{z - M(x,z)}{M(x,z) - x} & (x \in I',\ x \neq z), \\ 1 & (x = z). \end{cases} \tag{15.26}$$

(The positivity of the values of p is a consequence of (15.15).) Then it follows from (15.25), (15.26), and the symmetry of M that (15.23) holds in the cases $x = y,\ x = z$, and $y = z$. It remains to verify (15.23) for $x \neq y$, both being different from z. Therefore, (15.24), (15.26), and the symmetry of M imply

$$\frac{y - M(x,y)}{M(x,y) - x} = \frac{p(x)}{p(y)},$$

which gives (15.23) in this case. □

Remarks

(i) In the proof of the theorem, we did not use that I' is an interval. It is enough to assume that I' has at least three different elements.

(ii) The functional equation (15.24) has solutions that are not of the form (15.23); e.g.,

$$M(x,y) = x, \quad M(x,y) = y, \quad M(x,y) = \max\{x,y\}, M(x,y) = \min\{x,y\}.$$

15.2.3 Stolarsky Mean

Stolarksy introduced in [778] a mean

$$S_{p,q}(x,y)=\left(\frac{q(x^p-y^p)}{p(x^q-y^q)}\right)^{\frac{1}{p-q}},\quad \text{for } x,y\in\mathbb{R}_+^*,\ p\neq q\in\mathbb{R}^*, \tag{15.27}$$

called the *Stolarsky mean* or *Stolarsky's difference mean*, that lies between x and y. He showed that this mean can be extended continuously to $\{(p,q,x,y):p,q\in\mathbb{R},\ x,y\le 0\}$. This extension is given below.

$$S_{p,q}(x,y)=\begin{cases}\left(\frac{q(x^p-y^p)}{p(x^q-y^q)}\right)^{\frac{1}{p-q}} & \text{if } pq(p-q)(x-y)\neq 0,\\ \left(\frac{x^p-y^p}{p(\ln x-\ln y)}\right)^{\frac{1}{p}} & \text{if } p(x-y)\neq 0,\ q=0,\\ \left(\frac{q(\ln x-\ln y)}{(x^q-y^q)}\right)^{-\frac{1}{q}} & \text{if } q(x-y)\neq 0,\ p=0,\\ \exp\left(-\frac{1}{p}+\frac{x^p\ln x-y^p\ln y}{x^p-y^p}\right) & \text{if } q(x-y)\neq 0,\ p=q,\\ \sqrt{xy} & \text{if } x-y\neq 0,\ p=q=0,\\ x & \text{if } x-y=0.\end{cases}$$

When $q=1$, (15.27) takes the form

$$S_p(x,y)=\left[\frac{x^p-y^p}{p(x-y)}\right]^{\frac{1}{p}-1}\quad \text{for } x,y\in\mathbb{R}^*. \tag{15.27a}$$

For $x,y>0$, $S_p(x,y)$ is a continuous increasing function of p.

The Stolarsky mean includes a large class of well-known means such as

power mean $A_p(x,y):=\left(\frac{x^p+y^p}{2}\right)^{\frac{1}{p}}=S_{2p,p}(x,y)$,

with $A_0(x,y)=\lim_{p\to 0}A_p(x,y)=(xy)^{\frac{1}{2}}=G(x,y)=S_{0,0}(x,y)$,

arithmetic mean $A(x,y):=\frac{x+y}{2}=S_{2,1}(x,y)$,

harmonic mean $H(x,y):=\frac{2xy}{x+y}=S_{-2,-1}(x,y)$,

logarithmic mean $L(x,y):=\frac{x-y}{\ln x-\ln y}=\lim_{p\to 0}S_{p,1}(x,y)=:S_{0,1}(x,y)$
$=S_0(x,y)=\lim_{p\to 0}S_p(x,y)$,

and identric mean $i(x,y):=\frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}}$
$=\lim_{p\to 1}S_{p,1}(x,y)=:S_{1,1}(x,y)=S_1(x,y)$.

Characterizations of Stolarsky Means

We consider (Lukacs [616]) the functional mean of positive $x,\ y$,

$$M_f(x,y) = f^{-1}\left(\int_0^1 f(\lambda x + (1-\lambda)y)d\lambda\right),$$

where $f : \mathbb{R}_+^* \to \mathbb{R}$, f'' is continuous in $\mathbb{R}_+^*$ and f is strictly monotonic in $\mathbb{R}$.

Lemma. *For a twice continuously differentiable function $f : \mathbb{R}_+^* \to \mathbb{R}$ and $x, y > 0$, set*

$$g(x,y) := \int_0^1 f(\lambda x + (1-\lambda)y)d\lambda.$$

Then

$$g_{xx}(\xi,\xi) = \frac{1}{3}f''(\xi),$$

where ξ is an arbitrarily fixed positive number.

Theorem 15.8. [616].

(a) *Let $p \neq 0, 1$. Then $M_f(x,y) = S_p(x,y)$ holds for all positive $x,\ y$ if and only if $f(t) = at^{p-1} + b$, where $a\ (\neq 0),\ b$ are arbitrary real constants.*
(b) *$M_f(x,y) = S_0(x,y)$ holds for all positive $x,\ y$ if and only if $f(t) = \frac{a}{t} + b$, where $a\ (\neq 0),\ b$ are arbitrary real constants.*
(c) *$M_f(x,y) = S_1(x,y)$ holds for all positive $x,\ y$ if and only if $f(t) = a\log t + b$, where $a\ (\neq 0),\ b$ are arbitrary real constants.*

Proof.

(a) We present the proof of (a) only [616]. Suppose $M_f(x,y) = S_p(x,y)$ holds for all positive $x,\ y$. Then

$$f^{-1}\left(\int_0^1 f(\lambda x + (1-\lambda)y)d\lambda\right) = \left(\frac{x^p - y^p}{p(x-y)}\right)^{\frac{1}{p-1}}$$

and so

$$\int_0^1 f(\lambda x + (1-\lambda)y)d\lambda = f\left(\left(\frac{x^p - y^p}{p(x-y)}\right)^{\frac{1}{p-1}}\right) = g(x,y)$$

for all positive x, y.

Differentiating both sides with respect to x twice, setting $x = \xi,\ y = \xi$, where ξ is an arbitrarily fixed positive number, in the resulting equality, and using L'Hopital's rule, we obtain $g_{x,x}(\xi,\xi)$. By the lemma, we obtain

$$\frac{1}{3}f''(\xi) = \frac{1}{4}f''(\xi) + \frac{p-2}{12\xi}f'(\xi),$$

and hence

$$f''(\xi) - \frac{p-2}{\xi} f'(\xi) = 0.$$

This implies

$$f(t) = at^{p-1} + b$$

in $\mathbb{R}_+^*$, where a, b are real constants with $a \neq 0$.
On the contrary, suppose $f(t) = at^{p-1} + b$, where a $(\neq 0)$, b are real constants and $t = \lambda x + (1-\lambda)y$. Then

$$f^{-1}(t) = \left(\frac{t-b}{a}\right)^{\frac{1}{p-1}},$$

and therefore

$$\begin{aligned} M_f(x,y) &= f^{-1}\left[a\int_0^1 (\lambda x + (1-\lambda)y)^{p-1} d\lambda + b\right] \\ &= f^{-1}\left[a\frac{x^p - y^p}{p(x-y)} + b\right] \\ &= \left(\frac{x^p - y^p}{p(x-y)}\right)^{\frac{1}{p-1}} \\ &= S_p(x,y), \end{aligned}$$

for all positive x, y. □

15.3 Some Comments about the Logarithmic Function

1. Suppose $f : \mathbb{R}_+^* \to \mathbb{R}$ and satisfies

(i) $$f(xy) - f(x) = -f\left(\frac{1}{y}\right) \quad \text{for } x, y \in \mathbb{R}_+^*.$$

Now $y = 1$ yields $f(1) = 0$ and $x = 1$ yields

$$f\left(\frac{1}{y}\right) = -f(y),$$

so f satisfies the logarithmic equation (L).

Remark. Note that f satisfies (15.28). For

$$\begin{aligned} f\left(\frac{1}{x}+\frac{1}{y}\right) &= f\frac{\left(\frac{1}{xy}\right)}{x+y} \\ &\overset{\text{by (i)}}{=} -f\left((x+y)\cdot\frac{xy}{x+y}\right)+f(x+y) \\ &= f(x+y)-f(xy) \\ &\overset{\text{by (i)}}{=} f(x+y)-f(x)+f\left(\frac{1}{y}\right) \\ &= f(x+y)-f(x)-f(y), \quad \text{which is (15.28).} \end{aligned}$$

2. Suppose $f:\mathbb{R}_+^*\to\mathbb{R}$ and satisfies

(ii) $$f(xy)=-f\left(\frac{1}{x}\right)-f\left(\frac{1}{y}\right) \quad \text{for } x,y\in\mathbb{R}_+^*.$$

Note that $f(1)=0$. $y=\frac{1}{x}$ gives $f(\frac{1}{x})=-f(x)$; that is, f is logarithmic (L).

Remark. Note that f satisfies (15.28):

$$\begin{aligned} f\left(\frac{1}{x}+\frac{1}{y}\right) &= f\left((x+y)\cdot\frac{1}{xy}\right) \\ &\overset{\text{by (ii)}}{=} -f\left(\frac{1}{x+y}\right)-f(xy) \\ &\overset{\text{by (ii)}}{=} -f\left(\frac{1}{x+y}\right)+f\left(\frac{1}{x}\right)+f\left(\frac{1}{y}\right) \\ &= f(x+y)-f(x)-f(y), \quad \text{which is (15.28).} \end{aligned}$$

3. Suppose $f:\mathbb{R}_+^*\to\mathbb{R}$ satisfies the functional equation

(15.28) $$f(x+y)-f(x)-f(y)=f\left(\frac{1}{x}+\frac{1}{y}\right) \quad \text{for } x,y\in\mathbb{R}_+^*.$$

The functional equation (15.28) is equivalent to the logarithmic equation (L) (Heuvers [367]). (This is (1.69); see Theorem 1.48.)

Now we consider more functional equations related to (L).

Summary. Let $f:\mathbb{R}_+^*\to\mathbb{R}$ be a real-valued function on the set of positive reals. Then the functional equations

(15.29) $$f(x+y)-f(xy)=f\left(\frac{1}{x}+\frac{1}{y}\right) \quad \text{for } x,y\in\mathbb{R}'_+,$$

(15.28), and (L) are equivalent to each other.

If $f, g, h : \mathbb{R}_+^* \to \mathbb{R}$ are real-valued functions on the set of positive reals, then

$$f(x+y) - g(xy) = h\left(\frac{1}{x} + \frac{1}{y}\right) \tag{15.30}$$

is the Pexider generalization of (15.29). We find the general solution to this Pexider equation.

If $f, g, h, k : \mathbb{R}_+^* \to \mathbb{R}$ are real-valued functions on the set of positive reals, then

$$f(x+y) - g(x) - h(y) = k\left(\frac{1}{x} + \frac{1}{y}\right) \tag{15.31}$$

is the Pexider generalization of (15.28). We find the twice differentiable solution to this Pexider equation.

There is another functional equation that is equivalent to (L).

Theorem 15.9. (Heuvers and Kannappan [368]). *The functional equation*

$$f(x+y) - f(xy) = f\left(\frac{1}{x} + \frac{1}{y}\right), \quad \textit{for } x, y > 0, \tag{15.29}$$

where $f : \mathbb{R}_+^* \to \mathbb{R}$ *is equivalent to the logarithmic equation* (L).

Proof. First replace x and y by x^{-1} and y^{-1} in (15.29) to obtain

$$f(x+y) + f\left(\frac{1}{xy}\right) = f\left(\frac{1}{x} + \frac{1}{y}\right)$$

and

$$f(xy) = -f\left(\frac{1}{xy}\right) \quad \text{or} \quad f\left(\frac{1}{t}\right) = -f(t), \quad \text{for } t > 0.$$

Set $x + y = s$ and $t = \frac{1}{xy}$ so that (15.29) becomes

$$f(s) + f(t) = f(st) \quad \text{with } \frac{4}{s^2} \leq t. \tag{15.29a}$$

Here, $y = \frac{1}{tx}$, so $x + \frac{1}{tx} = s$ or $tx^2 - stx + 1 = 0$. Thus

$$x = \frac{st \pm \sqrt{s^2t^2 - 4t}}{2t}.$$

Let $s, t > 0$. Choose $u > 0$ such that $\frac{4}{s^2t^2} < u$, $\frac{4}{s^2t} < u$, and $\frac{4}{t^2} < u$. Now,

$$\begin{aligned} f(s(tu)) &= f(s) + f(tu) && \text{since } \frac{4}{s^2} < tu \text{ (by (15.29a))} \\ &= f(s) + f(t) + f(u) && \text{since } \frac{4}{t^2} < u \text{ (by (15.29a)).} \end{aligned}$$

But

$$f(s(tu)) = f((st)u) = f(st) + f(u) \quad \text{since } \frac{4}{s^2t^2} < u \text{ (by (15.29a))}.$$

Thus $f(st) = f(s) + f(t)$ for all $s, t > 0$. Obviously, any logarithmic solution satisfies (15.29). This completes the proof. □

The Pexider generalization of (15.29) is the equation

$$f(x+y) - g(xy) = h\left(\frac{1}{x} + \frac{1}{y}\right) \quad \text{for } x, y > 0, \tag{15.30}$$

where $f, g, h : \mathbb{R}_+^* \to \mathbb{R}$ are real-valued functions on the set of positive reals.

Theorem 15.10. [368]. *The general solution of* (15.30) *is given by*

$$f(x) = a + L(x),\ g(x) = b + L(x),\ h(x) = a - b + L(x), \quad x \in \mathbb{R}_+^*, \tag{15.30a}$$

where a and b are arbitrary constants and L is logarithmic on $]0, \infty[$.

Proof. First set

$$f_1(t) = f(t),\ g_1(t) = g(t) - g(1),\ h_1(t) = g(1) + h(t), \quad t > 0. \tag{15.32}$$

Then (15.30) transforms into

$$f_1(x+y) - g_1(xy) = h_1\left(\frac{1}{x} + \frac{1}{y}\right) \quad \text{for } x, y > 0, \tag{15.33}$$

where $g_1(1) = 0$. Replace $x,\ y$ by $\frac{1}{x},\ \frac{1}{y}$ in (15.33) to obtain

$$h_1(x+y) + g_1\left(\frac{1}{xy}\right) = f_1\left(\frac{1}{x} + \frac{1}{y}\right). \tag{15.34}$$

Add (15.33) and (15.34) to obtain

$$P_1(x+y) - Q_1(xy) = P_1\left(\frac{1}{x} + \frac{1}{y}\right) \quad \text{for } x, y > 0,$$

where $2P_1(x) = f_1(x) + h_1(x)$ and $2Q_1(x) = g_1(x) - g_1(\frac{1}{x}) = -2Q_1(\frac{1}{x})$ with $Q_1(1) = 0$. So we have

$$P_1(x+y) + Q_1\left(\frac{1}{xy}\right) = P_1\left(\frac{1}{x} + \frac{1}{y}\right) \quad \text{for } x, y > 0. \tag{15.35}$$

As before, set $x + y = s$ and $\frac{1}{xy} = t$ so that (15.35) transforms into

$$P_1(st) = P_1(s) + Q_1(t) \quad \text{for } \frac{4}{s^2} \le t. \tag{15.36}$$

Let $s, t > 0$. Choose $u > 0$ so that $\frac{4}{t^2} < u$, $\frac{4}{s^2t^2} < u$, and $\frac{4}{t^2s} < u^2$. Then, $P_1((st)u) = P_1(st) + Q_1(u)$ since $\frac{4}{s^2t^2} < u$ (by (15.36)). But

$$\begin{aligned} P_1((tu)s) &= P_1(tu) + Q_1(s) && \text{since } \frac{4}{t^2u^2} < s \text{ (by (15.36))} \\ &= P_1(t) + Q_1(u) + Q_1(s) && \text{since } \frac{4}{t^2} < u \text{ (by (15.36)).} \end{aligned}$$

Hence $P_1(st) = P_1(t) + Q_1(s)$ for all $s, t > 0$, a Pexider (logarithmic) equation.

Thus there exists a logarithmic function L such that

$$P_1(t) = \frac{f_1(t) + h_1(t)}{2} = a + L(t) \quad \text{for } t > 0, \tag{15.37}$$

$$Q_1(t) = \frac{g_1(t) - g_1\left(\frac{1}{t}\right)}{2} = L(t) \qquad \text{for } t > 0. \tag{15.37a}$$

Note that, putting $y = \frac{1}{x}$ in (15.33) or (15.34), we get

$$h_1\left(x + \frac{1}{x}\right) = f_1\left(x + \frac{1}{x}\right);$$

that is,

$$P_1(t) = f(t) = f_1(t) = h_1(t) = a + L(t) \quad \text{for } t \geq 2. \tag{15.37b}$$

Subtracting (15.34) from (15.33), we obtain

$$P_2(x + y) - Q_2(xy) = -P_2\left(\frac{1}{x} + \frac{1}{y}\right) \quad \text{for } x, y > 0, \tag{15.38}$$

where

$$P_2(x) = f_1(x) - h_1(x) \quad \text{and} \quad Q_2(x) = g_1(x) + g_1\left(\frac{1}{x}\right) = Q_2\left(\frac{1}{x}\right). \tag{15.38a}$$

From (15.37b), it follows that $P_2(t) = 0$ for $t \geq 2$. Set $x + y = 2$ in (15.38), and it follows that

$$P_2(2) - Q_2(x(2 - x)) = -P_2\left(\frac{2}{x(2-x)}\right)$$

for $0 < x < 2$. But $2 \leq \frac{2}{x(2-x)}$ and $0 < x(2 - x) \leq 1$ for $0 < x < 2$, so it follows that $Q_2(t) = 0$ for $0 < t < 1$. Now (15.38a) implies that $Q_2(t) = 0$ for all $t > 1$ and thus $Q_2(t) = 0$ for all $t > 0$. Then it follows from (15.38) that

$$P_2(x + y) = -P_2\left(\frac{1}{x} + \frac{1}{y}\right)$$

for $x, y > 0$. Here set $x = y = \frac{t}{2}$, and we get $P_2(t) = -P_2(\frac{4}{t})$, which implies that $P_2(t) = 0$ for $0 < t < 2$. Thus, $P_2(t) = 0$ for all $t > 0$.

Thus,

$$f_1(x) = h_1(x) \quad \text{and} \quad g_1\left(\frac{1}{x}\right) = -g_1(x) \quad \text{for } x > 0 \text{ (by (15.38a))}.$$

Now, by (15.37) and (15.37a),

$$f_1(x) = f(x) = h_1(x) = a + L(x), \quad \text{for } x > 0,$$

and

$$g_1(x) = L(x), \quad \text{for } x > 0.$$

In (15.32), if $b = g(1)$, it follows that the solution of (15.30) is given by (15.30a). This proves the theorem. □

Remark. If $P_2, Q_2 : \mathbb{R}_+^* \to \mathbb{R}$ are real-valued functions defined on the positive reals with $Q_2(1) = 0$, then the only solutions of

$$P_2(x+y) - Q_2(xy) = -P_2\left(\frac{1}{x} + \frac{1}{y}\right) \tag{15.38}$$

are the trivial solutions P_2 and Q_2 both equal to the zero function.

By Theorem 15.10, there is a logarithmic function L and constants a and $b \in \mathbb{R}$ such that $P_2 = a + L$, $Q_2 = b + L$, $-P_2 = a - b + L$. Since $Q_2(1) = 0 = L(1)$, it follows that $b = 0$ and $P_2 = a + L = -a - L = 0$. Thus, $L = -a =$ constant and therefore $L = 0$ and $a = 0$; i.e., $P_2 = Q_2 = 0$.

The Pexider generalization of

$$f(x+y) - f(x) - f(y) = f\left(\frac{1}{x} + \frac{1}{y}\right)$$

is the equation

$$f(x+y) - g(x) - h(y) = k\left(\frac{1}{x} + \frac{1}{y}\right), \tag{15.31}$$

where $f, g, h, k :]0, \infty[\to \mathbb{R}$ are real-valued functions on the set of positive reals.

Theorem 15.11. [368]. *The twice differentiable solution of* (15.31) *is given by*

$$\begin{aligned} f(x) &= -a \log x + bx + c_1, \\ g(x) &= -a \log x + bx - \frac{d}{x} + c_1 + c_3, \\ h(x) &= -a \log x + bx - \frac{d}{x} - c_2 - c_3, \\ k(x) &= -a \log x + dx + c_2, \end{aligned} \tag{15.31a}$$

where a, b, c_1, c_2, c_3, *and* d *are arbitrary constants.*

Proof. Differentiate (15.31) with respect to x and y to obtain

$$f''(x+y) = \left(\frac{1}{xy}\right)^2 k''\left(\frac{1}{x}+\frac{1}{y}\right).$$

Now set $s = x + y$ and $t = \frac{1}{x} + \frac{1}{y}$ to obtain $f''(s) = \frac{t^2}{s^2}k''(t)$ or

$$s^2 f''(s) = t^2 k''(t) \quad \text{for } st \geq 4. \tag{15.39}$$

Set $s = 4$ in (15.39) to obtain $t^2k''(t^2) = a =$ constant for $t \geq 1$ and set $t = 4$ in (15.39) to obtain $s^2 f''(s) = a$ for $s \geq 1$.

For $0 < t < 1$, choose $s > 4$ so that $st \geq 4$. Then, from (15.39) it follows that f and k satisfy the differential equations $s^2 f''(s) = a$ and $t^2k''(t) = a$ for $s, t > 0$. After two integrations for each function, it follows that

$$f(x) = -a\log x + bx + c_1 \quad \text{and} \quad k(x) = -a\log x + dx + c_2$$

as in (15.31a). Putting these into (15.31), we obtain

$$g(x) + h(y) = -a\log x - a\log y + bx + by - \frac{d}{x} - \frac{d}{y} + c_1 - c_2.$$

Separate the x and y terms in the above to get

$$g(x) + a\log x - bx + \frac{d}{x} - c_1 = -\left[h(y) + a\log y - by + \frac{d}{y} + c_2\right] = c_3,$$

with c_3 a constant. Thus

$$g(x) = -a\log x + bx - \frac{d}{x} + c_1 + c_3$$

and

$$h(x) = -a\log x + bx - \frac{d}{x} - c_2 - c_3.$$

Thus the twice differentiable solution of (15.31) is given by (15.31a). This proves the theorem. □

Remark. Let L be an arbitrary logarithmic function; i.e., a general solution of (L). Let A and A_2 be arbitrary additive functions, general solutions of (A). Then, if c_1, c_2, and c_3 are arbitrary constants, it follows that a solution of (15.31) is given by

$$\begin{aligned}
f(x) &= L(x) + A(x) + c_1,\\
g(x) &= L(x) + A(x) - A_2\left(\frac{1}{x}\right) + c_1 + c_3,\\
h(x) &= L(x) + A(x) - A_2\left(\frac{1}{x}\right) - c_2 - c_3,\\
k(x) &= L(x) + A_2(x) + c_2.
\end{aligned}$$

15.4 D'Alembert's Equation Revisited

D'Alembert's equation (C) is solved over all finite groups (see [213] for this entire section). We introduce the notion of a *basic* d'Alembert function, one for which $f(xy) = f(x)$ for all x implies that $y = 1$. It is shown that every d'Alembert function factors through a basic d'Alembert function. Then it is shown that the only finite groups that support a basic d'Alembert function are the cyclic groups (the classical case) and the *binary groups*: $\langle 2, m, n\rangle = \langle R, S, T\rangle : R^2 = S^m = T^n = RST$ in Coxeter's notation. Conversely, each of these groups supports a nonclassical d'Alembert function.

A function $f : G \to \mathbb{C}$ (G a group) is a d'Alembert function if f satisfies (C) with $f(e) = 1$. The functions cos, cosh are d'Alembert functions (solutions of (C)) on the additive group $\mathbb{R}$ of real numbers. Indeed they are, up to automorphisms of $\mathbb{R}$, the only continuous d'Alembert functions on $\mathbb{R}$ (see Chapter 3). In studying d'Alembert functions, the idea is to make similar categorical assertions. When G is Abelian, Kannappan's result [424] gives a definitive characterization: If f is a d'Alembert function on G, then there is a homomorphism $h : G \to \mathbb{C}^*$ such that, for all x in G, (3.37) holds. It is easy to verify that if G is a general group and $h : G \to \mathbb{C}^*$ is a homomorphism, then the function $\hat{h} : G \to \mathbb{C}$ given by

$$\hat{h}(x) = \frac{h(x) + h^*(x)}{2} \tag{15.40}$$

is a d'Alembert function of G. However, it is "really" a d'Alembert function on $\frac{G}{[G,G]}$, the Abelianized version of G, since for all x, y, z in G,

$$\hat{h}(x[y, z]) = \hat{h}(x),$$

where $[y, z] := yzy^{-1}z^{-1}$ is the commutator of y and z. To capture this sort of decomposition/factorization, the following definition is introduced: The d'Alembert function f is *basic* if

$$(f(xy) = f(x)\ \forall\, x \in G) \to y = e. \tag{15.41}$$

We see that, on $\mathbb{R}$, cosh is basic, while cos is not.

We show in Theorem 15.14 that every d'Alembert function is the composition of a group epimorphism followed by a basic d'Alembert function. So we need only characterize basic d'Alembert functions. Consequently, we say the group G is a *d'Alembert group* if there is a basic d'Alembert function with domain G. Thus $\mathbb{R}$ is a d'Alembert group (with cosh as the basic d'Alembert function) and $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is a d'Alembert group with $\tilde{\cos} : z \to \frac{1}{2}(z + \overline{z})$ as the basic d'Alembert function; note that $\cos(x) = \tilde{\cos}(e^{ix})$. Indeed it is easy to see (Proposition 15.13) from Kannappan's Theorem that the only Abelian d'Alembert groups are those isomorphic to subgroups of $\mathbb{C}^*$.

The main result is the characterization of the finite d'Alembert groups as follows.

Result 15.12. (Davison [213]). *The finite group G is a d'Alembert group if, and only if, it is isomorphic to one of the groups below:*

(i) *the cyclic group* $\mathbb{Z}_n$ *of order* n,
(ii) *the binary dihedral group* $\langle 2, 2, n\rangle$ *of order* $4n$,
(iii) *the binary tetrahedral group* $\langle 2, 3, 3\rangle$ *of order* 24,
(iv) *the binary octahedral group* $\langle 2, 3, 4\rangle$ *of order* 48,
(v) *the binary octahedral group* $\langle 2, 3, 5\rangle$ *of order* 120.

Here we use Coxeter's notation $\langle 2, m, n\rangle$ (see [178, Ch. 6.5]) to denote the group with generators R, S, T and relations

$$R^2 = S^m = T^n = RST.$$

That each of the groups in the theorem is, in fact, a d'Alembert group is proved. We show, using results of Wolf [833], that a finite solvable d'Alembert group must be one of those listed, and using Suzuki's result [781], we show that the only finite nonsolvable d'Alembert group is the binary icosahedral group.

After we prove Theorem 15.14, we provide some of its consequences, as well as pointing to possible future developments of this approach.

15.4.1 Basic d'Alembert Functions

In this section, we characterize the classical (given by equation (15.40)) d'Alembert functions that are basic and deduce some consequences [213]. We also introduce a subgroup called the *nub* of F that yields the basic theorem. We then relate certain standard subgroups to well-known subgroups of G. Finally, we give a simple characterization of the nub of f when f is assumed to be a bounded function. Throughout this section, and later, we find Stetkaer's work [772, 771] of inestimable importance.

Proposition 15.13. *Let* $h : G \to \mathbb{C}^*$ *be a homomorphism. The d'Alembert function* (15.40) *is basic if, and only if,* $\ker(h) = \langle e\rangle$.

Proof. Referring to (15.41), $h(xy) = \hat{h}(x)$ if, and only if,

$$h(x)[h(y) - 1] = h(x^{-1})h(y^{-1})[h(y) - 1].$$

So $\hat{h}(xy) = \hat{h}(x)$ for all x in G if, and only if, $h(x) = 1$; that is, if, and only if, $y \in \ker(h)$.

Using this, we can prove the following.

Proposition 15.13a. [213]. *Let* G *be an Abelian group and let* $f : G \to \mathbb{C}$ *be a d'Alembert function on* G. *If* f *is basic, then* G *is isomorphic to a subgroup of* $\mathbb{C}^*$. *In particular, if* f *is basic and* G *is finite, then* G *is a cyclic group.*

Proof. By Kannappan's Theorem [424], $f = \hat{h}$ for some h since G is Abelian. If f is basic, then $\ker(h) = \langle e\rangle$ by Proposition 15.13, so $h : G \to \mathbb{C}^*$ is injective (a monomorphism) and thus G is isomorphic to a subgroup of $\mathbb{C}^*$. Since finite subgroups of $\mathbb{C}^*$ are cyclic, the final statement is proved.

When G is non-Abelian, we need an analogue of $\ker(h)$. This is provided, as will be seen by the nub of f.

Definition. Let f be a d'Alembert function on the group G. The *nub* of f is

$$N(f) := \{y \in G : f(xy) = f(x)\ \forall\, x \in G\}.$$

We see, from this, that f is basic if and only if $N(f) = \langle e\rangle$. Reconsidering Proposition 15.13's proof, we see that $N(\hat{h}) = \ker(h)$. This leads naturally to our next result.

Proposition 15.13b. [213]. *$N(f)$ is a normal subgroup of G.*

Proof. First we observe that $e \in N(f)$ since $xe = x$. Now suppose $y, z \in N(f)$. Then, from $x(yz) = (xy)z$, we deduce that $yz \in N(f)$. Next, if $y \in N(f)$, then $f(x) = f(xe) = f(xy^{-1}y) = f(xy^{-1})$ for all $x \in G$. So $y^{-1} \in N(f)$. Thus $N(f)$ is a subgroup of G.

Finally, to show that $N(f)$ is a normal subgroup of G, suppose that $y \in N(f)$ and $z \in G$. Since d'Alembert functions are central [772, Lemma 5.1], we have for all $x \in G$

$$f(xzyz^{-1}) = f(z^{-1}xzy) = f(z^{-1}xz) = f(x).$$

Thus $zyz^{-1} \in N(f)$ and $N(f)$ is normal.

It is reasonable now to ask what happens when $N(f)$ is "factored out". Here is the answer.

Theorem 15.14. [213]. *Let f be a d'Alembert function on G. Let $\phi : G \to \frac{G}{N(f)} =: \tilde{G}$ be the canonical epimorphism of groups. Define $\tilde{f}$ on $\tilde{G}$ by the rule*

$$\tilde{f}(\phi(x)) := f(x) \quad \textit{for all } x \in G.$$

Then $\tilde{f}$ is a function on $\tilde{G}$. Indeed $\tilde{f}$ is a basic d'Alembert function. Thus

$$f = \tilde{f} \circ \phi$$

is a factorization of f into an epimorphism followed by a basic d'Alembert function.

Proof. First we show that $\tilde{f}$ is a function. So suppose that $\phi(x) = \phi(y)$ for $x,\ y$ in G; we have to show that $f(x) = f(y)$. Now $\phi(x) = \phi(y)$ if and only if $x^{-1}y \in N(f)$. Hence $f(x) = f(xx^{-1}y) = f(y)$, as claimed.

Next we show that $\tilde{f}$ is a d'Alembert function. For all $x,\ y$ in G,

$$\begin{aligned}\tilde{f}(\phi(x)\phi(y)) + \tilde{f}(\phi(x)\phi(y)^{-1}) &= \tilde{f}(\phi(xy)) + \tilde{f}(\phi(xy^{-1}))\\ &= f(xy) + f(xy^{-1})\\ &= 2f(x)f(y)\\ &= 2\tilde{f}(\phi(x))\tilde{f}(\phi(y)),\end{aligned}$$

and $\tilde{f}(\phi(e)) = f(e) = 1$.

Finally, we show that $\tilde{f}$ is basic. Suppose that $\tilde{f}(\phi(x)\phi(y)) = \tilde{f}(\phi(x))$ for all x in G. Then $f(xy) = f(x)$ for all x in G, so $y \in N(f)$ and $\phi(y) = \phi(e)$. Hence $N(\tilde{f}) = \langle\phi(e)\rangle = \langle\overline{e}\rangle$ and $\tilde{f}$ is basic.

We remark (see [772, Lemma 6.4]) that if h is a homomorphism from G to $\mathbb{C}^*$, then $\tilde{\tilde{h}} = \hat{\hat{h}}$, where $\tilde{h} : G/\ker(h) \to \mathbb{C}^*$.

For cosh, we see that $\phi(x) = \exp(x)$ is the canonical epimorphism that is an epimorphism of $\mathbb{R}$ with the positive real numbers, and similarly, for cos we have $\phi(x) = \exp(ix)$. We see further that $\tilde{\cosh}$ and $\tilde{\cos}$ are restrictions of the function

$$c : z \to \frac{z + z^{-1}}{2}$$

of $\mathbb{C}^*$ to $\mathbb{C}$ to subgroups. So this equation gives the "universal" Abelian d'Alembert function in the sense that every d'Alembert function on an Abelian group is of the form $c \circ h$, where h is a homomorphism into $\mathbb{C}^*$. □

We now turn to more general basic d'Alembert functions and see that they have some pleasant properties.

In analyzing Kannappan's proof with a view to extending the result to more general groups, the following has proved useful [771].

Definition. Let f be a d'Alembert function on G. Then

$$Z(f) : \{u \in G : f(xuy) = f(xyu),\ \forall\, x, y \in G\}.$$

Our principal result is the following.

Proposition 15.13c. [213]. *Suppose f is a basic d'Alembert function on G. Then $Z(f) = Z(G)$, the centre of G.*

Proof. Since $xuy = xyu$ for all $u \in Z(G)$, it is clear that $Z(G)$ is contained in $Z(f)$.

Assume now that $u \in Z(f)$. Then, for all $x, y \in G$,

$$f(x[y,u]) = f(xyuy^{-1}u^{-1}) = f(xyy^{-1}u^{-1}u) = f(x),$$

so $[y,u] \in N(f) = \langle e \rangle$ for all $y \in G$. Thus $u \in Z(G)$. This shows that $Z(f)$ is contained in $Z(G)$, and we are done.

The next result will be used frequently when G is finite.

Proposition 15.13d. [213]. *Suppose $f : G \to \mathbb{C}$ is a bounded d'Alembert function. Then*

$$N(f) = \{y \in G,\ \ f(y) = 1\}.$$

In particular, if f is basic,

$$f(y) = 1 \Longleftrightarrow y = e.$$

If f is basic and H is a subgroup of G, then $f|_H$ is a basic d'Alembert function on H.

That the basic d'Alembert function on G is bounded is sufficient to ensure that every subgroup of G is a d'Alembert group. However, it is not necessary, for every subgroup of $\mathbb{C}^*$ is a d'Alembert group with function $z \to \frac{z+z^{-1}}{2}$ even though this function is not bounded on $\mathbb{C}^*$.

15.4.2 D'Alembert Groups: Examples

In this section, we give explicit examples of d'Alembert groups as groups of 2×2 matrices, showing in particular that the binary dihedral groups are d'Alembert groups, as are the binary polyhedral groups (corollaries to Proposition 15.13e).

Our results all depend on the fact that the rescaled trace function when restricted to matrices of determinant 1 is a d'Alembert function. (It is convenient, though nonstandard, to denote the identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

by i.) This we show now.

Lemma. *Define* $c : SL_2(\mathbb{C}) \to \mathbb{C}$ *by*

$$c(X) := \frac{1}{2}\text{trace}\left(\frac{X + X^{-1}}{2}\right)$$

for all $X \in SL_2(\mathbb{C})$. *Then* c *is a d'Alembert function on* $SL_2(\mathbb{C})$.

Remark. We see from the definition of c that $c(X^{-1}) = c(X)$ for all X; also c is central since trace is. These are properties shared by all d'Alembert functions. We now use our function c and its restriction to produce d'Alembert groups.

Proposition 15.13e. [213]. *The group* $SU_2(\mathbb{C})$ *given by*

$$\left\{\begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1\right\}$$

and all of its subgroups are d'Alembert groups.

Proof. Let G be a subgroup of $SU_2(\mathbb{C})$. Define $f : G \to \mathbb{C}$ by $f(X) = c(X)$ for all $x \in \mathbb{C}$. We claim that f is a basic d'Alembert function on G. Since c is a d'Alembert function, so is f. Now suppose

$$Y = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

is in $N(f)$. Then $f(Y) = 1$, so $a + \overline{a} = 2$. But $|a| \leq 1$, so we must have $a = 1$. Thus $Y = i$ (identity). This shows that $N(f) = \langle i \rangle$, so f is basic and G is a d'Alembert group.

Remark. Since $|c(X)| \leq 1$ for all $x \in SU_2(\mathbb{C})$, we see that $c : SU_2(\mathbb{C}) \to \mathbb{C}$ is an example of a bounded d'Alembert function.

Corollary (i). *The cyclic group* Z_n *of order* n *is a d'Alembert group.*

Proof. Let $w \in \mathbb{C}$ be a primitive nth root of unity. Put

$$G = \left\langle \begin{bmatrix} w & 0 \\ 0 & \overline{w} \end{bmatrix} \right\rangle.$$

Then G is a subgroup of $SU_2(\mathbb{C})$ and $G \simeq Z_n$.

Corollary (ii). *The binary dihedral group* $\langle 2, 2, n\rangle$ *of order* $4n$ *is a d'Alembert group.*

Proof. Let $\omega \in \mathbb{C}^*$ be a primitive $(2n)$th root of unity, so $\omega^n = -1$, and put

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & \omega \\ -\overline{\omega} & 0 \end{bmatrix}, \quad T = \begin{bmatrix} \omega & 0 \\ 0 & \overline{\omega} \end{bmatrix}.$$

Then

$$R^2 = S^2 = T^n = RST = -i = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence $\langle R, S, T\rangle$ is a homomorphic image of $\langle 2, 2, n\rangle$. But $|\langle T\rangle| = 2n$ and $R \notin \langle T\rangle$, so $\langle R, S, T\rangle$ has order at least $4n$. Now $\langle 2, 2, n\rangle$ has order $4n$ [178, Ch. 6.5], so $\langle R, S, T\rangle \simeq \langle 2, 2, n\rangle$. Hence $\langle 2, 2, n\rangle$ is a d'Alembert group.

Corollary (iii). *The binary polyhedral groups* $\langle 2, 3, n\rangle$ *for* $n = 3, 4, 5$ *are d'Alembert groups.*

Proof. We choose generators in $SU_2(\mathbb{C})$ to satisfy the relations for $\langle 2, 3, n\rangle$ as follows. Choose $t \in \mathbb{C}$ such that when $n = 3$, $t = 1$; $n = 4$, $t^2 = 2$; and $n = 5$, $t^2 = t + 1$. Put $\tau = t + i\sqrt{3 - t^2}$. Then

$$R := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad S = \frac{1}{2}\begin{bmatrix} 1 & \tau \\ -\overline{\tau} & 1 \end{bmatrix}, \quad T = \frac{1}{2}\begin{bmatrix} \tau & 1 \\ -1 & \overline{\tau} \end{bmatrix}$$

satisfy, as can be verified, the relations

$$R^2 = S^3 = T^n = RST = -i = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So the subgroup $\langle R, S, T\rangle$ of $SU_2(\mathbb{C})$ satisfies the relations of $\langle 2, 3, n\rangle$. Again it is straightforward to check that $\langle R, S, T\rangle$ has the correct order, namely 24 for $n = 3$, 48 for $n = 4$ and 120 for $n = 5$.

At this point, we remark that another realization of $\langle 2, 3, 5\rangle$ is used by group theorists, specifically Suzuki [781]. Let $\mathbb{F}_p$ denote the finite field with p elements, where p is a prime. Then

$$SL_2(p) := SL_2(\mathbb{F}_p) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{F}_p,\ ad - bc = 1 \right\}.$$

We have an isomorphism $\langle 2, 3, 5\rangle \cong SL_2(5)$. Just take

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 3 & 1 \\ 3 & 3 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

in $SL_2(5)$. Then

$$R^2 = S^3 = T^5 = RST = -i = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

15.4.3 D'Alembert Groups: Generalities

We will show later that a finite group is a d'Alembert group only if it has a very particular structure. In this section, we develop tools to delimit the structure of finite d'Alembert groups.

Our first result concerns the centre $Z(G)$ of a d'Alembert group (not necessarily finite). We know from Proposition 15.13c that $Z(f) = Z(G)$ when f is basic. We will use this fact in quoting results from Stetkaer [772, 771].

Proposition. [213]. *Let G be a non-Abelian d'Alembert group. Then*

$$Z(G) = \{u \in G : u^2 = e\},$$

and $Z(G)$ has at most two elements.

Proposition. *Let G be a finite d'Alembert group. Then, for all $x, y \in G$, if $yxy^{-1} \in \langle x \rangle$, then $yxy^{-1} \in \{x, x^{-1}\}$.*

Proposition. *Let G be a finite d'Alembert group. Then*

(i) *every Abelian subgroup of G is cyclic,*
(ii) *$Z(G) = G$ or $Z(G) \leq 2$, and*
(iii) *if $yxy^{-1} = x^k$, then $yxy^{-1} \in \{x, x^{-1}\}$.*

15.4.4 Solvable Finite d'Alembert Groups

Wolf [833] has provided us with a convenient and easy-to-use formulation of Zassenhaus's results characterizing the solvable finite groups that have every Abelian subgroup cyclic. He divides them into four types (I, II, III, IV), giving generators and relations as well as arithmetical conditions. We will use these types to determine necessary conditions for a finite solvable group to be a d'Alembert group.

Proposition 15.15. [213]. *Let G be a Type* I *group. Thus G has order mn, $G = \langle a, b \rangle$ with $a^m = e$, $b^n = e$, $bab^{-1} = a^r$, where $m \geq 1$, $n \geq 1$, m and $n(r-1)$ are relatively prime, and $r^n \equiv 1 \mod m$.*

G is a d'Alembert group only if either $m = 1$ or m is odd, $m \geq 3$, $n = 4$, and $r = -1$. In the first case, $G \cong Z_n$, and in the second, $G \cong \langle 2, 2, m \rangle$.

Proposition 15.15a. [213]. *Let G be a Type* II *group. Thus G has order $2mn$, $G = \langle a, b, C \rangle$ with $\langle a, b \rangle$ a Type* I *group as in* Proposition 15.15, *and the additional generator C satisfying $C^2 = b^{\frac{n}{2}}$, $CaC^{-1} = a^\ell$, $CbC^{-1} = b^k$, where $n = 2^u v$ with v odd and $u \geq 2$, as well as $\ell^2 \equiv r^{k-1} \equiv 1 \mod m$, $k \equiv -1 \mod 2^u$, and $k^2 \equiv 1 \mod n$.*

G is a d'Alembert group only if it is isomorphic to $\langle 2, 2, \frac{mn}{2} \rangle$, where n is divisible by 4.

Proposition 15.15b. [213]. *Let G be a Type* III *group. Thus G is of order $8mn$, $G = \langle a, b, p, q \rangle$, where $\langle a, b \rangle$ is of Type* I, *and the additional generators satisfy*

$$p^4 = e, \quad p^2 = q^2 = (pq)^2, \quad ap = pa,$$
$$aq = qa, \quad bpb^{-1} = q, \quad bqb^{-1} = pq,$$

with n odd and divisible by 3. *G is a d'Alembert group only if it is isomorphic to* $\langle 2, 2, 3\rangle$, *the binary tetrahedral group of order* 24.

Finally, we turn to Type IV groups.

Proposition 15.15c. [213]. *Let G be a Type* IV *group. Thus G is of order* $16mn$, $G = \langle a, b, p, q, C\rangle$, *where* $\langle a, b, p, q\rangle$ *is a Type* III *group, and the additional generator C satisfies*

$$C^2 = p^2,\ CpC^{-1} = qp,\ CqC^{-1} = q^{-1},\ CaC^{-1} = a^{\ell},\ CbC^{-1} = b^k,$$

with $k^2 \equiv 1 \mod n$, $k \equiv -1 \mod 3$, *and* $r^{k-1} \equiv \ell^2 \equiv 1 \mod m$.

G is a d'Alembert group only if it is isomorphic to $\langle 2, 3, 4\rangle$, *the binary octahedral group of order* 48.

It must be noted that Wolf [833], at the end of the first section of his Chapter 6, mentions all of our groups—the cyclic, binary dihedral, binary tetrahedral, and binary octahedral—as occurring in the Type I, II, III, IV classification. In particular, he mentions that $\langle 2, 3, 3\rangle$ is Type III with $m = 1$ and $n = 3$, and $\langle 2, 3, 4\rangle$ is Type IV with $m = 1$, $n = 3$, and $k = -1$.

Proposition. *Let G be a finite solvable d'Alembert group. Then $Z(G) = G$ or $Z(G) \cong Z_2$.*

Corollary. *If $|G|$ is not divisible by* 4, *then G is Abelian.*

15.4.5 Nonsolvable Finite d'Alembert Groups

We show in this section that, up to isormorphism, there is only one nonsolvable finite d'Alembert group—the binary icosahedral group $\langle 2, 3, 5\rangle$, or equivalently $SL_2(5)$.

It is the matrix version of $\langle 2, 3, 5\rangle$ that is employed in this section because the group-theoretic result that we quote, Suzuki's Theorem, uses it in a significant way to describe the action of an involute automorphism.

Suzuki's Theorem. [781, Thm. E]. *Every Abelian subgroup of a nonsolvable group G of finite order is cyclic if, and only if, either*

(a) *$G \cong H \times SL_2(p)$, $p > 3$ is prime, H is a metacyclic group, and $|H|$ and p are relatively prime or*
(b) *G is generated by $\ell \cong SL_2(p)$, where $p > 3$, and three elements R, S, T, where $R^m = S^m = T^4 = i$, $RSR^{-1} = S^r$, $TRT^{-1} = R$, $TST^{-1} = S^{-1}$, and $RX = XR$, $SX = XS$, $TXT^{-1} = \theta(X)$ for all $X \in \ell$, $T^2 \in Z(\ell)$, and the involute automorphism $\theta : \ell \to \ell$ is given by*

$$\theta \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -\omega^{-1}c \\ -\omega b & a \end{bmatrix},$$

where $w \in \mathbb{F}_p^*$ is a generator of $\mathbb{F}_p^*$. Here m, n, r are integers with m and n relatively prime, mn is prime to $p(p^2-1)$, and $r^m \equiv 1 \mod n$.

We recall that H (in (a) *above) is metacyclic if, and only if, all Sylow subgroups are cyclic* [178, Ch. 1, Sect. 9]. *Note that a metacyclic group is solvable. Since $SL_2(p)$ is central to Suzuki's Theorem, it is convenient to determine necessary conditions on the prime p such that $SL_2(p)$ is a d'Alembert group.*

Proposition. [213]. *Let p be a prime. Then $SL_2(p)$ is a d'Alembert group only if $p \in \langle 3, 5\rangle$.*

Proposition 15.15d. [213]. *Let G be a nonsolvable finite group all of whose Abelian subgroups are cyclic. G is a d'Alembert group only if G is isomorphic to $\langle 2, 3, 5\rangle$—the binary icosahedral group.*

Proof of Theorem 15.14; Consequences

We state Theorem 15.14 again in a slightly condensed form to remind us of what we have to prove.

Theorem 15.14. [213]. *The finite group G is a d'Alembert group if, and only if, it is cyclic, binary dihedral, or binary polyhedral.*

Proof. The sufficiency is proved in the three corollaries to Proposition 15.13e, where all groups mentioned are exhibited as subgroups of $SU_2(\mathbb{C})$.

The necessity is proved in Propositions 15.15, 15.15a, 15.15b, and 15.15c for solvable groups and Proposition 15.15d for nonsolvable groups.

One easy consequence of Theorem 15.14 is the following.

Proposition. *The finite subgroups of $SL_2(\mathbb{C})$ are precisely the finite d'Alembert groups.*

Stetkaer [772, Theorem 7.2] characterized the d'Alembert functions on step 2 nilpotent groups. Theorem 15.14 shows that it is sufficient to characterize the basic d'Alembert functions, so the critical result is (cf. [772, Lemma 8.4]) the following.

Proposition 15.16. (Corovei [169]). *If G is a step 2 nilpotent d'Alembert group, then G is isomorphic to the binary dihedral group of order 8, the quaternion group.*

Corollary. [169, Theorem 7.2]. *If G is a step 2 nilpotent group and f is a d'Alembert function on G, then either f is classical or there is an epimorphism $\varphi : G \to \langle 2, 2, 2\rangle$ such that $f = \tilde{f} \circ \varphi$, where $\tilde{f}$ is the unique basic d'Alembert function on $\langle 2, 2, 2\rangle$.*

Proof. As in Theorem 15.14, let φ denote the canonical epimorphism from G to $\frac{G}{N(f)} =: \tilde{G}$. Since G is a step 2 nilpotent group, $\tilde{G}$ is either step 1 (Abelian) or step 2 nilpotent.

In the first case, Kannappan's Theorem yields $h : \tilde{G} \to \mathbb{C}^*$ such that $f = \widehat{h \circ \varphi}$. In the second case, $h : \tilde{G} \cong \langle 2, 2, 2\rangle$ by our proposition.

It seems likely that if G is a step n ($n \geq 2$) nilpotent d'Alembert group, then $h : \tilde{G} \cong \langle 2, 2, 2^{n-1} \rangle$, a generalized quaternion group, so nilpotency is a strong finiteness condition.

Finally, it may be possible to generalize Kannappan's Theorem as follows. Let f be a d'Alembert function on G. Then there is a homomorphism $h : G \to SL_2(\mathbb{C})$ such that, for all $x \in G$,

$$f(x) = \frac{1}{2}\text{trace}\left(\frac{h(x) + h(x)^{-1}}{2}\right).$$

This truly extends Kannappan via the embedding of $\mathbb{C}^*$ in $SL_2(\mathbb{C})$ by

$$z \to \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}.$$

We conclude by commenting that if g satisfies the "long" d'Alembert functional equation, we can define $N(g)$, the nub of g, and prove $Z(g) = Z(G)$ when g is basic. However, the noncentrality of g so far poses a barrier to further results.

15.5 Polynomials Revisited

We studied the mean value theorem and some of its generalizations in Chapter 7. Here we consider more generalizations of the mean value theorem.

Theorem 15.17. *Let $f, \phi, h, \lambda : \mathbb{R} \to \mathbb{R}$ such that*

$$\frac{f(x) - f(y)}{\lambda(x) - \lambda(y)} = \phi(x) + h(y), \quad \text{for } x, y \in \mathbb{R},\ x \neq y, \tag{15.42}$$

holds with $\lambda(x) \neq \lambda(y)$ for $x \neq y$. Then

$$\begin{cases} f(x) = \lambda(x)(a + b - c\lambda(x)) + a', \\ h(x) = a - c\lambda(x), \\ \phi(x) = b - c\lambda(x), \quad x \in \mathbb{R}, \end{cases} \tag{15.42a}$$

and $\lambda(x)$ is arbitrary and a, b, c are real constants.

Proof. It is easy to see that $f - f(0)$ and $\lambda - \lambda(0)$ is also a solution of (15.42). So, without loss of generality, assume that $f(0) = 0 = \lambda(0)$.

Setting $y = 0$ and $x = 0$ in (15.42), we have

$$f(x) = \lambda(x)(\phi(x) + a), \quad (a = h(0)) \quad \text{for } x \in \mathbb{R},\ x \neq 0, \tag{15.43}$$

$$f(y) = \lambda(y)(h(y) + b), \quad (b = \phi(0)) \quad \text{for } y \in \mathbb{R},\ y \neq 0. \tag{15.43a}$$

Note that (15.43) and (15.43a) hold for $x = 0$, $y = 0$ and that $\lambda(x) \neq 0$ for $x \neq 0$ and λ is 1–1.

Substituting these values of $f(x)$ and $f(y)$ in (15.42) yields

$$\lambda(x)(-h(y)+a) = \lambda(y)(-b+\phi(x)) \quad \text{for } x, y \in \mathbb{R}^*. \tag{15.44}$$

If $h(x) = a$, then $\phi(x) = b$ (since $\lambda(y) \neq 0$ for $y \neq 0$) and $f(x) = \lambda(x) + (a+b)$ (use either (15.43) or (15.43a)), which is (15.42a) with $c = 0$.

Otherwise, from (15.44) there follows

$$\lambda(x) = c_1(a - h(x)) \quad \text{for } x \in \mathbb{R}. \tag{15.45}$$

Interchange x and y in (15.42) to get

$$\phi(x) + h(y) = \phi(y) + h(x);$$

that is,

$$\phi(x) = c_2 + h(x) \quad \text{for } x \in \mathbb{R}. \tag{15.45a}$$

Now (15.45), (15.45a), and (15.43) give (15.42a), the solution sought. □

Remark. It is easy to see that (15.42) has many solutions dependent on λ and the solutions are obtained without assuming any regularity condition on $f,\ \lambda,\ \phi,\ h$. The mean value theorem is one among the solutions.

The mean value theorem is obtained from Theorem 15.17 by taking $\lambda(x) = x$. Then

$$h(x) = a - \frac{x}{c}, \quad \phi(x) = c_2 + a - \frac{x}{c_1}, \quad f(x) = x\left(2a + c_2 - \frac{x}{c_1}\right).$$

Even though

$$\lambda(x) = x^2, \quad h(x) = a - \frac{x^2}{c_1},$$
$$\phi(x) = c_2 + a - \frac{x^2}{c_1}, \quad \text{and} \quad f(x) = x^2\left(2a + c_2 - \frac{x^2}{c_1}\right)$$

satisfy (15.42), it does not constitute a solution since $\lambda(x) = \lambda(-x)$ for $x \neq -x$.

We can create many more solutions of (15.42) by choosing $\lambda(x)$ suitably with $(\lambda(0) = 0),\ \lambda(x) \neq \lambda(y)$, for $x \neq y$. For example,

$$\lambda(x) = e^x - 1, \quad f(x) = (e^x - 1)(a + b - c(e^x - 1)),$$
$$h(x) = a - c(e^x - 1), \quad \phi(x) = b - c(e^x - 1)$$

constitutes a solution of (15.42).

Another generalization is considered in the following.

Theorem 15.18. *Suppose* $f, g, h, \phi, \lambda : \mathbb{R} \to \mathbb{R}$ *satisfy*

$$\frac{f(x) - g(y)}{\lambda(x) - \lambda(y)} = \phi(x) + h(y) \quad \text{for } x \neq y,\ x, y \in \mathbb{R}, \tag{15.46}$$

with $\lambda(x) \neq \lambda(y)$ *for* $x \neq y$. *Then* $f,\ g,\ h,\ \phi$ *are given by*

$$\begin{aligned} f(x) &= \lambda(x)(a + b + c\lambda(x)) + a' = g(x), \\ \phi(x) &= c\lambda(x) + a, \\ h(x) &= c\lambda(x) + b, \quad \text{for } x \in \mathbb{R}, \end{aligned} \tag{15.47}$$

where $\lambda(x)$ *is arbitrary with* $\lambda(x) \neq \lambda(y)$ *for* $x \neq y$ *and* $a,\ b,\ c,\ a'$ *are constants.*

Proof. Note that $\lambda - \lambda(0)$ is also a solution of (15.46). So let us assume that $\lambda(0) = 0$. Also note that $\lambda(x) \neq 0$ for $x \neq 0$ and λ is one to one.

Putting $y = 0$ and $x = 0$ separately in (15.46), we get

$$f(x) = \lambda(x)(\phi(x) + a) + c_1 \quad \text{for } x \neq 0, \tag{15.48}$$

$$g(y) = \lambda(y)(h(y) + b) + c_2 \quad \text{for } y \neq 0, \tag{15.48a}$$

where $f(0) = c_1,\ g(0) = c_2,\ h(0) = a,\ \phi(0) = b$.

Substitution of (15.48) and (15.48a) in (15.46) gives

$$\lambda(x)(\phi(x)+a)+c_1-\lambda(y)(h(y)+b)-c_2 = (\lambda(x)-\lambda(y))(\phi(x)+h(y)), \tag{15.49}$$

$x, y \neq 0,\ x \neq y$. Interchange x and y in (15.49).

(i) First add the resultant to (15.49) to have

$$\lambda(x)(\alpha(y) - \alpha(0)) = \lambda(y)(\alpha(x) - \alpha(0)),$$

where $\alpha(x) = \phi(x) + h(x)$, from which results (since $\lambda(x) \neq 0,\ x \neq 0$)

$$\phi(x) + h(x) = \alpha(x) = c_3\alpha(x) + \alpha(0) \quad \text{for } x \neq 0 \tag{15.50}$$

(which also holds for $x = 0$).

(ii) Subtracting the resultant from (15.49), we obtain

$$\lambda(x)(\beta(y) - \beta(0)) + \lambda(y)(\beta(x) - \beta(0)) + 2(c_2 - c_1) = 0,$$

for $x \neq 0,\ y \neq 0,\ x \neq y$, where $\beta(x) = \phi(x) - h(x),\ x \in \mathbb{R}$, or

$$\phi(x) - h(x) = \beta(x) = c_4\lambda(x) + c_5, \quad x \neq 0 \tag{15.50a}$$

(use $\lambda(y) \neq 0,\ y \neq 0$).

From (15.50) and (15.50a), we have

$$\phi(x) = b_1\lambda(x) + b_2, \quad h(x) = b_3\lambda(x) + b_4, \quad x \neq 0, \tag{15.50b}$$

where b_i ($i = 1$ to 4) are constants.

Utilizing (15.47) and (15.50b), (15.46) yields

$$\lambda(x)(a - b_4) + \lambda(y)(b_2 - b) = c_2 - c_1 + (b_3 - b_1)\lambda(x)\lambda(y), \quad x, y \neq 0.$$

Fixing y ($\neq 0$) as y_0, we get

$$\lambda(x)(a - b_y - \lambda(y_0)(b_3 - b_1)) = c_2 - c_1 - \lambda(y_0)(b_2 - b).$$

Since λ is not constant, we get

$$\begin{aligned} a - b_4 &= \lambda(y_0)(b_3 - b_1) \\ &= \lambda(y_1)(b_3 - b_1) \quad (y_1 \neq y_0). \end{aligned}$$

Since λ is one to one, this leads to $b_3 = b_1$, $a = b_4$ and then $c_2 = c_1$, $b_2 = b$, giving the asserted solution (15.47) (cf. (15.42a)). □

Remark. As before, note that (15.46) has many solutions dependent on λ, and the mean value theorem is among them.

Now we consider the functional equation

$$\frac{f(x) - g(y)}{\lambda(x) - \lambda(y)} = h(xy), \quad x, y \in \mathbb{R},\ x \neq y, \tag{15.51}$$

where $f, g, h, \lambda : \mathbb{R} \to \mathbb{R}$ with $\lambda(x) \neq \lambda(y)$ for $x \neq y$, and show that

$$f(x) = c\lambda(x) + b = g(x), \quad \lambda \text{ arbitrary}. \tag{15.51a}$$

Notice that $\lambda - \lambda(0)$ is also a solution of (15.51), so take $\lambda(0) = 0$. Interchanging x and y in (15.51) results in

$$f(x) - g(y) = g(x) - f(y);$$

that is, $f = g$. Set $y = 0$ in (15.51) to have $f(x) = b + a\lambda(x)$, where $b = f(0) = g(0)$ and $h(0) = a$. This proves this result.

15.6 Inner Products Revisited

First we prove the following theorem, which is used in Theorem 15.20.

Theorem 15.19. *The functions $t_i : \mathbb{R} \to \mathbb{R}$ ($i = 1, 2, \ldots, n$) ($n \geq 3$) satisfy the equation*

$$\sum_{i=1}^{n} t_i(y_i) = d, \quad y_i \in \mathbb{R}, \tag{15.52}$$

with $\sum_{i=1}^{n} y_i = 0$, *provided there exist an additive* A *and constants* a_i *such that*

$$t_i(x) = A(x) + a_i, \quad \textit{for } x \in \mathbb{R}, \tag{15.52a}$$

with $\sum_{i=1}^{n} a_i = d$.

Proof. Let $y_1 = x,\ y_2 = y,\ y_3 = -x - y,\ y_4 = 0 = \cdots = y_n$ to get

$$t_1(x) + t_2(y) + t_3(-x - y) = c_1 \quad \text{for } x, y \in \mathbb{R}, \tag{15.53}$$

which is a Pexider equation.

Interchange x and y in (15.53) to have

$$t_2(x) - t_1(x) = \text{constant} = c_2, \tag{15.53a}$$

so that (15.53) becomes

$$t_1(x) + t_1(y) + t_3(-x - y) = c_3. \tag{15.53b}$$

Now $y = 0$ yields

$$t_1(x) + t_3(-x) = c$$

and

$$t_1(x) + t_1(y) = t_1(x + y) + c_4 \quad \text{for } x, y \in \mathbb{R}.$$

Thus there is an additive $A : \mathbb{R} \to \mathbb{R}$ such that

$$t_1(x) = A(x) + a_1.$$

From (15.53a), (15.53b), and (15.52), we get (15.52a) the solution sought.

Remark. Compare this result with Theorem 1.76.

In Chapter 5, we considered the functional equation (Result 5.7a)

$$\sum_{i=1}^{n} \phi(x_i - \overline{x}) = \sum_{i=1}^{n} \phi(x_i) - n\psi(\overline{x}) \tag{5.25a}$$

for x_i in an n.l.s. K. Here we consider the most general equation of this type,

$$\sum_{i=1}^{n} \phi_i(x_i - \overline{x}) = \sum_{i=1}^{n} \psi_i(x_i) + \psi(\overline{x}), \tag{15.54}$$

for $x_i \in K$, an n.l.s.,

$$\overline{x} = \frac{x_1 + \cdots + x_n}{n}.$$

Theorem 15.20. (See Kannappan [479].) *Functions* $\phi_i, \psi_i, \psi : K \to \mathbb{R}$ $(i = 1, 2, \ldots, n)$ *satisfy the functional equation* (15.54) *for* $n \geq 3$ *over a real n.l.s.* K *are given by*

$$\begin{aligned} \phi_i(x) &= B(x,x) + (A_i + A_i')(x) + b_i, \\ \psi_i(x) &= B(x,x) + A_i(x) + c_i, \\ \psi(x) &= -nB(x,x) - \sum_{i=1}^{n} A_i(x) - \sum_{i=1}^{n}(b_i - c_i), \end{aligned} \tag{15.55}$$

where A_i, A_i' *are additive,* B *is biadditive, and* b_i, c_i *are constants.*

Proof. We present a proof different from that in [479]. Putting $x_1 = x = x_2 = \cdots = x_n$ in (15.54) yields

$$\psi(x) = \sum_{i=1}^{n} c_i - \sum_{i=1}^{n} \psi_i(x), \quad x \in K, \tag{15.56}$$

where $\phi_i(0) = c_i$.

We have to treat the cases where n is even or n is odd.

Case (i). $n = 2k$, n even.

Set $x_1 = \cdots = x_k = 2x$, $x_{k+1} = \cdots = x_{2k} = 2y$ in (15.54) to get

$$(\phi_1 + \cdots + \phi_k)(x - y) + (\phi_{k+1} + \cdots + \phi_{2k})(y - x) = \beta(2x) + \delta(2y) + \psi(x + y), \tag{15.57}$$

where

$$\beta(x) = (\psi_1 + \cdots + \psi_k)(x), \quad \delta(x) = (\psi_{k+1} + \cdots + \psi_{2k})(x). \tag{15.57a}$$

Case (ii). $n = 2k + 1$, n odd.

Put

$$x_1 = \cdots = x_k = 2x, \quad x_{k+1} = \cdots = x_{2k} = 2y, \quad x_{2k+1} = x + y,$$

so that (15.54) becomes

$$\begin{aligned} (\phi_1 + \cdots + \phi_k)(x - y) + (\phi_{k+1} + \cdots + \phi_{2k})(y - x) + \phi_{2k+1}(0) \\ = \beta(2x) + \delta(2y) + \psi_{2k+1}(x + y) + \psi(x + y), \end{aligned}$$

which is the same as (15.57). So we solve (15.57) setting

$$2q(x) = (\phi_1 + \cdots + \phi_k)(x) + (\phi_{k+1} + \cdots + \phi_{2k})(-x), \quad x \in K. \tag{15.57b}$$

Equation (15.57) can be written as

$$2q(x - y) = \beta(2x) + \delta(2y) + \psi(x + y), \quad x, y \in K. \tag{15.57c}$$

With $x + y = u,\ x - y = v$, this equation becomes

$$\beta(u+v) + \delta(u-v) = 2q(v) - \psi(u). \tag{15.58}$$

Put $u = 0$ in (15.58) to get

$$\beta(v) + \delta(-v) = 2q(v) - \psi(0);$$

that is, by (15.56) and (15.57a),

$$\begin{aligned} \beta(u+v) + \delta(u-v) &= \beta(v) + \delta(-v) + \psi(0) + (\beta+\delta)(u) - \sum_1^n c_i \\ &= (\beta+\delta)(u) + \beta(v) + \delta(-v) - (\beta+\delta)(0), \\ &\quad u, v \in K. \end{aligned} \tag{15.59}$$

Change v to $-v$ in (15.59) to have

$$\beta(u-v) + \delta(u+v) = (\beta+\delta)(u) + \beta(-v) + \delta(v) - (\beta+\delta)(0). \tag{15.59a}$$

First adding (15.59) and (15.59a) results in

$$\begin{aligned} (\beta+\delta)(u+v) + (\beta+\delta)(u-v) = 2(\beta+\delta)(u) + (\beta+\delta)(v) \\ + (\beta+\delta)(-v) - 2(\beta+\delta)(0), \end{aligned}$$

which is the same as (4.22) (see [230, 241], Remark 4.3).

There exists an additive A and a symmetric biadditive B such that

$$(\beta+\delta)(x) = B(x,x) + A(x) + c' \quad \text{for } x \in K. \tag{15.60}$$

Second subtract (15.59a) from (15.59) to get

$$\begin{aligned} &(\beta-\delta)(u+v) - (\beta-\delta)(u-v) \\ &\quad = (\beta-\delta)(v) - (\beta-\delta)(-v) \\ &\quad = (\beta-\delta)(-u+v) - (\beta-\delta)(-u-v); \end{aligned} \tag{15.60a}$$

that is,

$$(\beta-\delta)(u+v) + (\beta-\delta)(-u-v) = (\beta-\delta)(u-v) - (\beta-\delta)(-u+v)$$

or

$$(\beta-\delta)(x) + (\beta-\delta)(-x) = \text{constant} \quad \text{for } x \in K.$$

So (15.60a) becomes

$$(\beta-\delta)(v+u) + (\beta-\delta)(v-u) = 2(\beta-\delta)(v),$$

which is Jensen (J), so that

$$(\beta - \delta)(x) = A'(x) + c'' \quad \text{for } x \in K. \tag{15.60b}$$

From (15.60) and (15.60b) results (add)

$$(\psi_1 + \cdots + \psi_k)(x) = \beta(x) = \frac{1}{2}B(x,x) + \frac{1}{2}(A + A')(x) + \frac{1}{2}(c' + c''). \tag{15.61}$$

This division of ψ_i's is arbitrary. Similarly,

$$(\psi_2 + \cdots + \psi_{k+1})(x) = \frac{1}{2}B(x,x) + \frac{1}{2}(A_1' + A_1'')(x) + c'''$$

(B is the same by considering $\psi_1 + \cdots + \psi_{2k}$ in both cases); that is,

$$\psi_{k+1}(x) = \psi_1(x) + A_{k+1}'(x) + c_{k+1}' \quad \text{for } x \in K \tag{15.62}$$

($k = 1, 2, \ldots, 2k-1$).

From (15.56) and (15.60), we obtain

$$\psi(x) = -\beta(x,x) - A'(x) + d_1, \tag{15.63}$$

with d some constant.

Substituting $\psi_k(x)$ from (15.62) into (15.57c) and using (15.63), we get

$$\begin{aligned} 2q(x-y) = {} & k\psi_1(2x) + (A_2 + \cdots + A_k)(2x) + c_2 + \cdots + c_k + k\psi_1(2y) \\ & + (A_{k+1} + \cdots + A_{2k})(2y) + c_{k+1} + \cdots + c_{2k} \\ & - B(x+y, x+y) - A(x+y) + d_1. \end{aligned} \tag{15.64}$$

Let $y = 0$ and $x = 0$ separately in (15.64) to have

$$\begin{aligned} 2q(x) = {} & k\psi_1(2x) + (A_2 + \cdots + A_k)(2x) + c_2 + \cdots + c_k \\ & - B(x,x) - A(x) + d_1, \end{aligned} \tag{15.64a}$$

$$\begin{aligned} 2q(-y) = {} & k\psi_1(2y) + (A_{k+1} + \cdots + A_{2k})(2y) + c_{k+1} + \cdots + c_{2k} \\ & - B(y,y)A(y) + d_1. \end{aligned} \tag{15.64b}$$

Hence we have

$$2q(x-y) = 2q(x) + 2q(-y) - B(x,y) - B(y,x) - d_1.$$

Replace y by $-y$ in the equation above and add the resultant to obtain

$$q(x+y) + q(x-y) = 2q(x) + q(y) + q(-y) - d_1. \tag{4.22}$$

Hence, as before, we see that

$$q(x) = B'(x,x) + A'(x) + d_1'$$

for some additive A' and biadditive B'. This $q(x)$ in (15.64a) yields

$$\psi_1(x) = B_1(x,x) + A_1(x) + c_1$$

and then

$$\psi_j(x) = B_1(x,x) + A_j(x) + c_k, \quad j = 1 \text{ to } n, \tag{15.65}$$

where B_1 is biadditive and A_j's are additive.

Substituting of ψ_j in (15.54) and using (15.56) yields

$$\begin{aligned}
\psi(x) &= \sum_1^n \phi_i(0) - nB_1(x,x) \\
&= \sum_{i=1}^n A_i(x) - \sum_1^n c_i, \\
\sum_{i=1}^n \phi_i(x_i - \overline{x}) &= \sum_{i=1}^n [B_1(x_i,x_i) + A_i(x_i) + c_i]n + \sum \phi_i(0) - nB_1(\overline{x},\overline{x}) \\
&\quad - \sum_{i=1}^n A_i(\overline{x}) - \sum_{i=1}^n c_i \\
&= \sum_{i=1}^n [B_1(x_i - \overline{x}, x_i - \overline{x}) + A_i(x_i - \overline{x}) + \phi_i(0)];
\end{aligned}$$

that is,

$$\sum_{i=1}^n [\phi_i(x_i - \overline{x}) - B_1(x_i - \overline{x}, x_i - \overline{x}) - A_i(x_i - \overline{x})] = \sum_{i=1}^n \phi_i(0)$$

or

$$\sum_{i=1}^n t_i(y_i) = d, \quad y_i = x_i - \overline{x}, \quad i = 1 \text{ to } n,$$

where

$$t_i(x) = \phi_i(x) - B_1(x,x) - A_i(x), \quad x \in K.$$

Then Theorem 15.19 shows that

$$\phi_i(x) - B_1(x,x) - A_i(x) = t_i(x) = A(x) + a_i,$$

where A is additive or

$$\phi_i(x) = B_1(x,x) + (A_i + A)(x) + a_i. \tag{15.66}$$

Thus (15.56), (15.65), and (15.66) give the asserted solution (15.55). This proves the theorem. □

16

General Inequalities

This chapter is devoted to functional inequalities. Functional inequalities occur in several fields, such as information theory, inner product spaces, geometry, complex analysis, trigonometry, and Cauchy, gamma, and beta equations. Classical A.M. $\geq$ G.M., logarithmic inequality, multiplicative inequality, trigonometric functional inequality, parallelogram identity, quadratic inequality, inequalities for the gamma and beta functions, Simpson's inequality, inequalities from information theory and mixed theory, and reproducing scoring systems are treated.

16.1 Cauchy Functional Inequalities

First we start off with a familiar classical inequality, A.M. (arithmetic mean) $\geq$ G.M. (geometric mean).

Several proofs are known in the literature. The following simple proof is due to Polya [343] (see also Beckenbach and Bellman [108, p. 4–15]).

Let a_k $(k = 1 \text{ to } n) \in \mathbb{R}_+$. Then

$$\text{A.M.} = \sum_{k=1}^{n} \frac{a_k}{n}$$

and

$$\text{G.M.} = \left(\prod_{k=1}^{n} a_k \right)^{1/n}.$$

In the inequality $e^x \geq 1 + x$, set $x = \frac{a_k}{\text{A.M.}} - 1$ $(k = 1 \text{ to } n)$ to get

$$e^{\frac{a_k}{\text{A.M.}} - 1} \geq \frac{a_k}{\text{A.M.}} \quad \text{and} \quad e^{\sum_1^n \left(\frac{a_k}{\text{A.M.}} - 1 \right)} \geq \prod_{k=1}^{n} \left(\frac{a_k}{\text{A.M.}} \right);$$

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that is,

$$e^0 \geq \frac{\prod_1^n a_k}{(\text{A.M.})^n},$$

which proves the inequality.

The role of the exponential function in this argument could be played as well by any function f with the properties

$$f(x+y) \geq f(x)f(y), \tag{16.1}$$

$$f(x) \geq 1 + x, \tag{16.2}$$

where f is a real-valued function.

It is shown in the following theorem that the *only* function f defined on an interval about 0 that satisfies both (16.1) and (16.2) on that interval is the function $f(x) = e^x$.

Theorem 16.1. (Wetzel [828].) *Let U be an open interval about* 0 *and $f : U \to \mathbb{R}$ satisfy* (16.1) *and* (16.2) *whenever x, y and $x + y$ belong to U. Then $f(x) = e^x$ for each x in U.*

Proof. By (16.1), $f(x) = f((x/2)+(x/2)) \geq [f(x/2)]^2 \geq 0$ for each x in U. Thus, if $f(x_0) = 0$, then $f(x_0/2) = 0$, and by induction, $f(2^{-n}x_0) = 0$ for each positive integer n. But (16.2) demands that $f(x) > 0$ near 0. It follows that $f(x) > 0$ for every x in U.

Next we show that f must be differentiable at each point x of U and that $f'(x) = f(x)$. Using (16.1) and (16.2), we see that, for small h,

$$f(x+h) - f(x) \geq f(x)[f(h) - 1] \geq hf(x).$$

Since $f(x) = f(x+h-h) \geq f(x+h)f(-h) \geq f(x+h)(1-h)$, we have similarly, for small h,

$$f(x+h) - f(x) \leq f(x)\left[\frac{1}{1-h} - 1\right] = h\frac{f(x)}{1-h}.$$

Thus, for small, positive h,

$$f(x) \leq \frac{f(x+h) - f(x)}{h} \leq \frac{f(x)}{1-h},$$

and the reverse inequalities hold for small, negative h. Hence

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists and equals $f(x)$ for each x in U.

It follows that

$$\frac{d}{dx}(e^{-x}f(x)) = e^{-x}[f'(x) - f(x)] = 0$$

for every $x \in U$, and so $f(x) = Ce^x$ for some constant C. Since (16.1) implies $f(0) \leq 1$ and (16.2) implies $f(0) \geq 1$, we have $C = f(0) = 1$, proving the theorem. □

We can generalize this result somewhat by relaxing condition (16.2).

Result 16.2. [828]. *Let $f : U \to \mathbb{R}$ be such that* (16.1) *holds whenever x, y and $x + y$ belong to U. If f is bounded below by a function g that is differentiable at* 0 *and satisfies $g(0) = 1$, then $f(x) = e^{kx}$ for $x \in U$, where $k = f'(0)$.*

Corollary (i). *If f satisfies* (16.1) *on U, $f(0) = 1$, and f is differentiable at* 0, *then $f(x) = e^{kx}$ for $x \in U$.*

The assumption in the hypotheses of this corollary that f be differentiable at 0 cannot be weakened to continuity, for the continuous function $f(x) = (1 + |x|)^{-1}$ satisfies (16.1) for all x and y and fails to be differentiable only at $x = 0$.

The situation should be compared with the state of affairs for the functional equation $f(x + y) = f(x)f(y)$. If f satisfies this equality for all real x and y, then the function $g = \log f$ satisfies the additive functional equation $g(x+y) = g(x)+g(y)$, and if f is assumed to be continuous (even at just one point) or assumed to be bounded on some interval, or even if the graph of f is assumed not to be everywhere dense in the upper half-plane, then $g(x) = kx$ and $f(x) = e^{kx}$ for every x (see [12] and Chapter 1).

No result similar to that of Corollary (i) can be expected in the other extreme case $f(0) = 0$; *any* function f that vanishes for all nonpositive x, is monotonically increasing, and is bounded above by 1 satisfies (16.1). A more interesting example can be constructed as follows. Let h be any differentiable function that satisfies $0 \leq h(x) \leq 0.9$ and $h(5) = h'(5) = h(7) = h'(7) = 0$. Define

$$f(x) = \begin{cases} 0 & \text{for } x < 5, \\ h(x) & \text{for } 5 \leq x \leq 7, \\ 0 & \text{for } 7 < x < 8, \\ 1 - \exp\{-(8-x)^2\} & \text{for } x \geq 8. \end{cases}$$

Then f is a differentiable function with $f(0) = 0$ that satisfies (16.1) for all x and y.

Corollary (ii). *Let F be a function defined on an open interval U about* 0 *that satisfies*

$$F(x + y) \leq F(x) + F(y) \tag{16.3}$$

whenever x, y, and $x + y$ belong to U. If F is bounded above by a function G that is differentiable at 0 *and satisfies $G(0) = 0$ (and, in particular, if F is differentiable at* 0 *and satisfies $F(0) = 0$), then $F(x) = kx$ for all $x \in U$, where k is a constant.*

Proof. Apply Result 16.2 to $f(x) = \exp\{-F(x)\}$ with $g(x) = \exp\{-G(x)\}$.

F satisfying the inequality (16.3) is called a *subadditive function*, and f satisfying the inequality

$$f(x+y) \geq f(x) + f(y) \tag{16.3a}$$

is called a *superadditive function.*

16.2 Subadditive and Superadditive Functions

Obviously the range of subadditive and superadditive functions are real numbers. We will consider them on various domains.

Subadditive functions play an important role in the study of bounded additive transformations on linear spaces. They also play a basic role in the theory of semigroups. A subadditive function need not be continuous (measurable) anywhere. Positive homogeneous subadditive functions have important applications to the theory of convex sets [636] and to the uniqueness theory of differential equations (Hukachura [381]).

Facts (Hille and Phillips [376]).

(i) Let X be an n.l.s. and $F : X \to \mathbb{R}$ satisfying (16.3) be nonnegative outside of a sphere. Then $F(x) \geq 0$ for all x.
Proof. Suppose $F(x) \geq 0$ for all $x,\ \|x\| \geq k$. Then, for an $x,\ \|x\| < k$, there is a positive integer n with $n\|x\| \geq k$. So, $0 \leq F(nx) \leq nF(x)$, implying $F(x) \geq 0$.

(ii) $F : X \to \mathbb{R}$ satisfying (16.3) continuous at 0 with $F(0) = 0$ is continuous for all $x \in X$.
Proof. $F(x) = F(x+h-h) \leq F(x+h)+F(-h)$. Evidently, $F(x)-F(-h) \leq F(x+h) \leq F(x) + F(h)$ for all h. Continuity at 0 implies the continuity at all x.

(iii) Let $\{F_\alpha\},\ \alpha \in$ index set, be a family of subadditive functions defined on $\mathbb{R}$ or $\mathbb{R}^*_+$ or $\mathbb{R}^*_-$. Then $F(t) = \sup_\alpha F_\alpha(t)$ is also subadditive.

(iv) Let $\{F_n\}$ be a pointwise convergent sequence of subadditive functions on $\mathbb{R}$ or $\mathbb{R}^*_+$ or $\mathbb{R}^*_-$. Then $F(t) = \lim_{n\to\infty} F_n(t)$ is also subadditive.

(v) (Rathore [690]). If $g'(x)$ exists on $]a, \infty[$ $(a \geq 0)$, then g is subadditive if $g'(x) < \frac{g(x)}{x}$ and g is superadditive if $g'(x) > \frac{g(x)}{x}$. Let F be subadditive on $\mathbb{R}$ and $F'(x)$ exist on $]a, \infty[$ $(a \geq 0)$. If $F(x) + F(-x) \leq 0$ for $x \in]a, \infty[$, then $F'(x)$ is monotonically nonincreasing on $]a, \infty[$.

(vi) Let F be subadditive and be monotonic for sufficiently large positive values of x. Then $\frac{F(x)}{x} \to$ a finite limit as $x \to +\infty$. If $F(x)$ is monotonic for sufficiently large negative x, then $\frac{F(x)}{x} \to$ a finite limit as $x \to -\infty$.

(vii) Discontinuous solutions of (16.3) (subadditivity). Unlike the additive equation (A), (16.3) (subadditivity) has discontinuous solutions that can be constructed without the aid of the axiom of choice. As an example, define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = m + k$ when $m \leq x \leq m + 1,\ m \in Z$.

For $n \le y < n+1$, $F(x+y) = m+n+k$ or $m+n+1+k$, so that $F(x+y) \le F(x)+F(y)$ if $k \ge 1$.
This solution is a measurable discontinuous solution (cf. (A)).

(a) If $F : \mathbb{R} \to \mathbb{R}$ satisfies (16.3), is odd, and is measurable, then $F(x) = cx$ for some constant c, $x \in \mathbb{R}$. Replace x, y in (16.3) by $-x$ and $-y$, and use F odd to get (A).

(b) Every solution of (16.3) that is even is nowhere negative. $x = 0$ in (16.3) yields $F(0) \ge 0$. Use F even to have $2F(x) = F(x) + F(-x) \ge F(0) \ge 0$.

(viii) (Rosenbaum [710]). Let $\mathbb{R}_+^n = \{(x_1, \ldots, x_n) : x_k > 0\}$.
$F : \mathbb{R}_+^n \to \mathbb{R}$ defined by $F(x) = \left(\sum_1^n a_k x_k\right)^r$, $a_k \ge 0$, is subadditive for $r > 1$ and superadditive for $r > 1$.
$F : \mathbb{R}_+^* \to \mathbb{R}$ defined by $F(t) = \log(t+1)$ is subadditive.
$F : \mathbb{R}_n \to \mathbb{R}$ given by $F(t) = 3 + \sin t$ is subadditive.
$F : \mathbb{R}_+^n \to \mathbb{R}$ given by $F(x) = (x^r)^{1/r}$ is subadditive for $r > 1$ and superadditive for $r < 1$ (Minkowski's inequality).

Theorem 16.3. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy superadditivity* (16.3a) *with $f(1) = 1$. Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, increasing function with $g(0) = 0$, $g(1) = 1$, such that*

$$f(g(|x|)) \ge g(|f(x)|) \quad \text{for } x \in \mathbb{R}. \tag{16.4}$$

Then $f(x) = x$ for $x \in \mathbb{R}$.

Proof. First claim that $f(x) \ge 0$ for $x \in \mathbb{R}_+$. Choose $x \in \mathbb{R}_+$. Then $x = ny$ for some $n \in Z_+$ and $0 \le y \le 1$. Moreover, $y = g(t)$ for $t \ge 0$ since g is increasing and continuous with $g(0) = 0$, $g(1) = 1$. From (16.3a) and (16.4), we get

$$f(x) = f(ny) \ge nf(y) = nf(g(t)) \ge ng(|f(t)|) \ge 0.$$

It follows from (16.3a) that f is a monotone nondecreasing function with

$$f(0) = 0 \quad \text{and} \quad f(-x) \le -f(x) \quad \text{for } x \in \mathbb{R}. \tag{16.5}$$

Next we prove $|f(x)| \le |x|$ for $x \in \mathbb{R}$. Indeed $g(1) = 1 = f(1) = f(g(1)) = f(g|-1|) \ge g(|f(-1)|)$ and thus $|f(-1)| = 1$. But $f(-1) \le -1$ by (16.5). Hence $f(-1) = -1$.

Use (16.3a) and (16.5) to get, for $n \in Z_+$,

$$f(n) \le -f(-n) = -f(n(-1)) \le -nf(-1) = -n.$$

But $f(n) \ge n(f(1)) = n$, so that $f(n) = n$, $n \in Z_+$. Let $p, q \in Z_+^*$. Then $p = f(p) = f\left(q\frac{p}{q}\right) \ge qf\left(\frac{p}{q}\right)$. Since f is a monotonic function, we have

$$f(x) \le x \quad \text{for } x \in Z. \tag{16.6}$$

Choose $x \in \mathbb{R}$. Use (16.5) and (16.6) to obtain

$$g(|x|) \geq f(g(|x|)) \geq g(|f(x)|)$$

and

$$|f(x)| \leq |x|, \quad \text{for } x \in \mathbb{R}. \tag{16.6a}$$

Finally, claim $f(x) = x$. First let $x = -\frac{p}{q}$ for $p, q \in Z_+^*$. From (16.3a), (16.5), and $f(n) = n$ for $n \in Z_+$, we have

$$qf\left(-\frac{p}{q}\right) \leq f\left(q\left(-\frac{p}{q}\right)\right) = f(-p) \leq -f(p) = -p,$$

which with (16.6a) yields

$$f\left(-\frac{p}{q}\right) = -\frac{p}{q}.$$

Since f is monotonic,

$$f(x) = x \quad \text{for } x \leq 0.$$

Let $x \geq 0$. There exists $n \in Z_+^*$ such that $x - n < 0$:

$$f(x) = f(x - n + n) \geq f(x - n) + f(n) = (x - n) + n = x.$$

Thus we obtain $f(x) = x$ for all $x \in \mathbb{R}$. This completes the proof. □

16.3 Logarithmic Inequality

Theorem 16.4. *For functions $f : G \to X$ from a group G to a real or complex inner product space X, the inequality*

$$\|f(xy)\| \geq \|f(x) + f(y)\| \quad (x, y \in G) \tag{16.7}$$

implies

$$f(xy) = f(x) + f(y) \quad (x, y \in G). \tag{L}$$

Proof. Let e denote the identity of the group G. With $x = y = e$ in (16.7), we get $f(e) = 0$, and with $y = x^{-1}$ we then have $f(x^{-1}) = -f(x)$ for all $x \in G$. Inequality (16.7) can be rewritten as

$$\|f(x)\|^2 + 2\,\mathrm{Re}\langle f(x), f(y)\rangle + \|f(y)\|^2 \leq \|f(xy)\|^2 \quad \text{for } x, y \in G. \tag{16.8}$$

Replacing x and y by xy and y^{-1}, respectively, in the inequality above, we get

$$\|f(xy)\|^2 + 2\,\mathrm{Re}\langle f(xy), f(y^{-1})\rangle + \|f(y^{-1})\|^2 \leq \|f(x)\|^2;$$

that is,

$$\|f(xy)\|^2 - 2\,\mathrm{Re}\langle f(xy),\ f(y)\rangle + \|f(y)\|^2 \le \|f(x)\|^2.$$

Adding the inequalities above and dividing by 2, we have

$$\mathrm{Re}\langle f(x) + f(y) - f(xy),\ f(y)\rangle \le 0. \tag{16.9}$$

Replacing x and y by x^{-1} and xy, respectively, in (16.8), we obtain

$$\|f(x^{-1})\|^2 + 2\,\mathrm{Re}\langle f(x^{-1}),\ f(xy)\rangle + \|f(xy)\|^2 \le \|f(y)\|^2;$$

that is,

$$\|f(x)\|^2 - 2\,\mathrm{Re}\langle f(x),\ f(xy)\rangle + \|f(xy)\|^2 \le \|f(y)\|^2.$$

Also, combining this inequality with (16.8) yields

$$\mathrm{Re}\langle f(x) + f(y) - f(xy),\ f(x)\rangle \le 0. \tag{16.9a}$$

Here replacing x and y by xy and y^{-1}, respectively, we get

$$\mathrm{Re}\langle f(xy) - f(y) - f(x),\ f(xy)\rangle \le 0. \tag{16.9b}$$

Finally, adding (16.9), (16.9a), and (16.9b), we obtain

$$\|f(x) + f(y) - f(xy)\|^2 \le 0,$$

which implies (L). The proof is complete. □

Remark. The same result is proved with the additional condition

$$f(xyz) = f(xzy) \quad \text{for } x, y, z \in G. \tag{K}$$

Here the result is proved by reduction to the quadratic equation (Q).

It is pointed out that if f satisfies subadditivity $\|f(xy)\| \le \|f(x) + f(y)\|$ instead of the superadditivity (16.7) with (K), $f(x^{-1}) = -f(x)$, $|f(x^2)| = 2|f(x)|$. Then f satisfies (L). $f(x) = \log|x|$ satisfies all these conditions on $\mathbb{R}^*$. If $G = X = \mathbb{R}$ and $f(x) = 1$, then f satisfies (16.3) but not (L).

It is also clear that the theorem does not hold for arbitrary semigroups G. Consider the interval $[0, +\infty[$ with the operation $+$. The function $f : [0, +\infty[\to \mathbb{R}$ given by

$$f(x) = x^2 \quad (x \ge 0)$$

satisfies $|f(x+y)| \ge |f(x) + f(y)|$, superadditivity, but we do not have $f(x+y) = f(x) + f(y)$ for all $x \ge 0,\ y \ge 0$.

16.4 Multiplicative Inequality

Result 16.5. *If f is continuous and increasing in $x \geq 0$, is somewhere less than unity, and satisfies $f(xy) \leq f(x)f(y)$, then either*

(i) *for $x \leq \delta,\ y \leq \delta$, where $\delta \neq 0$, or*
(ii) *for all values of $x,\ y$,*

we have respectively

(i) $f(x) \leq ax^{\alpha}$ *for* $x \leq 1$ *or*
(ii) $ax^{\alpha} \geq f(x) \geq f(1)x^{\beta}/b$ *for* $x \leq 1$ *and* $bx^{\beta} \geq f(x) \geq f(1)\frac{x^{\alpha}}{a}$ *for* $x \geq 1$, *where* $a, b > 0,\ 0 < \alpha \leq \beta$.

16.5 Convexity

Definition. A subset $C \subseteq \mathbb{R}^n$ is said to be *convex* provided, that for $x, y \in C$, all points of the line segment joining x, y belong to C; that is,

$$\alpha x + (1-\alpha)y \subseteq C \quad \text{for } 0 < \alpha < 1.$$

A real-valued function defined on an interval in $\mathbb{R}$ satisfying the inequality (Jensen)

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}[f(x) + f(y)] \tag{16.10}$$

is called a *Jensen convex* or J*-convex* function. The function is *concave* if the inequality is reversed. J-convex functions need not be continuous. However, if a J-convex function is bounded above on a sufficiently large set, it must be continuous.

A convex function partially monotonic at a point is continuous (Bereanu [119]). A measurable J-convex function is continuous (Hille and Phillips [376]) and satisfies the inequality (Hukachura [381])

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y), \quad 0 < \alpha < 1. \tag{16.11}$$

f satisfying (16.11) is known as a *convex function* and obviously satisfies (16.10), that is, it is J-convex. A convex function (16.10); that is discontinuous at an interior point of an interval is necessarily nonmeasurable. For finite measurable functions, (16.10) and (16.11) are equivalent. Equation (16.10) means geometrically that, for J-convex functions, the midpoint of the chord joining two points of the graph of the function lies "above" the graph of the midpoint of two points.

The function f is concave if $-f$ is convex.

Result 16.6. [376]. *Let $f : \mathbb{R}_+^* \to \mathbb{R}$ be such that* (i) *if $\frac{f(x)}{x}$ is decreasing, then f is subadditive;* (ii) *if f is J-convex and subadditive, then $\frac{f(x)}{x}$ is decreasing; and* (iii) *a necessary and sufficient condition that a measurable, concave function f is subadditive is that $f(0+) \geq 0$.*

Let ω be a positive number, and let ϕ be a real-valued function of a real variable. The function ϕ is said to be *ω-increasing* in an interval U if and only if the inequality

$$\phi(t+\omega) \geq \phi(t)$$

holds for all t such that $t,\ t+\omega \in U$; and ϕ is said to be *ω-decreasing* in U if and only if the inequality

$$\phi(t+\omega) \leq \phi(t)$$

holds for all t such that $t,\ t+\omega \in U$.

Let Ω be a set of positive numbers. The function ϕ is said to be *Ω-monotonic* in U if and only if it is either ω-increasing in U for every $\omega \in \Omega$ or ω-decreasing in U for every $\omega \in \Omega$.

Result 16.7. [555]. *Let $f : C \to \mathbb{R}$ be a convex function in an open convex set $C \subset \mathbb{R}^n,\ n \geq 1$, let x_0 be a point in C, let $a_i,\ i = 1, \ldots, n$, be n linearly independent n-dimensional vectors, let $\varepsilon_i,\ i = 1, \ldots, n$, be positive numbers, and let $\Omega_i \subset (0, \varepsilon_i),\ i = 1, \ldots, n$, be sets from the class θ_1. If, for each $i = 1, \ldots, n$, the function*

$$\phi_i(t) = f(x_0 + ta_i)$$

is Ω_i-monotonic in $(-\varepsilon_i, \varepsilon_i)$, then f is continuous in C.

Lemma (i). (Haruki [348]). *If $f,\ g$ are entire functions of z and $|f(z)| \leq m|g(z)|$ in $|z| < +\infty$, where $m\ (\geq 0)$ is a real constant, then $f(z) = cg(z)$ in $|z| < +\infty$, where c is a complex constant with $|c| \leq m$.*

Lemma (ii). [348]. *If H is an entire function of z and $g(t) = |H(te^{i\phi})|^2$ (t, ϕ real, ϕ fixed), then*

$$g''(0) = 2\,\mathrm{Re}(e^{2i\phi}H''(0)\overline{H(0)}) + 2|H'(0)|^2.$$

From the Jensen equation (J), we obtain the inequality

$$\left| f\left(\frac{x+y}{2}\right)\right| \leq \frac{|f(x)| + |f(y)|}{2}, \tag{16.12}$$

where f is an entire function on $\mathbb{C}$.

Now we shall determine all the entire functions that satisfy (16.12) (see [348, 346, 347]).

Theorem 16.8. (Haruki [348, 346]). *If $f(x)$ is an entire function of x, then the functions that satisfy* (16.12) *are $(\alpha x+\beta)^n$ and $\exp(\alpha x+\beta)$, where α, β are arbitrary complex constants and n is an arbitrary natural number, and only these.*

Proof. Since $\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$, where a, b are real, by $2|f(x)| \leq |f(x+y)| + |f(x-y)|$, which is obtained from (16.12), we have

$$2|f(x)|^2 \leq |f(x+y)|^2 + |f(x-y)|^2. \tag{16.13}$$

Using a real parameter, t, $p(t) = |f(\gamma + te^{i\theta})|^2 + |f(\gamma - te^{i\theta})|^2$ has a minimum $p(0) = 2|f(\gamma)|^2$ by (16.13). Here γ is an arbitrary complex constant and θ is an arbitrary real constant. Hence we have

$$p''(0) \geq 0.$$

By this and Lemma (ii), we have

$$\mathrm{Re}(e^{2i\theta} f''(\gamma)\overline{f(\gamma)}) + |f'(\gamma)|^2 \geq 0.$$

Choosing θ_0 properly, we have

$$\mathrm{Re}(e^{2i\theta_0} f''(\gamma)\overline{f(\gamma)}) = -|f''(\gamma)\overline{f(\gamma)}|.$$

Hence we have

$$|f(\gamma) f''(\gamma)| \leq |f'(\gamma)|^2.$$

Hence we have the following inequality at every point x:

$$|f(x) f''(x)| \leq |f'(x)|^2.$$

By Lemma (i), we have in $|x| < +\infty$

$$f(x) f''(x) = cf^2(x),$$

where c is a complex constant with $|c| \leq 1$. By solving this differential equation, the theorem is proved. □

Remark. If $f(x)$ is a meromorphic function and satisfies (16.12) in $|x| < +\infty$, then we can easily prove that $f(x)$ is an entire function of x.

Applications. By the theorem above, we can solve the following functional equations under the hypothesis that $f(x)$, and $g(x)$ are entire functions of x:

(i) $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$,
(ii) $f(x+y) = f(x) + f(y)$,
(iii) $f(x+y) = f(x) f(y)$,
(iv) $f(xy) = f(x) f(y)$,
(v) $f(x+y) f(x-y) = f^2(x)$,
(vi) $|f(x+y)| + |f(x-y)| = 2|f(x)| + 2|g(y)|$,
(vii) $|f(x+y)| + |f(x-y)| = 2|f(x)| + 2|f(y)|$.

Solution of (i). $f(z) = \alpha z + \beta$. It is clear.
Solution of (ii). $f(z) = \alpha z$. It is also clear.
Solution of (iii). We have

$$\begin{aligned}\left|f\left(\frac{x+y}{2}\right)\right| &= \left|f\left(\frac{x}{2}\right)\right| \left|f\left(\frac{y}{2}\right)\right| \\ &\leq \frac{\left|f^2\left(\frac{x}{2}\right)\right| + \left|f^2\left(\frac{y}{2}\right)\right|}{2} \\ &= \frac{|f(x)| + |f(y)|}{2}.\end{aligned}$$

Hence $f(z)$ satisfies the condition of the theorem. Hence we have $f(z) \equiv 0$, or $f(z) = \exp(\alpha z)$, where α is an arbitrary complex constant.

Solution of (iv). Putting $g(z) = f(e^z)$, by (iv) we have

$$g(x+y) = g(x)g(y).$$

By the result above, we have $f(z) \equiv 0$ or $f(z) \equiv 1$ or $f(z) = z^n$, where n is an arbitrary natural number.

Solution of (v). (v) has solutions 0, $\exp(\alpha x + \beta)$, where α and β are arbitrary constants.

Solution of (vi). By (vi) we have

$$|f(x+y)| + |f(x-y)| \geq 2|f(x)|,$$

which implies

$$\left| f\left(\frac{x+y}{2}\right)\right| \leq \frac{|f(x)| + |f(y)|}{2}.$$

Hence $f(x)$ satisfies the condition of our theorem, and we can conclude that $f(x) = (\alpha z+\beta)^2$, $g(z) = e^{i\theta}\alpha^2 z^2$, where α, β are both arbitrary complex constants.

Solution of (vii). By the result above, we have $f(z) = \alpha z^2$, where a is an arbitrary complex constant.

16.6 Trigonometric Functional Inequality

(a) The functional inequality

$$|f(x+y)| \leq |f(x)g(y)| + |f(y)g(x)|, \tag{16.14}$$

where f, g are entire functions (cf. (4.73)).

Result 16.9. [348]. *If f and g are entire functions with $f(0) = 0$, $f'(0) \neq 0$, $g(0) = 1$, and satisfy the inequality* (16.14)*, then the solutions are given by*

$$f(z) = \alpha e^{\beta z}\sin\delta z, \quad g(z) = e^{\beta z}\cos\delta z, \quad \alpha, \delta \neq 0,$$

and

$$f(z) = \alpha z e^{\beta z}, \quad g(z) = e^{\beta z}, \quad \alpha \neq 0.$$

From the result above, the solutions of the following functional equations (A) and (E) can be deduced:

(A) $$f(x+y) = f(x) + f(y),$$

(E) $$f(x+y) = f(x)f(y),$$

(4.73) $$f(x+y) = f(x)g(y) + f(y)g(x).$$

Solutions

(A): ax, where a is an arbitrary complex constant.

(E): $0, \exp(ax)$, where a is an arbitrary complex constant. (Put $F(x) = xf(x)$.)

(i) $\begin{cases} f(x) \equiv 0, \\ g(x) \text{ arbitrary}; \end{cases}$

(ii) $\begin{cases} f(x) = a\exp(bx), \\ g(x) = \frac{1}{2}\exp(bx); \end{cases}$ where a, b are arbitrary complex constants

(iii) $\begin{cases} f(x) = a\exp(bx)\sin cx, \\ g(x) = \exp(bx)\cos cx; \end{cases}$ where a, b, c are arbitrary complex constants

(iv) $\begin{cases} f(x) = ax\exp(bx), \\ g(x) = \exp(bx). \end{cases}$ where a, b are arbitrary complex constants

(b) The functional inequality $|f(x+y)f(x-y)| \le |f(x)|^2 + |f(y)|^2$

Considering Cauchy's additive functional equation

$$f(x+y) = f(x) + f(y),$$

where $f(x)$ is an entire function of x, we have

$$f(x-y) = f(x) - f(y),$$

and then

$$f(x+y)f(x-y) = f^2(x) - f^2(y).$$

Hence we have

$$|f(x+y)f(x-y)| \le |f(x)|^2 + |f(y)|^2. \tag{16.15}$$

Now we shall determine all the entire functions that satisfy (16.15).

Theorem 16.10. [348]. *If $f(x)$ is an entire function of x, then all the functions that satisfy* (16.15) *are* $\exp(\alpha x + \beta)$, $\alpha\sin\beta x$, αx, *where α, β are arbitrary complex constants, and only these.*

Proof. Case (i). $f(0) \neq 0$. Putting $y = x$ in (16.15), we have $|f(0)f(2x)| \le 2|f(x)|^2$ in $|x| < \infty$. By Lemma (i), we have in $|x| < +\infty$

$$f(0)f(2x) = Cf^2(x),$$

where C is a complex constant with $|C| \le 2$. Putting $x = 0$, by $f(0) \neq 0$, we have $C = 1$. Hence we have in $|x| < +\infty$

$$f(0)f(2x) = f^2(x). \tag{16.16}$$

Since $f(0) \neq 0$, choosing a positive constant r properly, we have $f(x) \neq 0$ in $|x| < r$. Hence there exists a regular function $\phi(x)$ in $|x| < r$ such that $f(x) = \exp(\phi(x))$ in $|x| < r$. By (16.16), we have in $|x| < r$

$$\phi(0) + \phi(2x) = 2\phi(x).$$

Using the power series $\phi(x) = \sum_{n=0}^{+\infty} a_n x^n$ in $|x| < r$ and equating the coefficients on both sides, we have

$$2^n a_n = 2a_n \quad (n = 1, 2, 3, \dots).$$

Hence we have $a_n = 0$ $(n = 2, 3, 4, \dots)$. Hence we have $\phi(x) = a_0 + a_1 x$. Hence, if $f(0) \neq 0$, the solution of (16.15) is $\exp(\alpha x + \beta)$, where α, β are arbitrary complex constants, and only this.

Case (ii). $f(0) = 0$. We may assume that $f(x) \not\equiv 0$. Using a real parameter, t,

$$p(t) = (|f(\gamma)|^2 + |f(te^{i\phi})|^2)^2 - |f(\gamma + te^{i\phi}) f(\gamma - te^{i\phi})|^2$$

has a minimum $p(0) = 0$ by (16.15). Here γ is an arbitrary complex constant that satisfies $f(\gamma) \neq 0$ and ϕ is an arbitrary real constant. Hence we have

$$p''(0) \geq 0.$$

By Lemma (ii), we have

$$\mathrm{Re}(e^{2i\phi}(f(\gamma) f''(\gamma) - f'^2(\gamma))\overline{f^2(\gamma)}) \leq |f'(0) f(\gamma)|^2.$$

Choosing ϕ_0 properly, we have

$$\mathrm{Re}(e^{2i\phi_0}(f(\gamma) f''(\gamma) - f'^2(\gamma))\overline{f^2(\gamma)}) = |(f(\gamma) f''(\gamma) - f'^2(\gamma))\overline{f^2(\gamma)}|.$$

Hence, by $f(\gamma) \neq 0$, we have

$$|f(\gamma) f''(\gamma) - f'^2(\gamma)| \leq |f'(0)|^2.$$

Hence we have the following inequality at every point x:

$$|f(x) f''(x) - f'^2(x)| \leq |f'(0)|^2.$$

By Liouville's Theorem, we have in $|x| < +\infty$

$$f(x) f''(x) - f'^2(x) = C, \tag{16.17}$$

where C is a complex constant with $|C| \leq |f'(0)|^2$. Differentiating both sides of (16.17), we have in $|x| < +\infty$

$$f(x) f'''(x) - f'(x) f''(x) = 0. \tag{16.18}$$

Since $f(x) \not\equiv 0$, choosing a neighbourhood V properly, we have $f(x) \neq 0$ in V. Hence, by (16.18) we have in V

$$\left(\frac{f''(x)}{f(x)}\right)' = 0.$$

Hence we have

$$\frac{f''(x)}{f(x)} = k,$$

where k is a complex constant.
Solving this differential equation, if $f(0) = 0$, then the solutions of (16.15) are $\alpha \sin \beta x,\ \alpha x$, where $\alpha,\ \beta$ are arbitrary complex constants, and only these. This proves the result. □

Applications. By the theorem above, we can solve the following functional equations under the hypothesis that f is an entire function of x:

(i) $f(x+y) = f(x) + f(y)$,
(ii) $f(x+y) = f(x)f(y)$,
(iii) $f(x+y)f(x-y) = f^2(x)$,
(iv) $|f(x+y)|^2 + |f(x-y)|^2 = 2|f(x)|^2 + 2|f(y)|^2$,
(v) $f(x+y)f(x-y) = f^2(x) - f^2(y)$,
(vi) $f(x+y)f(x-y) = (f(x) - f(y))^2$,
(vii) $f(x+y) + f(x-y) = 2f(x)f(y)$.

Solution of (iv). Since $2ab \leq a^2 + b^2$, where a, b are real, we have

$$2|f(x+y)f(x-y)| \leq |f(x+y)|^2 + |f(x-y)|^2.$$

Hence, by (iv) we have

$$|f(x+y)f(x-y)| \leq |f(x)|^2 + |f(y)|^2.$$

Hence $f(x)$ satisfies the condition of our result. Hence we have $f(x) = \alpha x$, where α is an arbitrary complex constant and the solution of (iv) is only this.
((v) $\alpha \sin \beta x,\ \alpha x$, where α, β are arbitrary complex constants. (vi) $\alpha \sin^2 \beta x$, ax^2, where α, β are arbitrary complex constants. (vii) $0, \cos \alpha x$, where α is an arbitrary complex constant.)

(c) The functional inequality $2|f(x)f(y)| \leq |f(0)|(|f(x+y)| + |f(x-y)|)$

Considering the functional equation

$$f(0)(f(x+y) + f(x-y)) = 2f(x)f(y),$$

where $f(x)$ is an entire function of x, we have the following functional equation:

$$2|f(x)f(y)| \leq |f(0)|(|f(x+y)| + |f(x-y)|). \tag{16.19}$$

Now we shall determine all the entire functions that satisfy (16.19).

Result 16.11. [348]. *If $f(x)$ is an entire function of x, then the functions that satisfy* (16.19) *are $\alpha\cos^n\beta x$, $\alpha\exp(\beta x^2)$, where α, β are arbitrary complex constants and n is an arbitrary natural number, and only these.*

Applications. By using Result 16.11, the solutions of (A), (E), (Q), (C), and $f(x+y)f(x-y) = f^2(x)f^2(y)$ can be deduced (see [348, p. 14]).

(d) The functional inequality $|f^2(x)f(y+z)f(y-z)| \le |f^2(y)f(x+z)f(x-z)| + |f^2(z)f(x+y)f(x-y)|$

Considering the functional equation concerning the sigma function in the theory of elliptic functions

$$f^2(x)f(y+z)f(y-z)+f^2(y)f(z+x)f(z-x)+f^2(z)f(x+y)f(x-y)=0,$$

where f is an entire function of x, under the consideration that f is an odd function of x that follows from the equation above, we have

$$|f^2(x)f(y+z)f(y-z)| \le |f^2(y)f(x+z)f(x-z)| + |f^2(x)f(x+y)f(x-y)|. \tag{16.20}$$

Now we shall determine all the entire functions that satisfy (16.20).

Result 16.12. [348]. *If $f(x)$ is an entire function of x, then the functions that satisfy* (16.20) *are the following and only these:*

(i) $f(x) = \exp(\alpha_1 x^2 + \alpha_2 x + \alpha_3)$, *where α_1, α_2, α_3 are arbitrary complex constants,*

(ii) $f(x) = \exp(\alpha_1 x^2 + \alpha_2)\sigma(x)$,
$f(x) = x\exp(\alpha_1 x^2 + \alpha_2)$,
$f(x) = \exp(\alpha_1 x^2 + \alpha_2)\sin(\alpha_3 x)$,
where α_1, α_2, α_3 are arbitrary complex constants.

Applications. By using Result 16.12, the functional equations (A), (E), (Q), (S), $f(x+y)f(x-y) = f^2(x)f^2(y)$, $f(x+y)f(x-y) = \phi(x)\psi(y)$, $f(x+y)f(x-y) = f^2(x)$, $f(x+y)+f(x-y) = 2\phi(x)+2\psi(y)$ can be solved (see [348, p. 18]).

(e) The functional inequality $2|f(x)g(y)| \le |f(x+y)| + |f(x-y)|$

If f, g are entire functions with $g(0) = 1$ and satisfy the functional inequality

$$2|f(x)g(y)| \le |f(x+y)| + |f(x-y)|, \tag{16.21}$$

then $f(x) = a\cos^n(bx+c)$ or $f(x) = \exp(ax^2+bx+c)$ or $f(x) = (ax+b)^n$, where a, b, c are arbitrary complex constants and n is an arbitrary natural number. The solutions of (16.21) are only these.

Proof. Putting $h(y) = \frac{1}{2}(g(y)+g(-y))$, we have

$$2|f(x)h(y)| \le \frac{|f(x)g(y)| + |f(x)g(-y)|}{2}.$$

From (16.21), we have

$$2|f(x)g(-y)| \le |f(x-y)| + |f(x+y)|.$$

Thus we have

$$2|f(x)h(y)| \le |f(x+y)| + |f(x-y)|, \tag{16.22}$$

where $h(0) = 1$.
Since $\frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}}$, where $a,\ b$ are real, by (16.22) we have

$$2|f(x)h(y)|^2 \le |f(x+y)|^2 + |f(x-y)|^2. \tag{16.23}$$

Since $h(0) = 1$, there exists a positive number δ such that $h(y) \ne 0$ in $|y| < \delta$. Let us assume that γ is an arbitrary complex constant and put $y = te^{i\phi}$, where t, ϕ are real with $|t| < \delta$.
Putting $x = y,\ y = te^{i\phi}$ in (16.23), we have in $|t| < \delta$

$$2|f(\gamma)|^2 \le \left|\frac{f(\gamma + te^{i\phi})}{h(te^{i\phi})}\right| + \left|\frac{f(\gamma - te^{i\phi})}{h(te^{i\phi})}\right|^2.$$

From this inequality, we have the following differential equation in the same way as in Result 16.11 (considering that $h(y)$ is an even function of y and that $h(0) = 1,\ h'(0) = 0$):

$$f(x)f''(x) - h''(0)f^2(x) = Cf'^2(x),$$

where C is a complex constant with $|C| \le 1$. Solving this differential equation, the theorem is proved. □

Applications. By the theorem above, we can solve the following functional equations and functional inequalities under the hypothesis that $f,\ p,\ q$ are entire functions.

(i) $f(x+y) + f(x-y) = 2p(x)q(y)$,
(ii) $f(x+y) - f(x-y) = 2p(x)q(y)$,
(iii) $f(x+y)f(x-y) = f^2(x) - f^2(y)$,
(iv) $f(x+y)f(x-y) = f^2(x) + f^2(y) - 1$,
(v) $f(x+y)f(x-y) = p(x)q(y)$,
(vi) $f(x+y) + f(x-y) = 2p(x) + 2q(y)$,
(vii) $\left|f\left(\frac{x+y}{2}\right)\right| \le \frac{|f(x)|+|f(y)|}{2}$,
(viii) $2|f(x)f(y)| \le |f(0)|(|f(x+y)| + |f(x-y)|)$,
(ix) $|f(x+iy)|^2 = \phi(x) + \psi(y)$, where $x,\ y$ are real and $\phi(x),\ \psi(y)$ are real-valued functions,
(x) $|f(x+iy)|^2 = |f(x)|^2 + |f(iy)|^2$ ($x,\ y$ real),
(xi) $|f(x+iy)|^2 = |f(x)|^2 + |f(iy)|^2 - 1$ ($x,\ y$ real).

Solution of (i). We may assume that $q(0) = 1$. Putting $y = 0$ in (i), we have $p(x) = f(x)$. Hence, by (16.21), we have

$$f(x+y) + f(x-y) = 2f(x)q(y).$$

Hence we have

$$2|f(x)q(y)| \leq |f(x+y)| + |f(x-y)|,$$

where $q(0) = 1$.

By the theorem above, we have $f(x) = a\sin\alpha x + b\cos\alpha x$ or $f(x) = ax + b$, where a, b, c are arbitrary complex constants. The solutions of (i) are only these.

Solution of (ii). Differentiating both sides of (ii) with respect to y, we have

$$f'(x+y) + f'(x-y) = 2p(x)q'(y).$$

Hence, by the result of (i) above, we have $f(x) = a\sin\alpha x + b\cos\alpha x + c$ or $f(x) = ax^2 + bx + c$, where a, b, c, α are arbitrary complex constants.

Solution of (iii). Putting $x+y$, $x-y$ in place of x, y, respectively, in (iii), we have

$$f^2(x+y) - f^2(x-y) = f(2x)f(2y).$$

Hence, by the result of (ii) above, we have $f(x) = C\sin\alpha x$ or $f(x) = Cx$, where C, α are arbitrary complex constants. The solutions of (iii) are only these.

Solution of (iv). Putting $x+y$ and $x-y$ in place of x and y, respectively, in (iv), we have

$$f^2(x+y) + f^2(x-y) - 1 = f(2x)f(2y).$$

Putting $F(x) = f^2(x)$, by (16.21) we have

$$F(x+y) + F(x-y) - 1 = f(2x)f(2y).$$

Differentiating both sides of the equation above with respect to y, we have

$$F'(x+y) - F'(x-y) = 2f(2x)f'(2y).$$

Hence, by the result of (ii) above, we have $f(x) = \pm\cos\alpha x$, where α is an arbitrary complex constant. The solutions of (iv) are only these.

Solution of (v). We may assume that $q(0) = 1$ and $q(y) \neq 0$ in $|y| < +\infty$. Putting $y = 0$ in (v), we have $p(x) = f(x)^2$. Hence, by (v), we have

$$\sqrt{|f(x+y)f(x-y)|} = |f(x)\sqrt{q(y)}|,$$

where $\sqrt{q(y)}$ stands for an entire function of y that satisfies $\sqrt{q(y)^2} = q(y)$ in $|y| < \infty$. Since

$$2\sqrt{|f(x+y)f(x-y)|} \le |f(x+y)| + |f(x-y)|,$$

we have

$$2|f(x)\sqrt{q(y)}| \le |f(x+y)| + |f(x-y)|,$$

where $\sqrt{q(0)} = 1$.
Finally, we have $f(x) \equiv 0$ or $f(x) = \exp(ax^2 + bx + c)$, where $a,\ b,\ c$ are arbitrary complex constants. The solutions of (v) are only these.
Solution of (vi). Putting $F(x) = \exp(f(x))$, $\phi(x) = \exp(p(x))$, $\psi(x) = \exp(q(x))$, by (vi) we have

$$F(x+y)F(x-y) = \phi^2(x)\psi^2(y).$$

By the result of (v) above, we have $f(x) = ax^2 + bx + c$, where $a,\ b,\ c$ are arbitrary complex constants. The solution of (vi) is only this.
Solution of (ix). By (ix) we have

$$|f(x+y)|^2 + |f(x-y)|^2 = |f(x+\bar{y})|^2 + |f(x-\bar{y})|^2,$$

where $x,\ y$ are complex and

$$f(x+y) + f(x-y) = 2p(x)q(y),$$

where $p(x),\ q(y)$ are entire functions.

Hence, by the result of (i) above, we have $f(x) = a\sin\alpha x + b\cos\alpha x$ or $f(x) = a\sin h\alpha x + b\cos h\alpha x$ or $f(x) = ax + b$, where $a,\ b$ are arbitrary complex constants and α is an arbitrary real constant. The solutions of (ix) are only these.
Solution of (x). By the result of (ix) above, we have $f(x) = C\sin\alpha x$ or $f(x) = C\sin h\alpha x$ or $f(x) = Cx$, where C is an arbitrary complex constant and α is an arbitrary real constant. The solutions of (x) are only these.
Solution of (xi). By the result of (ix) above, we have $f(x) = e^{i\theta}\cos\alpha x$ or $f(x) = e^{i\theta}\cos h\alpha x$, where θ, α are arbitrary real constants. The solutions of (xi) are only these.

16.7 Cosine and Sine Functional Inequalities

Result 16.13. (Badora and Ger [83]).

(i) *Let G be an Abelian group and let $f : G \to \mathbb{C}$ and $\phi : G \to \mathbb{R}$ satisfy the inequality*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \le \phi(y), \quad \text{for } x, y \in G, \tag{16.24}$$

or

$$|f(x+y)+f(x-y)-2f(x)f(y)|\le\phi(x),\quad \text{for } x,y\in G. \tag{16.24a}$$

Then f is either bounded or satisfies (C)*, the cosine equation.*

(ii) *Let G be an Abelian group, H a semisimple commutative Banach algebra, and $f: G\to H$ and $\phi: G\to\mathbb{R}$ satisfy* (16.24) *or* (16.24a)*. Then f is a solution of* (C) *provided that, for an arbitrary linear multiplicative functional $x^*\in B^*$, $x^*\circ f$ fails to be bounded.*

Remark. It would be desirable to get rid of the somewhat awkward hypothesis that all the superpositions $x^*\circ f$ are unbounded, assuming simply that f itself fails to be bounded. However, as shown by the counterexample below, the assumption that f yields an unbounded solution of inequality (16.24) (resp. (16.24a)) is insufficient to force f to satisfy equation (C). In fact, consider a Banach algebra H of all diagonal 2×2 matrices with complex entries and a function $f:\mathbb{R}\to H$ given by the formula

$$f(x):=\begin{bmatrix}\cos\alpha x & 0\\ 0 & b\end{bmatrix}\quad (x\in\mathbb{R}),$$

where $\alpha\in\mathbb{C}\backslash\mathbb{R}$ and $b\in\mathbb{C}\backslash\{0,1\}$ are arbitrarily fixed. Then f yields an unbounded solution to both the inequalities (16.24) and (16.24a) with an arbitrary majorizing function $\phi:\mathbb{R}\to\mathbb{R}$ enjoying the property $\inf\{\phi(t): t\in\mathbb{R}\}\ge 2|b|\cdot|1-b|$. Nevertheless, f fails to satisfy d'Alembert's equation (C).

What is missing? The superposition of f with the linear multiplicative functional $x^*: H\to\mathbb{C}$ given by

$$x^*\left(\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}\right):=b\quad\left(\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}\in H\right)$$

is constant and hence bounded.

Corollary. *A function $f:\mathbb{R}\to\mathbb{C}$ is of the form*

$$f(x)=\cos\alpha x\quad (x\in\mathbb{R}),$$

with some constant $\alpha\in\mathbb{C}\backslash\mathbb{R}$, if and only if

(i) *f is Lebesgue measurable;*
(ii) *f is unbounded; and*
(iii) *there exists a function $\phi:\mathbb{R}\to\mathbb{R}$ such that*

$$|f(x+y)+f(x-y)-2f(x)f(y)|\le\phi(y),\quad \text{for all } x,y\in\mathbb{R},$$

or

$$|f(x+y)+f(x-y)-2f(x)f(y)|\le\phi(x),\quad \text{for all } x,y\in\mathbb{R}.$$

Result 16.14. [83].

(i) *Let G be a uniquely 2-divisible Abelian group and let $f : G \to \mathbb{C}$ and $\phi : G \to \mathbb{R}$ satisfy the inequality*

$$\left| f(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \le \phi(x) \tag{16.25}$$

or

$$\left| f(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \le \phi(y) \tag{16.25a}$$

for $x, y \in G$. Then either f is bounded or satisfies (S)*, the sine equation.*

(ii) *Let G be an Abelian group, H be a semisimple commutative Banach algebra, and $f : G \to H$ and $\phi : G \to \mathbb{R}$ satisfy either* (16.25) *or* (16.25a)*. Then f satisfies* (S)*, provided that, for an arbitrary linear functional $x^* \in H^*$, $x^* \circ f$ fails to be bounded.*

16.8 Functional Equation Concerning the Parallelogram Identity—Quadratic Inequality

Considering Cauchy's functional equation (A), where f is an entire function, we have the following functional inequality:

$$|f(x+y)| + |f(x-y)| \le 2|f(x)| + 2|f(y)|. \tag{16.26}$$

All the entire functions that satisfy (16.26) are determined in the following theorem.

Theorem 16.15. [348]. *If f is an entire function, then all the functions that satisfy* (16.26) *are α, αx, αx^2, where α is an arbitrary complex constant, and only these.*

Proof. Putting $y = x$ in (16.26), we have in $|x| < +\infty$

$$|f(2x)| \le |4f(x)|.$$

By Lemma (i), we have in $|x| < +\infty$

$$f(2x) = Cf(x),$$

where C is a complex constant with $|C| \le 4$.

Using the power series

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n$$

in the equation above and equating the coefficients of both sides, we have

$$2^n a_n = Ca_n \quad (n = 0, 1, 2, \dots).$$

(i) If $C = 1$, then we have $a_n = 0$ $(n = 1, 2, 3, \dots)$.
(ii) If $C = 2$, then we have $a_n = 0$ $(n = 0, 2, 3, \dots)$.
(iii) If $C = 4$, then we have $a_n = 0$ $(n = 0, 1, 3, \dots)$.
(iv) If $C \neq 1, 2, 4$, then we have $a_n = 0$ $(n = 0, 1, 2, \dots)$.

We now consider the functional inequality

$$|f(x+y)|^2 + |f(x-y)|^2 \geq 2|f(x)|^2 + 2|f(y)|^2, \tag{16.27}$$

where f is an entire function.

Inequality (16.27) is connected with the parallelogram inequality (parallelogram identity when $p = 1$)

$$|x+y|^{2p} + |x-y|^{2p} \geq 2|x|^{2p} + |2|y|^{2p},$$

where x, y are complex variables and p is an arbitrary natural number.

Inequality (16.27) implies the functional inequality

$$|f(x+y)|^2 + |f(x-y)|^2 \geq 2|f(x)^2 - f(y)^2|, \tag{16.28}$$

where f is an entire function.

Result 16.16. (Haruki [351]). *If f is an entire function of z and satisfies* (16.28)*, then and only then $f(z) = az^n$ or $f(z) = a\sin\alpha z$, where n is* 0 *or an arbitrary natural number and a, α are arbitrary complex constants.*

Corollary. *The entire solutions of* (16.27) *are obtained from that of* (16.28)*. The only entire solution of* (16.27) *is ax^n, where $n \in Z_+$ and a is an arbitrary complex constant.*

16.9 Inequalities for the Gamma and Beta Functions via some Classical Results

16.9.1 Inequalities via Chebychev's Inequality

The following result is well known in the literature as Chebychev's integral inequality for synchronous (asynchronous) mappings.

Let $f, g, h : I \subseteq \mathbb{R} \to \mathbb{R}$ be such that $h(x) \geq 0$ for $x \in I$ and h, hfg, hf and hg are integrable on I. If f, g are synchronous (*asynchronous*) *on I (i.e., we recall*

$$(f(x) - f(y))(g(x) - g(y)) \geq (\leq)\, 0 \quad \text{for all } x, y \in I),$$

then we have the inequality

$$\int_I h(x)dx \int_I h(x)f(x)g(x)dx \geq (\leq) \int_I h(x)f(x)dx \int_I h(x)g(x)dx. \tag{16.29}$$

A simple proof of this result can be obtained using Korkine's identity

$$\int_I h(x)dx \int_I h(x)f(x)g(x)dx - \int_I h(x)f(x)dx \int_I h(x)g(x)dx = \frac{1}{2}\int_I \int_I h(x)h(y)(f(x)-f(y))(g(x)-g(y))dxdy. \tag{16.30}$$

The following result holds.

Theorem 16.17. [228]. *Let $m,\ n,\ p,\ q$ be positive numbers with the property that*

$$(p-m)(q-n) \leq (\geq)\ 0.$$

Then

$$\beta(p,q)\beta(m,n) \geq (\leq)\ \beta(p,n)\beta(m,q) \tag{16.31}$$

and

$$\Gamma(p+n)\Gamma(q+m) \geq (\leq)\ \Gamma(p+q)\Gamma(m+n). \tag{16.31a}$$

Proof. Define the mappings $f, g, h : [0,1] \to [0,\infty)$ by

$$f(x) = x^{p-m},\ g(x) = (1-x)^{q-n}, \quad \text{and} \quad h(x) = x^{m-1}(1-x)^{n-1}.$$

Then

$$f'(x) = (p-m)x^{p-m-1},\ g'(x) = (n-q)(1-x)^{q-n-1}, \quad x \in (0,1).$$

As, by (16.30), $(p-m)(q-n) \leq (\geq)\ 0$, then the mappings f and g are synchronous (asynchronous), having the same (opposite) monotonicity on $[0,1]$. Also, h is nonnegative on $[0,1]$.

Writing Chebychev's inequality for the selection of $f,\ g$, and h above, we get

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx \int_0^1 x^{m-1}(1-x)^{n-1}x^{p-m}(1-x)^{q-n}dx \geq (\leq) \int_0^1 x^{m-1}(1-x)^{n-1}x^{p-m}dx \int_0^1 x^{m-1}(1-x)^{n-1}(1-x)^{q-n}dx.$$

That is,

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx \int_0^1 x^{p-1}(1-x)^{q-1}dx \geq (\leq) \int_0^1 x^{p-1}(1-x)^{n-1}dx \int_0^1 x^{m-1}(1-x)^{q-1}dx,$$

which, by (14.36), is equivalent to (16.31).

Now, using (16.31) and (14.34), we can state

$$\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \cdot \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \geq (\leq) \frac{\Gamma(p)\Gamma(n)}{\Gamma(p+n)} \cdot \frac{\Gamma(m)\Gamma(q)}{\Gamma(m+q)},$$

which is (16.31a). This proves the result. □

The following corollary of Theorem 16.17 may be noted as well.

Corollary. *For any $p, m > 0$, we have the inequalities*

$$\beta(m, p) \geq [\beta(p, p)\beta(m, m)]^{\frac{1}{2}} \tag{16.32}$$

and

$$\Gamma(p+m) \geq [\Gamma(2p)\Gamma(2m)]^{\frac{1}{2}}. \tag{16.32a}$$

Proof. In Theorem 16.17, set $q = p$ and $n = m$. Then

$$(p-m)(q-n) = (p-m)^2 \geq 0$$

and thus

$$\beta(p, p)\beta(m, m) \leq \beta(p, m)\beta(m, p) = \beta^2(p, m),$$

and the inequality (16.32) is proved. The inequality (16.32a) follows from (16.32).

The following result employing Chebychev's inequality on an infinite interval holds.

Theorem 16.18. [228]. *Let m, p, and k be real numbers with $m, p > 0$ and $p > k > -m$. If*

$$k(p-m-k) \geq (\leq) 0, \tag{16.33}$$

then we have

$$\Gamma(p)\Gamma(m) \geq (\leq) \Gamma(p-k)\Gamma(m+k) \tag{16.34}$$

and

$$\beta(p, m) \geq (\leq) \beta(p-k, m+k), \tag{16.34a}$$

respectively.

Proof. Consider the mappings $f, g, h : [0, \infty) \to [0, \infty)$ defined by

$$f(x) = x^{p-k-m},\ g(x) = x^k,\ h(x) = x^{m-1}e^{-x}.$$

If the condition (16.33) holds, then we can assert that the mappings f and g are synchronous (asynchronous) on $(0, \infty)$ and then, by Chebychev's inequality on $[0, \infty)$, we can state

$$\int_0^\infty x^{m-1}e^{-x}dx \int_0^\infty x^{p-k-m}x^k x^{m-1}e^{-x}dx$$
$$\geq (\leq) \int_0^\infty x^{p-k-m}x^{m-1}e^{-x}dx \int_0^\infty x^k x^{m-1}e^{-x}dx;$$

i.e.,

$$\int_0^\infty x^{m-1}e^{-x}dx \int_0^\infty x^{p-1}e^{-x}dx$$
$$\geq (\leq) \int_0^\infty x^{p-k-1}e^{-x}dx \int_0^\infty x^{k+m-1}e^{-x}dx.$$

Using the integral representation (14.4), the inequality above provides the desired result (16.34). On the other hand, using (14.34) and

$$\beta(p-k, m+k) = \frac{\Gamma(p-k)\Gamma(m+k)}{\Gamma(p+m)},$$

we can easily deduce that (16.34a) follows from (16.34). This proves the result. □

The following corollary is of interest.

Corollary. *Let $p > 0$ and $q \in \mathbb{R}$ such that $|q| < p$. Then*

$$\Gamma^2(p) \leq \Gamma(p-q)\Gamma(p+q)$$

and

$$\beta(p, p) \leq \beta(p-q, p+q).$$

Proof. In Theorem 16.18, choose $m = p$ and $k = q$. Then

$$k(p-m-k) = -q^2 \leq 0,$$

and by (16.34) we get the first inequality,

$$\Gamma^2(p) \leq \Gamma(p-q)\Gamma(p+q).$$

The second inequality follows from it.

Definition. The positive real numbers a and b may be called similarly (oppositely) unitary if

$$(a-1)(b-1) \geq (\leq) 0. \tag{16.35}$$

Theorem. *Let $a, b > 0$ and be similarly (oppositely) unitary. Then*

$$\Gamma(a+b) \geq (\leq) ab\Gamma(a)\Gamma(b) \tag{16.36}$$

and

$$\beta(a,b) \geq (\leq) \frac{1}{ab}, \tag{16.36a}$$

respectively.

Proof. Consider the mappings $f, g, h : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$f(t) = t^{a-1},\ g(t) = t^{b-1}, \quad \text{and} \quad h(t) = te^{-t}.$$

If the condition (16.35) holds, then obviously the mappings f and g are synchronous (asynchronous) on $\mathbb{R}_+$, and by Chebychev's integral inequality we can state that

$$\int_0^\infty te^{-t}dt \int_0^\infty t^{a+b-1}e^{-t}dt \geq (\leq) \int_0^\infty t^a e^{-t}dt \int_0^\infty t^b e^{-t}dt$$

provided $(a-1)(b-1) \geq (\leq) 0$; i.e.,

$$\Gamma(2)\Gamma(a+b) \geq (\leq)\ \Gamma(a+1)\Gamma(b+1). \tag{16.37}$$

Since $\Gamma(a+1) = a\Gamma(a)$, $\Gamma(b+1) = b\Gamma(b)$, and $\Gamma(2) = 1$, (16.37) becomes (16.36). The inequality (16.36a) follows by (16.37). This proves the theorem. □

The following corollaries may be noted as well.

Corollary. *The mapping* $\log\Gamma(x)$ *is superadditive for* $x > 1$.

Proof. If $a, b \in [1, \infty)$, then, by (16.36),

$$\log\Gamma(a+b) \geq \log a + \log b + \log\Gamma(a) + \log\Gamma(b) \geq \log\Gamma(a) + \log\Gamma(b),$$

which is the superadditivity of the desired mapping.

Corollary. *For every* $n \in Z_+^*$, $n \geq 1$, *and* $a > 0$, *we have the inequality*

$$\Gamma(na) \geq (n-1)!a^{2(n-1)}[\Gamma(a)]^n. \tag{16.38}$$

Proof. Using the inequality (16.36) successively, we can state that

$$\Gamma(2a) \geq a^2\Gamma(a)\Gamma(a),\ \Gamma(3a) \geq 2a^2\Gamma(2a)\Gamma(a),\ \Gamma(4a) \geq 3a^2\Gamma(3a)\Gamma(a),$$
$$\ldots, \Gamma(na) \geq (n-1)a^2\Gamma[(n-1)a]\Gamma(a).$$

By multiplying these inequalities, we arrive at (16.38).

Corollary. *For any* $a > 0$, *we have*

$$\Gamma(a) \leq \frac{2^{2a-1}}{\sqrt{\pi}a^2}\Gamma\left(a + \frac{1}{2}\right).$$

Proof. Using (14.13) since $\Gamma(2a) \geq a^2\Gamma^2(a)$, we arrive at

$$2^{2a-1}\Gamma(a)\Gamma\left(a+\frac{1}{2}\right) \geq \sqrt{\pi}a^2\Gamma^2(a),$$

which is the desired inequality.

For a given $m > 0$, consider the mapping $\Gamma_m : \mathbb{R}_+ \to \mathbb{R}$,

$$\Gamma_m(x) = \frac{\Gamma(x+m)}{\Gamma(m)}.$$

The following result holds.

Theorem 16.19. [228]. *The mapping Γ_m is supermultiplicative on* $[0, \infty)$.

Proof. Consider the mappings $f(t) = t^x$ and $g(t) = t^y$, which are monotonically nondecreasing on $[0, \infty)$, and $h(t) := t^{m-1}e^{-t}$ is nonnegative on $[0, \infty)$.

Applying Chebychev's inequality for the synchronous mappings f, g and the weight function h, we can write

$$\int_0^\infty t^{m-1}e^{-t}dt \int_0^\infty t^{x+y+m-1}e^{-t}dt \geq \int_0^\infty t^{x+m-1}e^{-t}dt \int_0^\infty t^{y+m-1}e^{-t}dt.$$

That is,

$$\Gamma(m)\Gamma(x+y+m) \geq \Gamma(x+m)\Gamma(y+m),$$

which is equivalent to

$$\Gamma_m(x+y) \geq \Gamma_m(x)\Gamma_m(y),$$

and the result is proved.

16.9.2 Inequalities via Hölder's Inequality

Let $I_2 \subseteq \mathbb{R}$ be an interval in $\mathbb{R}$, and assume that $f \in L_p(I_2)$, $g \in L_q(I_2)$ $\left(p > 1,\ \frac{1}{p} + \frac{1}{q} = 1\right)$; i.e.,

$$\int_{I_2} |f(s)|^p ds, \quad \int_{I_2} |g(s)|^q (ds) < \infty.$$

Then $fg \in L_1(I_2)$ and the following inequality due to Hölder holds:

$$\left|\int_{I_2} f(s)g(s)ds\right| \leq \left(\int_{I_2} |f(s)|^p ds\right)^{\frac{1}{p}} \left(\int_{I_2} |g(s)|^q ds\right)^{\frac{1}{q}}. \tag{16.39}$$

Using Hölder's inequality, some functional properties of the mappings gamma and beta are given under the following result.

Theorem 16.20. [228]. *Let $a, b \geq 0$ with $a + b = 1$ and $x, y > 0$. Then*

$$\Gamma(ax + by) \leq [\Gamma(x)]^a [\Gamma(y)]^b; \tag{16.40}$$

i.e., the mapping Γ is logarithmically convex on $\mathbb{R}_+^*$.

Proof. We use the weighted version of Hölder's inequality

$$\left| \int_{I_2} f(s)g(s)h(s)ds \right| \leq \left(\int_{I_2} |f(s)|^p h(s)ds \right)^{\frac{1}{p}} \left(\int_{I_2} |g(s)|^q h(s)ds \right)^{\frac{1}{q}} \tag{16.39a}$$

for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, h nonnegative on I_2, and provided all the other integrals exist and are finite.

Choose

$$f(s) = s^{a(x-1)}, \quad g(s) = s^{b(y-1)} \quad \text{and} \quad h(s) = e^{-s}, \quad s \in (0, \infty)$$

in (16.39a) to get (for $I_2 = (0, \infty)$ and $p = \frac{1}{a}$, $q = \frac{1}{b}$)

$$\int_0^\infty s^{a(x-1)} \cdot s^{b(y-1)} e^{-s} ds \leq \left(\int_0^\infty s^{a(y-1)\cdot\frac{1}{a}} e^{-s} ds \right)^a \left(\int_0^\infty s^{b(y-1)\cdot\frac{1}{b}} e^{-s} ds \right)^b,$$

which is clearly equivalent to

$$\int_0^\infty s^{ax+by-1} e^{-s} ds \leq \left(\int_0^\infty s^{y-1} e^{-s} ds \right)^a \left(\int_0^\infty s^{y-1} e^{-s} ds \right)^b,$$

and inequality (16.40) is proved. □

Remark. Consider the mapping $g(x) := \log \Gamma(x)$, $x \in \mathbb{R}_+^*$. We have

$$g'(x) = \frac{\Gamma'(x)}{\Gamma(x)} \quad \text{and} \quad g''(x) = \frac{\Gamma''(x)\Gamma(x) - [\Gamma(x)]^2}{\Gamma^2(x)}$$

for $x \in (0, \infty)$. Using the inequality, we conclude that $g''(x) \geq 0$ for all $x \in (0, \infty)$, which shows that Γ is logarithmically convex on $(0, \infty)$.

We now prove a similar result for the beta function.

Theorem 16.20a. [228]. *The mapping β is logarithmically convex on $(0, \infty)^2$ as a function of two variables.*

Proof. Let $(p, q), (m, n) \in (0, \infty)^2$ and $a, b \geq 0$ with $a + b = 1$. We have

$$\begin{aligned}
\beta[a(p, q) + b(m, n)] &= \beta(ap + bm, aq + bn) \\
&= \int_0^1 t^{ap+bm-1}(1-t)^{aq+bn-1} dt \\
&= \int_0^1 t^{a(p-1)+b(m-1)}(1-t)^{a(q-1)+b(n-1)} dt \\
&= \int_0^1 \left[t^{p-1}(1-t)^{q-1} \right]^a \times \left[t^{m-1}(1-t)^{n-1} \right]^b dt.
\end{aligned}$$

Define the mappings

$$f(t) = \left[t^{p-1}(1-t)^{q-1}\right]^a, \quad t \in I_0,$$
$$g(t) = \left[t^{m-1}(1-t)^{n-1}\right]^b, \quad t \in I_0,$$

and choose $p = \frac{1}{a},\ q = \frac{1}{b}$ $(\frac{1}{p} + \frac{1}{q} = a + b = 1,\ p \geq 1)$.

Applying Hölder's inequality for these selections, we get

$$\int_0^1 \left[t^{p-1}(1-t)^{q-1}\right]^a \left[t^{p-1}(1-t)^{q-1}\right]^b dt \leq \left[\int_0^1 t^{p-1}(1-t)^{q-1}dt\right]^a \times \left[\int_0^1 t^{m-1}(1-t)^{n-1}dt\right]^b;$$

i.e.,

$$\beta[a(p,q) + b(m,n)] \leq [\beta(p,q)]^a[\beta(m,n)]^b,$$

which is the logarithmic convexity of β on $(0,\infty)^2$. This proves the result.

Closely associated with the derivative of the gamma function is the *logarithmic-derivative function*

$$\Psi(x) = \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x \neq 0, -1, -2, \ldots.$$

The function $\Psi(x)$ is also commonly called the *psi function.*

Remark. The digamma function is monotonically nondecreasing and concave on $(0,\infty)$.

16.10 Simpson's Inequality and Applications

Introduction

The following inequality is well known in the literature as *Simpson's inequality* (see [228]),

$$\left|\int_a^b f(x)dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^5, \tag{16.41}$$

where the mapping $f : [a,b] \to \mathbb{R}$ is assumed to be four times continuously differentiable on the interval (a,b) and the fourth derivative is assumed to be bounded on (a,b); that is,

$$\left\|f^{(4)}\right\|_\infty := \sup_{x\in(a,b)}\left|f^{(4)}(x)\right| < \infty.$$

Now, if we assume that $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ is a partition of the interval $[a, b]$ and f is as above, then we have the classical *Simpson quadrature formula*

$$\int_a^b f(x)dx = A_S(f, I_n) + R_S(f, I_n), \tag{16.42}$$

where $A_S(f, I_n)$ is the *Simpson rule*

$$A_S(f, I_n) =: \frac{1}{6}\sum_{i=0}^{n-1}[f(x_i) + f(x_{i+1})]h_i + \frac{2}{3}\sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)h_i$$

and the *remainder term* $R_S(f, I_n)$ satisfies the estimate

$$|R_S(f, I_n)| \leq \frac{1}{2880}\left\| f^{(4)}\right\|_\infty \sum_{i=0}^{n-1} h_i^5,$$

where $h_i := x_{i+1} - x_i$ for $i = 0, \ldots, n-1$.

When we have an equidistant partitioning of $[a, b]$ given by

$$I_n : x_i := a + \frac{b-a}{n} \cdot i, \quad i = 0, \ldots, n,$$

then we have the formula

$$\int_a^b f(x)dx = A_{S,n}(f) + R_{S,n}(f), \tag{16.42a}$$

where

$$A_{S,n}(f) := \frac{b-a}{6n}\sum_{i=0}^{n-1}\left[f\left(a + \frac{b-a}{n} \cdot i\right) + f\left(a + \frac{b-a}{n} \cdot (i+1)\right)\right]$$
$$+ \frac{2(b-a)}{3n}\sum_{i=0}^{n-1} f\left(a + \frac{b-a}{n} \cdot \frac{2i+1}{2}\right),$$

and the remainder satisfies the estimation

$$|R_{S,n}(f)| \leq \frac{1}{2880} \cdot \frac{(b-a)^5}{n^4}\left\| f^{(4)}\right\|_\infty.$$

We point out some very recent developments on Simpson's inequality for which the remainder is expressed in terms of derivatives lower than the fourth.

It is well known that if the mapping f is not four times differentiable and the fourth derivative $f^{(4)}$ is not bounded on (a, b), then we cannot apply the classical Simpson quadrature formula, which actually is one of the quadrature formulas used most often in practical applications.

Simpson's Inequality for Mappings of Bounded Variation

The following result holds.

Result 16.21. [228]. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequality*
(16.41a)

$$\left| \int_a^b f(x)dx - \frac{b-a}{3} \cdot \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \le \frac{1}{3}(b-a) \bigvee_a^b (f),$$

where $\bigvee_a^b (f)$ denotes the total variation of f on the interval $[a,b]$. The constant $\frac{1}{3}$ is the best possible.

Corollary. *Suppose $f : [a,b] \to \mathbb{R}$ is a continuous differentiable mapping with*

$$\| f' \|_1 = \int_a^b |f'(x)|dx < \infty.$$

Then the following inequality holds:
(16.41b)

$$\left| \int_a^b f(x)dx - \frac{b-a}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \le \frac{1}{3} \| f' \|_1 (b-a)^2.$$

16.11 Applications for Special Means

Let us recall the following means:

- *the arithmetic mean:*

$$A(a,b) := \frac{a+b}{2}, \quad a,b \ge 0;$$

- *the geometric mean:*

$$G(a,b) := \sqrt{ab}, \quad a,b \ge 0;$$

- *the harmonic mean:*

$$H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}, \quad a,b > 0;$$

- *the logarithmic mean:*

$$\ell(a,b) := \frac{b-a}{\ln b - \ln a}, \quad a,b > 0,\ a \ne b,\ \ell(a,a) = a;$$

- *the identric mean:*

$$I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad a,b > 0,\ a \ne b; \text{ and}$$

- *the p-logarithmic mean:*

$$\ell_p(a,b) := \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \quad p \in \mathbb{R}\backslash\{-1,0\},\ a,b>0,\ a\neq b.$$

It is well known that $\ell_p(a,b)$ is monotonically nondecreasing over $p \in \mathbb{R}$ with $\ell_{-1} := \ell$ and $\ell_0 := \ell(a,b)$. In particular, we have the inequalities

$$H \leq G \leq \ell \leq \ell(a,b) \leq A(a,b).$$

Using Result 16.21, some new inequalities are derived for the means above.
Let $f : [a,b] \to \mathbb{R}$ $(0<a<b)$, $f(x) = x^p$, $p \in \mathbb{R}\backslash\{-1,0\}$. Then

$$\frac{1}{b-a}\int_a^b f(x)dx = \ell_p(a,b),$$
$$\frac{f(a)+f(b)}{2} = A(a^p,b^p),$$
$$f\left(\frac{a+b}{2}\right) = A^p(a,b),$$

and

$$\|f'\|_1 = |p|(b-a)\ell_{p-1}^{p-1}, \quad p \in \mathbb{R}\backslash\{-1,0,1\}.$$

Using inequality (16.41b), we get

$$\left|\ell_p^p(a,b) - \frac{1}{3}A(a^p,b^p) - \frac{2}{3}A^p(a,b)\right| \leq \frac{|p|}{3}\ell_{p-1}^{p-1}(b-a)^2.$$

The following result holds.

Result 16.22. [228]. *Let $f : [a,b] \to \mathbb{R}$ be an L-Lipschitzian mapping on $[a,b]$. Then we have the inequality*

$$\left|\int_a^b f(x)dx - \frac{b-a}{3}\cdot\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{5}{36}\ell(a,b)(b-a)^2.$$

Applications for Special Means

Using Result 16.22, we now point out some new inequalities for the special means defined earlier.

(i) Let $f : [a,b] \to \mathbb{R}$ $(0<a<b)$, $f(x) = x^p$, $p \in \mathbb{R}\backslash\{-1,0\}$. Then

$$\|f'\|_\infty = \delta_p(a,b) := \begin{cases} pb^{p-1} & \text{if } p \geq 1, \\ |p|a^{p-1} & \text{if } p \in (-\infty,1)\backslash\{-1,0\}. \end{cases}$$

Then we get

$$\left|\ell_p^p(a,b) - \frac{1}{3}A(a^p,b^p) - \frac{2}{3}A^p(a,b)\right| \leq \frac{5}{35}\delta_p(a,b)(b-a).$$

(ii) Let $f:[a,b]\to\mathbb{R}$ $(0<a<b)$, $f(x)=\frac{1}{x}$. Then

$$\|f'\|_\infty=\frac{1}{a^2}.$$

Then we get

$$3H(a,b)A(a,b)-\ell(a,b)A(a,b)-2\ell(a,b)H(a,b)\le\frac{5}{12}\cdot\frac{b-a}{a^2}\ell(a,b)A(a,b)H(a,b).$$

Simpson's Inequality in Terms of the p-Norm

The following result holds.

Result 16.23. [228]. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping on $[a,b]$ whose derivative belongs to $L_p[a,b]$. Then we have the inequality*

$$\left|\int_a^b f(x)dx-\frac{b-a}{3}\cdot\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|\le\frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}}(b-a)^{1+\frac{1}{q}}\|f'\|_p,$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$.

16.12 Inequalities from Information Theory

The well-known Shannon entropy (SE)

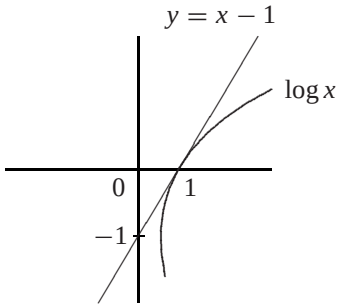
$$H_n(P)=-\sum_{i=1}^n p_i\log p_i$$

satisfies the *Shannon inequality*

$$-\sum_{k=1}^n p_k\log p_k\le-\sum_{k=1}^n p_k\log q_k, \tag{16.43}$$

where $(p_k),(q_k)\in\Gamma_n$.

Indeed, $\log x \leq x - 1$ for $x > 0$.



Put $x = \frac{q_k}{p_k}$, $k = 1$ to n so that

$$\sum_{k=1}^{n} p_k \cdot \log \frac{q_k}{p_k} \leq \sum p_k \left(\frac{q_k}{p_k} - 1 \right) = 0,$$

which is (16.43). The Shannon inequality plays an important role in coding theory, where it provides the best lower bound for the average code length.

Also, from directed divergence (dd),

$$D_n(P \| Q) = \sum_{k=1}^{n} p_k \log \frac{p_k}{q_k} \geq 0$$

follows (16.43). The inequality (16.43) above can be expressed in the functional form

$$\sum_{k=1}^{n} p_k f(p_k) \geq \sum_{k=1}^{n} p_k f(q_k), \tag{16.44}$$

where $f : I_0 \to \mathbb{R}$, $(p_k), (q_k) \in \Gamma_n^0$.

There are several interpretations of the Shannon functional inequality (16.44). One of the interpretations is connected with the problem "how to keep the expert honest" [15, 22]. I.J. Good considered the problem of how a firm can encourage its forecaster to give fair and accurate estimates of probabilities. In general, the problem is complicated, so he considers only a simple case. Suppose, for instance, an expert is asked to estimate a certain probability distribution regarding the occurrence of some events (outcome of an experiment, market situation, weather, etc.). He gives this as $(q_1, q_2, \ldots, q_n)$, while his (subjective) probabilities for the occurrence of the same events are $(p_1, p_2, \ldots, p_n)$. If he agrees to be paid only after the occurrence of the event (market situation, etc.) is known and his payoff will be (ideally) $f(q_k)$ if the kth event happens, then his expected earnings will be $\sum_{k=1}^{n} p_k f(q_k)$. In order to keep the expert honest, it is natural to choose the payoff function f so that the expert's expected earnings will be maximized if he has given his probabilities. Hence (16.44) holds. Inequality (16.44) can also be given a nice interpretation in connection with the reproducing scoring system.

The inequality (16.44) was extensively studied in [15, 46, 61, 62, 43, 273, 641, 505] assuming various regularity conditions on the unknown function. The general solution of (16.44) without assuming any regularity condition is

$$f(p) = a \log p + b \quad (a \geq 0,\ p \in]0, 1[). \tag{16.45}$$

Several proofs are known. We will present the proof given in [15] and treat generalizations of (16.44) later.

Solution of (16.44) from Aczél [15]

The proof is done showing first that f is monotonically increasing and then that f is differentiable.

Put in (16.44) $p_k = q_k$ $(k = 3, 4, \ldots, n)$, $p_1 = p$, $q_1 = q$, $p_1 + p_2 = r = q_1 + q_2$, $r \in]0, 1[$, so that it goes over to

$$p(f(p) - f(q)) \geq (r - p)(f(r - q) - f(r - p)) \tag{16.46}$$

for $p, q \in]0, r[$, $r \in]0, 1[$.

Interchanging p and q in (16.46), we get

$$q(f(q) - f(p)) \geq (r - q)(f(r - p) - f(r - q)). \tag{16.46a}$$

Multiply (16.46) by $(r - q)$, (16.46a) by $(r - p)$, and add to obtain

$$r(q - p)(f(q) - f(p)) \geq 0.$$

If $q > p$, then $f(q) \geq f(p)$; that is, f is monotonically nondecreasing.

From (16.46a) and (16.46), for $q > p$, we have

$$\begin{aligned} f(r - p) - f(r - q) &\leq \frac{q}{r - q}(f(q) - f(p)) \\ &\leq \frac{q}{r - q} \cdot \frac{r - p}{p}(f(r - p) - f(r - q)); \end{aligned}$$

that is,

$$\begin{aligned} \frac{r - q}{q} \frac{(f(r - p) - f(r - q))}{q - p} &\leq \frac{f(q) - f(p)}{q - p} \\ &\leq \frac{r - p}{p} \frac{(f(r - p) - f(r - q))}{q - p}. \end{aligned}$$

Suppose now that f is differentiable at $r - p$ in $]0, 1[$. As $q \to p$, the first and last terms of the inequality tend to

$$\frac{r - p}{p} f'(r - p),$$

and then we conclude that f is differentiable at p also and

$$pf'(p) = (r-p)f'(r-p) \quad \text{for } p \in]0,1[,\ r \in]p,1[. \tag{16.47}$$

Now we claim that f is differentiable everywhere on $]0,1[$. Indeed, if f were not differentiable at $p_0 \in]0,1[$, then f is not differentiable at $r - p_0$ ($r \in]p_{0,1}[$); that is, f would not be differentiable on the nondegenerate interval $]0, 1-p_0[$. But this would contradict the fact that f is a monotonic function that is almost everywhere differentiable. Thus f is differentiable everywhere, and (16.47) shows that $pf'(p) = 0$ (constant $a \geq 0$ since f is nondecreasing). Thus we obtain (16.45). This concludes the proof.

16.12.1 Generalization

The inequality

$$\sum_{k=1}^{n} p_k f_k(q_k) \leq \sum_{k=1}^{n} p_k f_k(p_k) \tag{16.44a}$$

is a generalization of (16.44). The general solution of (16.44a) was given in [46] without assuming any regularity conditions on f_k, and it was shown that the solutions f_k of the inequality (16.44a) are of the form

$$f_k(p) = c \log_2 p + b_k, \quad p \in I_0,\ k = 1, 2, \ldots, n,\ n > 2, \tag{16.45a}$$

where $c \geq 0$, b_k are arbitrary constants. The inequality

$$\sum_{k=1}^{n} g(p_k) f(q_k) \leq \sum_{k=1}^{n} g(p_k) f(p_k), \tag{16.44b}$$

a generalization of (16.44), was treated in [274], and some variations of (16.44) were investigated in [277]. Now we determine the general solution of the inequality

$$\sum_{k=1}^{n} g_k(p_k) f_k(q_k) \leq \sum_{k=1}^{n} g_k(p_k) f_k(p_k), \tag{16.44c}$$

when it holds for $n > 2$, without assuming any regularity conditions on the functions f_k. The inequalities (16.44), (16.44a), and (16.44b) are special cases of (16.44c). We will provide an application of this to find the solution of an inequality connected with the generalized problem of "how to keep the inset expert honest" [22].

Solution of the Inequality (16.44c)

We will prove the following theorem adopting the method found in [46].

Theorem 16.24. (Kannappan and Sahoo [505]). *Let* $f_k : [0, 1[\to \mathbb{R}$ *be real-valued functions and* $g_k :]0, 1[\to \mathbb{R}_+^*$ *be positive and strictly increasing continuous functions for every* k $(k = 1, 2, \ldots, n)$. *For all* $P, Q \in \Delta_n^0$, *suppose* f_k *and* g_k *satisfy the functional inequality* (16.44c) *for arbitrary but fixed* $n \geq 3$. *Then each* f_k *is a monotonically increasing function and differentiable everywhere. Further, there exist constants* $c \geq 0,\ a \in]0, 1[,\ b_1, b_2, \ldots, b_n$ *such that*

$$f_k(p) = c \int_a^p \frac{dt}{g_k(t)} + b_k \quad (k = 1, 2, \ldots, n). \tag{16.45c}$$

The converse is also true.

Proof. Let $p_1 = q,\ q_1 = q$, and $p_i = q_i$ for all $i > 2$ in (16.44c). Then, after simplification, we get

$$g_1(p)\, f_1(q) + g_2(r - p)\, f_2(r - q) \leq g_1(p)\, f_1(p) + g_2(r - p)\, f_2(r - p),$$

where $p + p_2 = r = q + q_2$, for $r \in]0, 1[$. The inequality above is equivalent to

$$g_1(p)[\, f_1(p) - f_1(q)] \geq g_2(r - p)[\, f_2(r - q) - f_2(r - p)] \tag{16.48}$$

for all $p, q \in]0, r[$ and $r \in]0, 1[$. The domain on which (16.48) holds is symmetric in p and q. Thus, interchanging p and q in the above, we obtain

$$g_1(q)[\, f_1(q) - f_1(p)] \geq g_2(r - q)[\, f_2(r - p) - f_2(r - q)]. \tag{16.48a}$$

Since g_2 is positive, multiplying (16.48) by $g_2(r - q)$ and (16.48a) by $g_2(r - p)$ and adding, we get

$$(g_1(p)g_2(r - q) - g_1(q)g_2(r - p)(\, f_1(p) - f_1(q)) \geq 0. \tag{16.49}$$

Let us choose p and q in $]0, r[$ such that $p > q$. Since g_1 and g_2 are strictly increasing, $g_1(p)g_2(r - q) - g_1(q)g_2(r - p) > 0$. Hence, from (16.49) we conclude that, for $r \in]0, 1[,\ p, q \in]0, r[$ whenever $p > q\ f_1(p) \geq f_1(q)$. That is, f_1 is increasing on the interval $]0, 1[$. Similarly, f_k is monotonically increasing on I_0.

Also from (16.48) and (16.48a), one obtains

$$\begin{aligned} \frac{g_2(r - p)}{g_1(p)} \frac{f_2(r - q) - f_2(r - p)}{(r - q) - (r - p)} &\leq \frac{f_1(p) - f_1(q)}{p - q} \\ &\leq \frac{g_2(r - q)}{g_1(q)} \frac{f_2(r - q) - f_2(r - p)}{(r - q) - (r - p)}. \end{aligned} \tag{16.50}$$

Thus, from (16.50), if f_2 is differentiable at $r - p$, then f_1 is differentiable at p. Hence (as $q \to p$)

$$g_1(p)\, f_1'(p) = g_2(r - p)\, f_2'(r - p). \tag{16.51}$$

In other words, if f_1 is not differentiable at p, then f_2 is not differentiable at any $r - p \in]0, 1 - p[$. But this is impossible: Since f_2 is monotonically increasing,

f_2 is almost everywhere differentiable. Therefore, f_1 is everywhere differentiable and (16.51) holds for all $p \in]0, r[$. Since r is arbitrary in I_0, it is easy to see that (16.51) holds for all $p \in I_0$. Similarly, f_2 and all f_k $(k > 2)$ are differentiable. Since (16.51) is true for every r in I_0, letting $s = r - p$ in (16.51), we get

$$g_1(p)\, f_1'(p) = g_2(s)\, f_2'(s) = c \quad \text{(constant)}.$$

Since g_1 is positive, this equation yields

$$f_1'(p) = \frac{c}{g_1(p)}.$$

Thus

$$f_1(p) = c \int_a^p \frac{dt}{g_1(t)} + b_1,$$

where c, a, and b_1 are constants. Similarly,

$$f_k(p) = c \int_a^p \frac{dt}{g_k(t)} + b_k, \tag{16.45c}$$

where c, a, and b_k are constants for $k = 1, 2, \ldots, n$. Since each f_k is increasing $c \geq 0$ and since $p \in I_0$, therefore $a > 0$. This completes the "only if" part of the theorem.

What follows next is the proof of the "if" part of the theorem. Consider

$$\begin{aligned}
\sum_{k=1}^{n} g_k(p_k)[\, f_k(p_k) - f_k(q_k)] &= \sum_{k=1}^{n} g_k(p_k) \int_{q_k}^{p_k} \frac{c}{g_k(t)} dt \quad \text{(using (16.45c))} \\
&= c \sum_{p_k > q_k} \int_{q_k}^{p_k} \frac{g_k(p_k)}{g_k(t)} dt + c \sum_{p_k < q_k} \int_{q_k}^{p_k} \frac{g_k(p_k)}{g_k(t)} dt \\
&= c \sum_{p_k > q_k} \int_{q_k}^{p_k} \frac{g_k(p_k)}{g_k(t)} dt - c \sum_{p_k < q_k} \int_{p_k}^{q_k} \frac{g_k(p_k)}{g_k(t)} dt.
\end{aligned}$$

Since g_k is monotonically increasing, $g_k(p_k) \geq g_k(t)$ for $p_k \geq t \geq q_k$ and $g_k(p_k) \leq g_k(t)$ for $p_k \leq t \leq q_k$. Thus, since $c \geq 0$, one obtains

$$\sum_{k=1}^{n} g_k(p_k)[\, f_k(p_k) - f_k(q_k)] \geq c \sum_{k=1}^{n} \int_{q_k}^{p_k} dt = 0.$$

Thus (16.45c) satisfies (16.44c). This completes the proof of the theorem. □

Remark (i). If g is strictly decreasing and does not change sign, then Theorem 16.24 is also valid (with $c \leq 0$).

Corollary (i). [46]. *The inequality*

$$\sum_{k=1}^{n} p_k\, f_k(q_k) \leq \sum_{k=1}^{n} p_k\, f_k(p_k) \tag{16.44d}$$

holds for fixed $n \geq 3$ *and for all* $P, Q \in \Gamma_n^0$ *if, and only if, there exist constants* $c \geq 0,\ b_1, b_2, \ldots, b_n$ *such that*

$$f_k(p) = c \log p + b_k. \tag{16.45d}$$

Furthermore, the right-hand side of inequality (16.44d) *is equal to*

$$c \sum_{k=1}^{n} p_k \log_2 p_k + \sum_{k=1}^{n} b_k p_k.$$

Proof. Letting $g_k(p) = p$ and using (16.45c), one obtains (16.45d). This completes the proof of the corollary.

Remark (ii). Since in (16.44c) it is required that g_k's be positive and strictly increasing continuous functions only, several choices are available for g_k's. Suppose we let $g_k(p) = p^\beta$ and the g_k's are strictly increasing for $\beta > 1$ and strictly decreasing for $\beta < 1$. Also $g_k(p) > 0$ for all p in I_0. Thus, the right-hand side of the inequality (16.44c) for $f = f_k$ $(k = 1, 2, \ldots, n)$ becomes $c + d H_n^\beta(P)$, where $c,\ d$ are constants and

$$H_n^\beta(P) = (1/(2^{1-\beta} - 1)) \left[\sum_{k=1}^{n} p_k^\beta - 1 \right], \tag{10.4}$$

the entropy of degree β (giving an inequality for the entropy of degree β).

16.12.2 Application to Mixed Theory of Information

In this section, we provide an application of Theorem 16.24 to determine the general solution of the inequality

$$\sum_{k=1}^{n} g_k(p_k)\, f_k(q_k, x_k) \leq \sum_{k=1}^{n} g_k(p_k)\, f_k(p_k, x_k). \tag{16.52}$$

The inequality

$$\sum_{k=1}^{n} p_k\, f_k(q_k, x_k) \leq \sum_{k=1}^{n} p_k\, f_k(p_k, x_k)$$

arises in connection with the generalized problem of "how to keep the (inset) expert honest" (Aczél [15, 22]). Inequality (16.52) is a generalization of the inequality above. Consider the randomized systems of events

$$\begin{pmatrix} x_1, x_2, \ldots, x_n \\ p_1, p_2, \ldots, p_n \end{pmatrix},$$

where p_i is the probability associated with the event x_i $(i = 1, 2, \ldots, n;\ n > 2)$. The events x_i are considered as pairwise disjoint elements of a ring of subsets $\mathcal{B}$ of

some given set Ω. Let $n \geq 3$ and $D_n^0 = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathcal{B} - \{0\},\ x_i \cap x_j = 0$ for $i \neq j;\ i, j = 1, 2, \ldots, n$, and $\Omega = \bigcup_{i=1}^n x_i\}$. We assume throughout this section that $\mathcal{B}$ is not an algebra (that is, $\mathcal{B}$ does not contain $\bigcup_{x \in \mathbb{B}} x$).

We now prove the following theorem.

Theorem 16.25. (Kannappan and Sahoo [505]). *Let $g_k : I_0 \to \mathbb{R}$ be positive and strictly increasing continuous functions and $f_k : I_0 \times (\mathcal{B} - \{0\}) \to \mathbb{R}$ be real-valued functions for $k = 1, 2, \ldots, n$ $(n \geq 3)$. Then f_k and g_k satisfy* (16.52) *for all $(p_1, p_2, \ldots, p_n)$ and $(q_1, q_2, \ldots, q_n) \in \Delta_n^0$ and all $(x_1, x_2, \ldots, x_n) \in D_n^0$ if, and only if, there exist constants $b \geq 0,\ a \in I_0$, and functions $\gamma_k : (\mathcal{B} - \{0\}) \to \mathbb{R}$ $(k = 1, 2, \ldots, n)$ such that*

$$f_k(p_k, x_k) = b \int_a^{p_k} \frac{dt}{g_k(t)} + \gamma_k(x_k) \tag{16.52a}$$

$(x_k \in \mathcal{B} - \{0\},\ p_k \in]0, 1[,\ k = 1, 2, \ldots, n)$.

Proof. Temporarily fixing x_k in (16.52) and using Theorem 16.24, one obtains

$$f_k(p, x_k) = b \int_a^{p} \frac{dt}{g_k(t)} + \gamma_k, \tag{16.53}$$

where $b,\ a$, and γ_k depend on the event $x_k \in \mathcal{B} - \{0\}$. As in (16.45c), α and a are the same for all f_k $(k = 1, 2, \ldots, n)$ and thus in (16.53), and thus we get

$$b(x_1) = b(x_i), \quad i = 1, 2, \ldots, n, \tag{16.54}$$

and

$$a(x_1) = a(x_i), \quad i = 1, 2, \ldots, n, \tag{16.54a}$$

where $(x_1, x_2, \ldots, x_n) \in D_n^0$.

Now we shall show that b and a are constants. Let us consider two nonzero arbitrary elements $x_1,\ y_1$ in $\mathcal{B}$, each contained in partitions $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ of Ω belonging to D_n^0. Without loss of generality, we may assume $x_1 \cap y_1 \neq 0$. If $x_1 \cap y_1 = 0$, then since $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are both the partitions of Ω, there exists an i_0 such that $x_{i_0} \cap y_1 \neq 0$. By renaming x_{i_0} as x_1 and x_1 as x_{i_0}, we obtain a partition satisfying $x_1 \cap y_1 \neq 0$. Next, we shall find two partitions of Ω having one set in common. It is not difficult to see that $(x_1 \cap y_1) \cap x_j = 0,\ ((x_1 - y_1) \cup x_2) \cap x_j = 0$ for all $j \geq 3$ and $(x_1 \cap y_1) \cap ((x_1 - y_1) \cup x_2) = 0$ with $(x_1 - y_1) \cup x_2,\ (y_1 - x_1) \cup y_2,\ x_1 \cap y_1$ are in $\mathcal{B} - \{0\}$. Thus, $(x_1 \cap y_1, (x_1 - y_1) \cup x_2, x_3, \ldots, x_n)$ and $(x_1 \cap y_1, (y_1 - x_1) \cup y_2, y_3, \ldots, y_n)$ also belong to D_n^0. Now, using (16.54a), we get

$$b(x_1 \cap y_1) = b(x_3) = \cdots = b(x_n) = b(y_3) = \cdots = b(y_n) = b(x_1) = b(y_1).$$

Thus b is a constant. Similarly, by (16.54a), a is found to be a constant. Hence we obtain (16.52a). This completes the proof of the theorem.

The following corollaries are obvious from Theorem 16.25.

Corollary (ii). [22]. *Suppose $\mathcal{B}$ is not an algebra. The inequality*

$$\sum_{k=1}^{n} p_k f(q_k, x_k) \leq \sum_{k=1}^{n} p_k f(p_k, x_k)$$

holds for a fixed $n \geq 3$ and for all $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n) \in \Delta_n^0$ if, and only if, there exists a constant $c \geq 0$ and a function $\gamma : \mathcal{B} - \{0\} \to \mathbb{R}$ such that

$$f(p_k, x_k) = c \log p_k + \gamma(x_k)$$

for all $x_k \in \mathcal{B} - \{0\}$, $p_k \in]0, 1[$, $k = 1, 2, \ldots, n$. Thus, the right-hand side of the inequality reduces to

$$c \sum_{k=1}^{n} p_k \log p_k + \sum_{k=1}^{n} p_k \gamma(x_k). \tag{16.55}$$

Remark (iii). The first term in (16.55) is a constant multiple of the Shannon entropy, and the second term is the sum of the expected value of a random variable [15, 22]. In a way, (16.55) has the form of an *inset* entropy.

Corollary (iii). *Suppose $\mathcal{B}$ is not an algebra. The inequality*

$$\sum_{k=1}^{n} p_k^{\beta} f(q_k, x_k) \leq \sum_{k=1}^{n} p_k^{\beta} f(p_k, x_k)$$

holds for a fixed $n \geq 3$, $\beta > 1$, and for all $(x_1, x_2, \ldots, x_n) \in D_n^0$; $(p_1, p_2, \ldots, p_n)$, $(q_1, q_2, \ldots, q_n) \in \Delta_n^0$ if, and only if, there exist constants d, c, and a function $\gamma : \mathcal{B} - \{0\} \to \mathbb{R}$ such that

$$f(p_k, x_k) = \frac{d}{1-\beta} p_k^{1-\beta} + \gamma(x_k) + c$$

for all $x_k \in \mathcal{B} - \{0\}$, $p_k \in]0, 1[$. Thus the right-hand side of the inequality reduces to

$$a + b H_n^{\beta}(P) + \sum_{k=1}^{n} p_k^{\beta} \gamma(x_k), \tag{16.56}$$

where $H_n^{\beta}(P)$ is the entropy of degree β and a, b are constants.

Remark (iv). Note that, in a way, (16.56) has the form of an *inset* entropy of degree β (see [51]).

Absolutely Continuous Solution of the Inequality (16.44e)

Kullback-Leibler directed divergence $D_n(P \| Q)$ defined by (dd) serves as a separability measure (Veglius, Janson, and Johansson [815]) between the distributions

P and Q. It is frequently used in statistics (Kullback [570]), pattern recognition (Ton and Gonzales [805]), coding theory, signal processing, and analysis of questionnaire data. This measure locally behaves like the square of the distance (Kas [528])

$$d(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \sqrt{i(\theta)}\, d\theta,$$

where $\sqrt{i(\theta)}$ is the positive definite square root of the information matrix $i(\theta)$, with θ being the parameter. Further, it is also connected with the χ-square discrepancy measure and is the derivative of the Chernoff coefficient [151, Chapter 10] with respect to α at $\alpha = 1$.

Definition. A function $\mu_n : \Gamma_n^0 \to \mathbb{R}$ is called a *separability measure* if and only if $\mu_n(P\|Q) \geq 0$, $\mu_n(P\|Q)$ attains the minimum if $P = Q$ for all $P, Q \in \Gamma_n^0$ with $n \geq 2$.

Definition. A separability measure $\mu_n : \Gamma_n^0 \to \mathbb{R}$ is a distance measure of Kullback-Leibler type if there exists a function $f : I_0 \to \mathbb{R}$ such that

$$\mu_n(P\|Q) = \sum_{k=1}^{n} p_k(f(p_k) - f(q_k))$$

for all $P, Q \in \Gamma_n^0$ with $n \geq 2$. The function f is called a generating function for the distance measure of Kullback-Leibler type.

In view of these definitions, it is easy to see that the distance measures with the generating function $f : I_0 \to \mathbb{R}$ satisfy (16.44) for $P, Q \in \Gamma_n^0$ (with $n \geq 2$). The most general inequality arising from (16.44) is (16.44c), studied in Theorem 16.24. Now we consider the absolutely continuous solution of (16.44c) holding for $n = 2$, namely

(16.44e) $g_1(x)\, f_1(y) + g_2(1-x)\, f_2(1-y) \leq g_1(x)\, f_1(x) + g_2(1-x)\, f_2(1-x)$,

for $x, y \in]0, 1[$.

Let $g_k : I_0 \to \mathbb{R}_+^*$ $(k = 1, 2)$ be positive, strictly increasing, and continuous, and $f_k : I_0 \to \mathbb{R}$ $(k = 1, 2)$ be absolutely continuous functions, and these functions satisfy the functional inequality (16.44e). Rewriting this equation, we obtain

$$g_1(x)\{f_1(y) - f_1(x)\} \leq g_2(1-x)\{f_2(1-x) - f_2(1-y)\} \tag{16.57}$$

for all $x, y \in I_0$. Interchanging x with y in the inequality above, we obtain

$$g_1(y)\{f_1(x) - f_1(y)\} \leq g_2(1-y)\{f_2(1-y) - f_2(1-x)\}. \tag{16.57a}$$

Since g_2 is positive, multiplying (16.57) by $g_2(1-y)$ and (16.57a) by $g_2(1-x)$ and then adding the resultant, we obtain

$$\{g_1(y)g_2(1-x) - g_1(x)g_2(1-y)\}\{f_1(x) - f_1(y)\} \leq 0. \tag{16.58}$$

Since the functions g_1 and g_2 are strictly increasing, if $x > y$, then

$$g_1(y)g_2(1-x) - g_1(x)g_2(1-y) \le 0$$

and hence (16.58) yields $f_1(x) \ge f_1(y)$. That is, f_1 is an increasing function on I_0. Similarly, f_2 is also an increasing function on I_0.

Remark. Note that the monotonicity of f_k's follow from the monotonicity of g_k's. Here the absolute continuity of f_k's is not used.

Also, from (16.57) and (16.57a), we get for $x > y$

$$\frac{g_2(1-x)}{g_1(x)}\,\frac{f_2(1-y) - f_2(1-x)}{x-y} \le \frac{f_1(x) - f_1(y)}{x-y} \le \frac{g_2(1-y)}{g_1(y)}\,\frac{f_2(1-y) - f_2(1-x)}{x-y}.$$

From this inequality, we conclude using the continuity of g_1 and g_2 that if $f_2'(1-x)$ exists at $1-x$, then $f_1'(x)$ exists at x; that is, if f_2 is differentiable at $1-x$, then so is f_1 at x. Since g_1 and g_2 are strictly increasing continuous functions and since f_1 and f_2 have derivatives almost everywhere, we get

$$g_1(x)\, f_1'(x) = g_2(1-x)\, f_2'(1-x) \tag{16.59}$$

for almost all $x \in I_0$ and (refer to Natanson [644, p. 255])

$$f_1(x) = f_1(a) + \int_a^x \frac{g_2(1-t)}{g_1(t)}\psi'(1-t)dt \tag{16.60}$$

and

$$f_2(x) = \psi(x), \tag{16.60a}$$

where ψ is an arbitrary absolutely continuous increasing function on I_0 and $a \in I_0$ is a constant. Thus we have proved the following theorem.

Theorem 16.26. (Kannappan and Sahoo [507]). *Let $g_k : I_0 \to \mathbb{R}_+^*$ $(k = 1, 2)$ be positive, strictly increasing, and continuous functions on I_0. Let $f_k : I_0 \to \mathbb{R}$ $(k = 1, 2)$ be real-valued functions and satisfy* (16.44e). *Then f_k's are increasing. In addition, if f_1 is differentiable at x, then f_2 is differentiable at $(1-x)$. Further, if f_k's are absolutely continuous, then f_k's satisfy* (16.44e) *if, and only if, f_k's have the form given in* (16.60) *and* (16.60a).

16.12.3 Continuous Solution of the Inequality (16.61)

Now we find the continuous solution of (16.61), which is a special case of (16.44e) for $g_1(x) = x = g_2(x)$, by making use of a result in [641],

$$x f_1(y) + (1-x) f_2(1-y) \le x f_1(x) + (1-x) f_2(1-x). \tag{16.61}$$

Theorem 16.27. [507]. *Let $f_1, f_2 : I_0 \to \mathbb{R}$ be real-valued continuous functions, and they satisfy* (16.61) *for all $x, y \in I_0$. Then f_1 and f_2 are of the form*

$$f_1(x) = (1-x)H\left(x - \frac{1}{2}\right) + \int_0^{x-\frac{1}{2}} H(t)dt + c_1, \quad x \in I_0, \tag{16.61a}$$

and

$$f_2(x) = (x-1)H\left(\frac{1}{2} - x\right) + \int_0^{\frac{1}{2}-x} H(-t)dt + c_2, \quad x \in I_0, \tag{16.61b}$$

where $H : I_{1/2} =]0, \frac{1}{2}[\to \mathbb{R}$ is an arbitrary continuous and increasing function and c_1, c_2 are arbitrary constants.

Proof. Let $f_1, f_2 : I_0 \to \mathbb{R}$ be continuous and satisfy (16.61). First of all, notice that if f_1 and f_2 are solutions of (16.61), so also are $f_1 + a_1$ and $f_2 + a_2$ for some arbitrary constants a_1 and a_2. Replacing x by $x + \frac{1}{2}$ and y by $y + \frac{1}{2}$ in (16.61), we obtain

$$\begin{aligned}\left(\frac{1}{2}+x\right) f_1\left(\frac{1}{2}+x\right) + \left(\frac{1}{2}-x\right) f_2\left(\frac{1}{2}-x\right) \\ \geq \left(\frac{1}{2}+x\right) f_1\left(\frac{1}{2}+y\right) + \left(\frac{1}{2}-x\right) f_2\left(\frac{1}{2}-y\right)\end{aligned}$$

for all $x, y \in I_{1/2}$. By defining

$$G(x) := \frac{1}{2}\left\{ f_1\left(\frac{1}{2}+x\right) + f_2\left(\frac{1}{2}-x\right)\right\}$$

and

$$H(x) := \left\{ f_1\left(\frac{1}{2}+x\right) - f_2\left(\frac{1}{2}-x\right)\right\},$$

we obtain from the inequality above

$$G(x) - G(y) \geq -x\{H(x) - H(y)\} \tag{16.62}$$

for all $x, y \in I_{1/2}$. From Theorem 16.26, we see that f_1 and f_2 are increasing. Thus H is also increasing on the interval $I_{1/2}$. As f_1 and f_2 are continuous, so are H and G on the interval $I_{1/2}$. Interchanging x and y in (16.62), we have

$$G(y) - G(x) \geq -y\{H(y) - H(x)\}. \tag{16.62a}$$

From these, we obtain

$$-x\{H(x) - H(y)\} \leq G(x) - G(y) \leq -y\{H(x) - H(y)\}.$$

If $H(x) = H(y)$ for all $x, y \in I_{1/2}$, then $G(x) = G(y)$, which in fact shows that G is identically a constant. Hence, from the definitions of G and H, we see that f_1 and f_2 are constants.

Suppose now that there exist x and y such that $H(x) \neq H(y)$. Then, as in [641], if we define $\Omega = \{H(x) \mid x \in I_{1/2}\}$ and $\phi : \Omega \to \mathbb{R}$ by $\phi(\omega) = G(x)$, where x is such that $H(x) = \omega$, it follows that

$$G(x) = \phi(H(x)) = -xH(x) + \int_0^x H(t)dt + c, \tag{16.63}$$

where c is an arbitrary constant. Since

$$f_1\left(\frac{1}{2} + x\right) = G(x) + \frac{1}{2}H(x) \quad \text{and} \quad f_2\left(\frac{1}{2} - x\right) = G(x) - \frac{1}{2}H(x),$$

we get from (16.63)

$$f_1\left(\frac{1}{2} + x\right) = \left(\frac{1}{2} - x\right)H(x) + \int_0^x H(t)dt + c$$

and

$$f_2\left(\frac{1}{2} - x\right) = -\left(\frac{1}{2} + x\right)H(x) + \int_0^x H(t)dt + c.$$

These can be rewritten as (16.61a) and (16.61b). This completes the proof of the theorem. □

Corollary. *The continuous solution of the functional inequality*

$$\sum_{k=1}^{2} p_k f(q_k) \leq \sum_{k=1}^{2} p_k f(p_k),$$

where $P, Q \in \Gamma_2^0$, *is given by*

$$f(x) = (1 - x)H\left(x - \frac{1}{2}\right) + \int_0^{x-\frac{1}{2}} H(t)dt + c,$$

where H *is a continuous, increasing, and odd function on the interval* $I_{1/2}$.

Remark. The distance measures of Kullback-Leibler type on Γ_2^0 with a continuous generating function are of the form

$$\mu_2(P, Q) = \sum_{k=1}^{2} p_k \left\{(1 - p_k)H\left(p_k - \frac{1}{2}\right) - (1 - q_k)H\left(q_k - \frac{1}{2}\right)\right\} + \sum_{k=1}^{2} p_k \int_{q_k - \frac{1}{2}}^{p_k - \frac{1}{2}} H(t)dt,$$

where $H : I_{1/2} \to \mathbb{R}$ is an arbitrary continuous, increasing, odd function.

16.13 Reproducing Scoring Systems

A direct approach to measuring a subject's knowledge concerning a test item is to attempt to elicit from him his subjective probability distribution over the options. We shall investigate scoring systems designed for this purpose. There are many techniques for assessing the present state of a subject's knowledge. In the case of objective testing and programmed instruction techniques, most of the information necessary to assess a subject's knowledge is contained in the subjective probabilities (of the subject) concerning the correctness of various possible answers. To measure these subjective probabilities, it is necessary to have a scoring system. In connection with weather forecasting, scoring systems were used long ago in [83]. For scoring systems and other related materials, [771] is a good source of references. Assume that a subject has a subjective probability distribution $P = (p_1, p_2, \ldots, p_n) \in \Gamma_n$ over the n options of a particular item. Let $Q = (q_1, q_2, \ldots, q_n) \in \Gamma_n$ be any probability distribution. A *scoring system* is a collection of n-place functions $\{f_1, f_2, \ldots, f_n\}$ for which $f_i(q_1, q_2, \ldots, q_n)$ is the score awarded to the subject for the ith option if he indicates that Q is his subjective probability distribution. The expected score of the subject from giving the distribution Q as his response to the item is

$$E_n(P, Q) = \sum_{k=1}^{n} p_k f_k(q_1, q_2, \ldots, q_n), \tag{16.64}$$

and the loss to the subject by giving Q as his subjective probability when it is actually P is

$$D_n(P \| Q) = E_n(P, P) - E_n(P, Q).$$

The collection of scoring functions $\{f_1, f_2, \ldots, f_n\}$ is a *reproducing scoring system* (RSS) if it encourages the subject to report his true probabilities (see Weber [826, p. 269]). That is, $\{f_1, f_2, \ldots, f_n\}$ is an RSS if

$$\operatorname*{Max}_{Q \in \Gamma_n} E_n(P, Q) = E_n(P, P). \tag{16.64a}$$

From (16.64) and (16.64a), we obtain the functional inequality

$$\sum_{k=1}^{n} p_k f_k(q_1, q_2, \ldots, q_n) \le \sum_{k=1}^{n} p_k f_k(p_1, p_2, \ldots, p_n), \tag{16.64b}$$

where $P, Q \in \Gamma_n$.

The solution of (16.64b) generates the scoring functions for the RSS for multiple-choice testing. The functional inequality above was solved in [735] for the two-option case (that is, for $n = 2$,) with the assumptions that $\psi_1(x) := f_1(x, 1-x)$ and $\psi_2(x) := f_2(1-x, x)$ are both differentiable and nondecreasing on $[0, 1]$.

Consider the inequality

$$\sum_{k=1}^{n} g_k(p_k) f_k(q_1, q_2, \ldots, q_n) \le \sum_{k=1}^{n} g_k(p_k) f_k(p_1, p_2, \ldots, p_n), \tag{16.65}$$

where $P, Q \in \Gamma_n^0$ and $g_k : I_0 \to \mathbb{R}$ $(k = 1, 2, \ldots, n)$ are continuous monotonically increasing and positive functions. In what follows, we assume once and for all that g_k's have the properties above. The functional inequality (16.65) is a generalization of (16.64b). For the two-option case, (16.65) yields

(16.65a)
$$g_1(p_1)\, f_1(q_1, q_2) + g_2(p_2)\, f_2(q_1, q_2) \le g_1(p_1)\, f_1(p_1, p_2) + g_2(p_2)\, f_2(p_1, p_2),$$

where $p_1 + p_2 = 1 = q_1 + q_2$. Denote $p_1 = p$ and $q_1 = q$ in (16.65a) to obtain

(16.66) $$g_1(p)\psi_1(q) + g_2(1-p)\psi_2(1-q) \le g_1(p)\psi_1(p) + g_2(1-p)\psi_2(1-p),$$

where $\psi_1(p) = f_1(p, 1-p)$, $\psi_2(p) = f_2(1-p, p)$, and $p, q \in I_0$.

Now we investigate the functional inequality (16.66). First of all, we will show that every solution of (16.66) is monotonically increasing. Second, we will determine the general solution of (16.66) under weaker regularity conditions on ψ_k than those assumed for solving (16.66) with $g_k(p) = p$ $(k = 1, 2)$ in [735]. Finally, we propose a generalization of the RSS.

16.13.1 Solution of the Functional Inequality (16.66)

Let $g_k : I_0 \to \mathbb{R}$ $(k = 1, 2)$ be positive, strictly increasing, and continuous, and let $\psi_k : I_0 \to \mathbb{R}$ $(k = 1, 2)$ be absolutely continuous functions (see Natanson [644, p. 243] and these functions satisfy the functional inequality (16.66).

Rewriting this, we obtain

(16.67) $$g_1(p)(\psi_1(q) - \psi_1(p)) \le g_2(1-p)(\psi_2(1-p) - \psi_2(1-q))$$

for all $p, q \in I_0$. Interchanging p with q in (16.67), the inequality becomes

(16.67a) $$g_1(q)(\psi_1(p) - \psi_1(q)) \le g_2(1-q)(\psi_2(1-q) - \psi_2(1-p)).$$

Since g_2 is positive, multiplying (16.67) by $g_2(1-q)$ and (16.67a) by $g_2(1-p)$, and then adding the resultant, we obtain

(16.67b) $$(g_1(q)g_2(1-p) - g_1(p)g_2(1-q))(\psi_1(p) - \psi_1(q)) \le 0.$$

Since the functions g_1 and g_2 are strictly increasing, if $p > q$, then $g_1(q)g_2(1-p) - g_1(p)g_2(1-q) \le 0$ and hence (16.67) yields $\psi_1(p) \ge \psi_1(q)$. That is, ψ_1 is an increasing function on I_0. Similarly, ψ_2 is also an increasing function on I_0.

Remark (i). Note that the monotonicity of ψ_k's follows from the monotonicity of g_k's. Here the absolute continuity of ψ_k's is not used.

Also from (16.67) and (16.67a), we get for $p > q$

$$\frac{g_2(1-p)}{g_1(p)} \frac{\psi_2(1-q) - \psi_2(1-p)}{p-q} \le \frac{\psi_1(p) - \psi_1(q)}{p-q} \le \frac{g_2(1-q)}{g_1(q)} \frac{\psi_2(1-q) - \psi_2(1-p)}{p-q}.$$

From the inequality above, we conclude using the continuity of $g_1,\ g_2$ that if $\psi_2'(1-p)$ exists at $1-p$, then $\psi_1'(p)$ exists at p; that is, if ψ_2 is differentiable at $1-p$, then so is ψ_1 at p. Since g_1 and g_2 are continuous and monotone functions have derivatives almost everywhere, we have

$$g_1(p)\psi_1'(p) = g_2(1-p)\psi_2'(1-p)$$

for almost all $p \in I_0$ and (see [644, p. 255])

$$\psi_1(p) = \psi_1(a) + \int_a^p \frac{g_2(1-t)}{g_1(t)}\psi'(1-t)dt \tag{16.68}$$

and

$$\psi_2(p) = \psi(p), \tag{16.68a}$$

where ψ is an arbitrary absolutely continuous and increasing function on I_0, and $a \in I_0$ is a constant. Thus we have proved the following theorem.

Theorem 16.28. *Let g_k's $(k = 1, 2)$ be positive, strictly increasing, and continuous functions on I_0. Let $\psi_k : I_0 \to \mathbb{R}$ $(k = 1, 2)$ be real-valued functions and satisfy* (16.66)*. Then ψ_k's are monotonically increasing. If ψ_1 is differentiable at p, then ψ_2 is differentiable at $1-p$. Suppose ψ_k's are absolutely continuous. Then ψ_k's satisfy* (16.66) *if, and only if, ψ_k's have the form given in* (16.68) *and* (16.68a)*.*

Remark (ii). If ψ_k's are differentiable, then solutions of (16.66) can be obtained from (16.68) and (16.68a). Assuming g_k's to be the identity function, Sholander [735] found the differentiable solutions of (16.64b) for $n = 2$. Some of the consequences of Theorem 16.28 are given below.

Corollary (i). *If $\psi_1(p) = f_1(p, 1-p)$ and $\psi_2(p) = f_2(1-p, p)$ are absolutely continuous and satisfy* (16.64b) *for $n = 2$ with $g_1(p) = g_2(p) = p$, then f_k's are of the form*

$$f_1(p, 1-p) = \psi_1(a) + \int_a^p \left(\frac{1}{t} - 1\right)\psi'(1-t)dt$$

and

$$f_2(p, 1-p) = \psi(1-p),$$

where $\psi(x)$ is an arbitrary but monotonically increasing and absolutely continuous function on I_0 and $a \in I_0$ is an arbitrary constant.

Example (i). Let $g_1(p) = g_2(p) = p$ in (16.66). Further, let $\psi_2(p) = -\frac{1}{2}(1-p)^2$ be a monotonically increasing function on I_0. Then, from (16.68), we get $\psi_1(p) = -\frac{1}{2}(1-p)^2$. Therefore, the scoring functions for the RSS are $f_1(p, 1-p) = f_2(1-p, p) = -\frac{1}{2}(1-p)^2$.

Example (ii). Let $g_1(p) = p$ and $g_2(p) = p^2$ in (16.66). Assume that $\psi_2(p) = -\frac{1}{2}(1-p)^2$. Then, by (16.68), we get

$$\psi_1(p) = \psi_1(a) + \int_a^p \frac{(1-t)^2}{t} t\, dt.$$

Hence the scoring functions are given by $f_1(p, 1-p) = -\frac{(1-p)^3}{3}$ and $f_2(p, 1-p) = -\frac{1}{2}p^2$.

Corollary (ii). *Let g_k's be positive, strictly increasing, and continuous functions on I_0. Let $\psi_1(p) = f_1(p, 1-p)$ and $\psi_2(p) = f_2(1-p, p)$ be absolutely continuous and satisfy* (16.66)*. If $f_1 = f_2 = f$, then $f_1(p, 1-p) = c$ (constant) for almost all $p \in I_0$.*

Proof. Now $f_1 = f_2$ on I_0 implies $\psi_1(p) = \psi_2(1-p)$ on I_0. Thus, from (16.67b), we obtain $(g_1(p) + g_2(1-p))\psi_1'(p) = 0$. Since $g_1(p) + g_2(1-p) > 0$, we have $\psi_1'(p) = 0$. Hence $\psi_1(p)$ is a constant for almost all $p \in I_0$.

Example (iii). For $g_1(p) = g_2(p) = p$ and $f_1 = f_2 = f$, then one of the solutions of (16.66) is $2p - p^2$ (cf. Weber [826]).

Example (iv). If we do not impose the condition of absolute continuity on ψ_k's, then there are solutions of the functional inequality (16.66) that are not of the form (16.68) and (16.68a). For example, if $g_1(p) = g_2(p) = p$ and $\psi_1(t) = \psi_2(t)$, then, for a positive constant c, the saltus function

$$f(p) = \begin{cases} 0, & \text{if } 0 < p < \frac{1}{2}, \\ c, & \text{if } p = \frac{1}{2}, \\ 2c, & \text{if } \frac{1}{2} < p < 1, \end{cases}$$

is a solution of (16.66).

16.13.2 Symmetric Reproducing Scoring Systems

A reproducing scoring system $\{f_1, f_2, \ldots, f_n\}$ is said to be a *symmetric reproducing scoring system* (SRSS) if, and only if,

$$f_{\pi(i)}(p_1, p_2, \ldots, p_n) = f_1(p_{\pi(1)}, p_{\pi(2)}, \ldots, p_{\pi(n)})$$

for every permutation π of $\{1, 2, \ldots, n\}$. For details regarding SRSS, refer to [735].

In the SRSS $\{f_i(p_1, p_2, \ldots, p_n) \mid i = 1, 2, \ldots, n\}$, the score awarded depends not only on the probability assigned to the correct option but also the distribution over the various incorrect options. An SRSS does not depend on the distribution of incorrect options if

$$f_{\pi(i)}(p_1, p_2, \ldots, p_n) = f(p_{\pi(i)}).$$

A functional inequality related to such a special kind of SRSS is the inequality (16.44) (Shannon's inequality). The general solution of (16.44) for $n \geq 3$ with no regularity assumptions on the unknown function f is given by (16.45). In [735] it was shown that, under the differentiability of the scoring function f, such a special

type of SRSS gives rise to a *logarithmic* scoring system. However, even without assuming any regularity condition on f, one may arrive at a logarithmic scoring system from (16.44) for $n > 2$.

Sholander [735] has discussed the construction of an RSS in which the subject's score depends only on the probability assigned to the correct answer and not on the probabilities assigned to the incorrect answers. For such a scoring system, the scoring functions would have the property that

$$f_k(p_1, p_2, \ldots, p_n) = h_k(p_k)$$

for all $P \in \Gamma_n^0$. Assuming h_k's to be continuous over the closed unit interval and continuously differentiable over its interior, it was shown in [735] that h_k's have the form

$$h_k(p) = a \log p + b_k, \quad p \in I_0,$$

where $a, b_1, b_2, \ldots, b_n$ are arbitrary constants. In this RSS, the loss to the subject when he gives Q as his subjective probability when it is actually P is (dd) $D_n(P \| Q)$. The aforementioned type of RSS with the special scoring functions h_k satisfies the functional inequality (16.44).

16.13.3 A Generalization of RSS

Consider that a subject has a subjective probability distribution $P = (p_1, p_2, \ldots, p_n) \in \Gamma_n^0$ over the n options of a particular item. Let $Q \in \Gamma_n^0$ be any probability distribution. Let $\{f_1, f_2, \ldots, f_n\}$ be a scoring system and $f_i(q_1, q_2, \ldots, q_n)$ be the score awarded to the subject for the ith option if he indicates that Q is his subjective probability distribution. In such a scoring system, the score awarded depends not only on the probability of the correct option but also on the probabilities of the incorrect options. To counterbalance the effect of the probabilities of the incorrect options, we award $g_i(p_i)\, f_i(q_1, q_2, \ldots, q_n)$ as his score instead of $f_i(q_1, q_2, \ldots, q_n)$. The function $g_i(p)$ can be regarded as the weighting function depending only on the true probability of the ith option. Then we define the total weighted score of the subject from giving the distribution Q as his response to the item as

$$T_n(P, Q) = \sum_{k=1}^{n} g_k(p_k)\, f_k(q_1, q_2, \ldots, q_n).$$

A collection of scoring functions $\{f_1, f_2, \ldots, f_n\}$ is a generalized reproducing scoring system if, and only if,

$$\operatorname*{Max}_{Q \in \Gamma_n^0} T_n(P, Q) = T_n(P, P).$$

It is easy to see that the reproducing scoring system is a special case of the generalized reproducing scoring system since

$$T_n(P, Q) = E_n(P, Q)$$

if $g_k(p) = p$. The following is an example of a generalized RSS.

Example (v). Let $g_k(p) = p^2$ $(k = 1, 2)$. Then the set of scoring functions $\{f_1, f_2, \ldots, f_n\}$, where

$$f_k(p_1, p_2, \ldots, p_n) = \frac{p_k^2}{\left(\sum_{j=1}^n p_j^4\right)^{1/2}},$$

defines a generalized RSS. To see this, notice that

$$\sum_{k=1}^n p_k^2 f_k(q_1, q_2, \ldots, q_n) = \frac{\sum_{k=1}^n p_k^2 q_k^2}{\left(\sum_{j=1}^n q_j^4\right)^{1/2}} \le \left(\sum_{k=1}^n p_k^4\right)^{1/2}$$
$$= \sum_{k=1}^n p_k^2 f_k(p_1, p_2, \ldots, p_n).$$

This inequality is obtained via Holder's inequality. Equality in the inequality above holds if, and only if, $q_k = a p_k$ $(k = 1, 2, \ldots, n)$, where a is a constant.

Since $(p_1, p_2, \ldots, p_n)$ and $(q_1, q_2, \ldots, q_n)$ are in Γ_n^0, $a = 1$, and hence, for $P \in \Gamma_n^0$, $\sum_{k=1}^n p_k^2 f_k(q_1, q_2, \ldots, q_n)$ is maximized when $P = Q$.

A special case of the generalized RSS that may be of some interest is the one that has the property that $f_k(p_1, p_2, \ldots, p_n) = h_k(p_k)$ $(k = 1, 2, \ldots, n)$ for all $P \in \Gamma_n^0$. Such a system gives rise to the functional inequality (16.44c).

16.14 More Inequalities from Information Theory

We are concerned with the functional inequality

$$\sum_{i=1}^n p_i \frac{f_i(p_i)}{f_i(q_i)} \le 1, \tag{16.69}$$

where $P, Q \in \Gamma_n^0$, $f_i : I_0 \to \mathbb{R}^*$ (that is, $f_i(p) \neq 0$) holding for fixed $n \ge 2$, connected with Rényi's entropy.

Result 16.29. (Kardos [526]). *Let $f_i : I_0 \to \mathbb{R}^*$ $(i = 1, 2)$ satisfy the inequality*

$$p \frac{f_1(p)}{f_1(q)} + (r - p) \frac{f_2(r - p)}{f_2(r - q)} \le r \tag{16.69a}$$

for all $p, q \in]0, r[$ for fixed $r \in I_0$. If the f_i do not change signs, then each of the following holds:

(i) *f_i is monotonically decreasing (increasing) on $]0, r[$ if f_i is positive (negative) on I_0.*

(ii) *$p \to p f_i(p)$ is increasing* (*decreasing*) *on* $]0, r[$ *if* f_i *is positive* (*negative*) *on* I_0.

(iii) *f_i is locally absolutely continuous on* $]0, r[$.

(iv) *If f_1 is differentiable at p, then f_2 is differentiable at $r - p$ and the following relation is valid:*

$$p\frac{f_1'(p)}{f_1(p)} = (r-p)\frac{f_2'(r-p)}{f_2(r-p)}.$$

All solutions f_i that do not change signs on I_0 $(i = 1, 2)$ of the inequality

$$p\frac{f_1(p)}{f_1(q)} + (1-p)\frac{f_2(1-p)}{f_2(1-q)} \le 1, \quad 0 < p < 1,\ 0 < q < 1, \tag{16.70}$$

are of the form

$$f_1(p) = a\exp\left(\int_c^p \frac{(1-t)g'(1-t)}{tg(1-t)}dt\right), \quad f_2(p) = bg(p), \quad p \in I_0,$$

where a, b, and c are arbitrary, $ab \neq 0$, $c \in I_0$, with g arbitrarily continuous, positive, and decreasing, and $p \to pg(p)$ increasing on I_0.

Corollary. *All solutions of* (16.70) *when $f_2 = f_1 = f$ are of the form*

$$f(p) = a\exp\left(\int_b^p \frac{G(t)}{t}dt\right), \quad p \in I_0,$$

where $a \neq 0$, $b \in I_0$ with G arbitrarily measurable on I_0 and satisfying, for almost all $p \in I_0$, $G(1-p) = G(p)$ and $-1 \le G(p) \le 0$.

Result 16.30. [526]. *If f_i $(i = 1, 2, \ldots, n)$ do not change signs, then the general solution to* (16.69) *for fixed $n \ge 3$ has the form*

$$f_i(p) = b_i p^a, \quad i = 1, 2, \ldots, n,$$

where $-1 \le a \le 0$ and $b_i > 0$ (< 0) if $f_i > 0$ (< 0).

Next in line is the inequality

$$\sum_{i=1}^{n} p_i \frac{f(p_i)}{f(q_i)} \ge 1, \tag{16.71}$$

for all $P, Q \in \Gamma_n^0$, holding for some integer $n \ge 2$.

Result 16.31. (Fischer [276]). *Let $f : I_0 \to \mathbb{R}_+^*$ satisfy* (16.71)*. If* (16.71) *holds for a fixed $n \ge 3$ and if f is monotonic, then f is differentiable and has the form $f(p) = ap^c$, where $a > 0$ and either $c \le -1$ or $c \ge 0$.*

From the measure (dd) $D_n(P\|Q)$ and other considerations, we studied (16.44). We considered symmetric distance measures in Chapter 10. From this symmetric divergence or J-divergence

$$\begin{aligned}\mu_n(P\|Q) &= D_n(P\|Q) + D_n(Q\|P)\\ &= \sum_{i=1}^{n} p_i \log p_i + q_i \log q_i - p_i \log q_i - q_i \log p_i\end{aligned}$$

there results the functional inequality

$$\sum_{i=1}^{n}[p_i f(p_i) + q_i f(q_i)] \geq \sum_{i=1}^{n}[p_i f(q_i) + q_i f(p_i)] \tag{16.72}$$

for $P, Q \in \Gamma_n^0$ (see [275]).

16.15 Inequalities from Inner Product Spaces

In Skof [752] it is shown that if $f : X \to \mathbb{R}$ and X a real linear space satisfies the equations

$$|f(x+y)| = |2f(x) + 2f(y) - f(x-y)|, \tag{16.73}$$

$$|f(x-y)| = |2f(x) + 2f(y) - f(x+y)|, \tag{16.73a}$$

$$|f(x+y) + f(x-y) - 2f(x)| = 2|f(y)|, \tag{16.73b}$$

$$|f(x+y) - f(x-y) - 2f(y)| = 2|f(x)|, \tag{16.73c}$$

$$|f(x+y) + f(x-y)| = |2f(x) + 2f(y)|, \tag{16.73d}$$

for $x, y \in X$, then f satisfies the Jordan-von Neumann parallelogram equation; that is, the quadratic equation (Q).

Remarks

(i) $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} cx^2, & x \geq 0,\\ -cx^2, & x < 0, \end{cases} \quad c \neq 0,$$

satisfies the inequality version of (16.73a),

$$|2f(x) + 2f(y) - f(x+y)| \leq |f(x-y)|,$$

but not the quadratic equation (Q).

(ii) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = cx^{2n}$, $c \neq 0$, $n \in Z_+^*$ (> 1), $x \in \mathbb{R}$ satisfies the inequalities

$$|2f(x)| \le |f(x+y) + f(x-y) - 2f(y)|,$$
$$|2f(y)| \le |f(x+y) + f(x-y) - 2f(x)|,$$
$$|2f(x) + 2f(y)| \le |f(x+y) + f(x-y)|,$$

for $x, y \in \mathbb{R}$, but not (Q).

The situation changes if we consider the inequality arising from (16.73), namely

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \le \|f(xy)\|, \tag{16.74}$$

for $x, y \in G$ a group. It is proved that if $f : G \to X$ satisfies (16.74), where G is a group, X is an i.p.s., and G is a 2-divisible Abelian group or is 2-divisible and f satisfies (K), then f satisfies (Q).

Here we present the proof from [695] and prove the following theorem.

Theorem 16.32. (Rätz [695]). *Let G be a group and X an i.p.s. over a field K ($= \mathbb{R}$ or $\mathbb{C}$), and $f : G \to X$ satisfies the inequality* (16.74)*,*

$$f(xy) = f(yx), \tag{16.74i}$$

and

$$f(xyxy^{-1}) = f(x^2), \quad \text{for } x, y \in G. \tag{16.74ii}$$

Then

$$f(e) = 0, \quad f(x) = f(x^{-1}), \quad f(x^2) = 4f(x), \tag{16.75}$$

$x \in G$, and f is quadratic (Q).

Proof. $x = e$ in (16.74) gives $f(e) = 0$, and e the identity in G. $y = x^{-1}$ in (16.74) implies

$$2f(x) + 2f(x^{-1}) = f(x^2) \quad \text{for } x \in G. \tag{16.76}$$

Let $y = x$ in (16.74) to get

$$4\|f(x)\| \le \|f(x^2)\| \quad \text{for } x \in G. \tag{16.77}$$

Now

$$\|4f(x)\| \le \|f(x^2)\| \le 2\|f(x)\| + 2\|f(x^{-1})\|$$

by (16.76) or

$$\|f(x)\| \le \|f(x^{-1})\|,$$

and in turn $\le \|f(x)\|$; that is,

$$\|f(x)\| = \|f(x^{-1})\| \tag{16.77a}$$

and

$$\|f(x^2)\| = 4\|f(x)\|. \tag{16.77b}$$

Now

$$\|f(x) - f(x^{-1})\|^2 = 2\|f(x)\|^2 + 2\|f(x^{-1})\|^2 - \|f(x) + f(x^{-1})\|^2$$

(parallelogram identity in an i.p.s.)

$$\begin{aligned} &= 4\|f(x)\|^2 - \frac{1}{4}\|f(x^2)\|^2 \quad \text{(by (16.77a) and (16.76))} \\ &= 4\|f(x)\|^2 - 4\|f(x)\|^2 \quad \text{(by (16.77b))} \\ &= 0. \end{aligned}$$

Thus $f(x) = f(x^{-1})$ and $f(x^2) = 4f(x)$. This proves (16.75).

Replace x by xy, and y by xy^{-1} in (16.74), and use (16.75), (16.74i), and (16.74ii) to obtain

$$\|2f(xy) + 2f(xy^{-1}) - 4f(y)\| \le 4\|f(x)\|.$$

Squaring both sides, we obtain

$$\|2f(xy)+2f(xy^{-1})\|^2+16\|f(y)\|^2-2\mathrm{Re}\langle 2f(xy)+2f(xy^{-1}), 4f(y)\rangle \le 16\|f(x)\|^2.$$

Interchange x and y and use f even from (16.75) to get

$$\|2f(xy)+2f(xy^{-1})\|^2+16\|f(x)\|^2-2\mathrm{Re}\langle 2f(xy)+2f(xy^{-1}), 4f(x)\rangle \le 16\|f(y)\|^2.$$

Adding, we have

$$\begin{aligned} \|f(xy) + f(xy^{-1})\|^2 &\le 2\mathrm{Re}\langle f(xy) + f(xy^{-1}),\ f(x) + f(y)\rangle \\ &\le 2\|f(xy) + f(xy^{-1})\| \cdot \|f(x) + f(y)\| \\ &\qquad \text{(Cauchy-Schwarz inequality)} \end{aligned} \tag{16.78}$$

or

$$\|f(xy) + f(xy^{-1})\| \le 2\|f(x) + f(y)\|.$$

Let $u, v \in G$. Put $x = uv,\ y = uv^{-1}$, and use (16.74), (16.74i), and (16.74ii) in the above to have

$$\|4f(u) + 4f(v)\| \le 2\|f(uv) + f(uv^{-1})\|,$$

so that

$$\|f(xy) + f(xy^{-1})\| = 2\|f(x) + f(y)\|. \tag{16.79}$$

Now

$$\begin{aligned}
&\|f(xy)+f(xy^{-1})-2f(x)-2f(y)\|^2\\
&\quad=\|f(xy)+f(xy^{-1})\|^2\\
&\qquad+4\|f(x)+f(y)\|^2-2\mathrm{Re}\langle f(xy)+f(xy^{-1}),2f(x)+2f(y)\rangle\\
&\quad=2\|f(xy)+f(xy^{-1})\|^2\\
&\qquad-2\mathrm{Re}\langle f(xy)+f(xy^{-1}),2f(x)+2f(y)\rangle \quad \text{by (16.79)}\\
&\quad\le 0 \quad \text{(by (16.78))}.
\end{aligned}$$

Thus f satisfies (Q). This completes the proof of this theorem. □

By some modification of parallelogram identity:

(i) $\|u+v\|^2+\|u-v\|^2\sim 4$, where $\sim$ denotes either $\le$ or $\ge$ for $\|u\|=\|v\|=1$ [214].

(ii) For $\|u\|=1=\|v\|$, there are $\alpha,\beta\neq 0$ such that

$$\|\alpha u+\beta v\|^2\|\beta u-\alpha v\|^2\sim 2(\alpha^2+\beta^2).$$

16.16 Miscellaneous Inequalities

16.16.1 More on Convex Functions

Let H be a convex open nonempty subset of $\mathbb{R}^n$ and $\mathcal{B}_n=\{T\subseteq\mathbb{R}^n$: every convex function bounded above on T is continuous$\}$. Which subsets belong to $\mathcal{B}_n$? Every nonempty open subset T of $\mathbb{R}^n$ is in ξ_n. Let $C\subseteq\mathbb{R}^n$ be an open connected set $(n\ge 2)$. If $f:C\to\mathbb{R}$ is a continuous function such that gr f (graph of f) contains $(n+1)$ affinely independent points, then gr $f\in\mathcal{B}_n$ (Jablanski [387]).

Suppose $f:C\to\mathbb{R}$ is convex (16.10) (concave), where $C\subseteq X$ (a real linear space) is a convex set. Then, for $x_1,x_2,\ldots,x_n\in C$, $p_1,\ldots,p_n\in\mathbb{R}_+$, $P_n=\sum_{i=1}^n p_i>0$, the following inequalities hold [227]:

(i)

$$f\left(\frac{1}{P}\sum_{i=1}^n p_i x_i\right)\le(\ge)\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i).$$

(ii)

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq (\geq)\frac{1}{P_n^{k+1}}\sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}} f\left(\frac{1}{k+1}\sum_{j=1}^{k+1} x_{i_j}\right)$$
$$\leq (\geq)\frac{1}{P_n^{k}}\sum_{i_1,\ldots,i_{k}=1}^{n} p_{i_1}\cdots p_{i_{k}} f\left(\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right) \leq (\geq)\cdots\leq(\geq)$$
$$\leq (\geq)\frac{1}{P_n^{2}}\sum_{i_1,i_2=1}^{n} p_{i_1} p_{i_2} f\left(\frac{x_{i_1}+x_{i_2}}{2}\right)$$
$$\leq (\geq)\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i),$$

where k is a positive integer such that $1 \leq k \leq n-1$.

(iii) Let $f : C \to \mathbb{R}$ be a convex mapping, $x_i \in C$, $p_i \geq 0$, and $P_n > 0$. Then the inequality

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)$$
$$\geq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - \frac{1}{P_n^{k}}\sum_{i_1,\ldots,i_{k}=1}^{n} p_{i_1}\cdots p_{i_{k}} f\left(\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right)$$
$$\geq \left|\frac{1}{kP_n^{k}}\sum_{i_1,\ldots,i_{k}=1}^{n} p_{i_1}\cdots p_{i_{k}}\left|\sum_{j=1}^{k} f(x_{i_j})\right|\right.$$
$$\left. -\frac{1}{P_n^{k}}\sum_{i_1,\ldots,i_{k}=1}^{n} p_{i_1}\cdots p_{i_{k}}\left|f\left(\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right)\right|\right| \geq 0$$

holds for every positive integer k such that $1 \leq k \leq n$.

Let $f : C \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function (16.10) on an interval C and $a, b \in C$ with $a < b$. The double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is known in the literature as Hadamard's inequalities. The inequalities are due to C. Hermite, who discovered them in 1883, ten years before J. Hadamard [333].

The following refinement of the first inequality holds [226]. Let f be as above and $\alpha, \beta : [a, b] \to \mathbb{R}_+$ be two continuous mappings so that $\alpha(x) + \beta(y) > 0$ for all $x, y \in [a, b]$. Then one has the inequalities

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy$$
$$\le \frac{1}{(b-a)^2}\int_a^b\int_a^b \frac{1}{2}\left[f\left(\frac{\alpha(x)x+\beta(y)y}{\alpha(x)+\beta(y)}\right)\right.$$
$$\left.+f\left(\frac{\beta(y)x+\alpha(x)y}{\alpha(x)+\beta(y)}\right)\right]dxdy$$
$$\le \frac{1}{b-a}\int_a^b f(x)dx.$$

For a given convex mapping (16.10) $f : [a, b] \to \mathbb{R}$:

(I) Let $H : I \to \mathbb{R}$ be defined by

$$H(t) := \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx.$$

The following hold:

(i) H is convex on I_0.
(ii) $\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{b-a}\int_a^b f(x)dx$.
(iii) H is monotonically nondecreasing on I.

(II) Let $F : I \to \mathbb{R}$ define

$$F(t) := \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)dxdy.$$

The following hold:

(i) $F\left(s+\frac{1}{2}\right) = F\left(\frac{1}{2}-s\right)$ for all s in $\left[0, \frac{1}{2}\right]$.
(ii) F is convex on I.
(iii) F is monotonically nonincreasing on $\left[0, \frac{1}{2}\right]$ and nondecreasing on $\left[\frac{1}{2}, 1\right]$.
(iv) One has the inequality

$$H(t) \le F(t) \quad \text{for all } t \in I.$$

16.16.2 Inequalities for Integrals

Inequalities of interest may be obtained by integrating, under appropriate conditions, known inequalities for functions. We consider explicitly two such inequalities that seem to be both simple and useful and are suggestive of further generalizations.

Let $f(x)$ be a nonnegative function of a real variable x, and let S be a set of real numbers such that the integral

$$I_k \equiv \int f^k \equiv \int_a^b \{f(x)\}^k dx,$$

taken over an appropriate range (a, b), with $b > a$, exists when $k \in S$. Then

$$\int f^m + \int f^n \geq 2 \int f^{(m+2)/2}, \tag{16.80}$$

$$\int f^m \int f^n \geq \left[\int f^{(m+n)/2} \right]^2, \tag{16.81}$$

where $m,\ n$, and $\frac{1}{2}(m+n)$ belong to the set S.

Inequality (16.80) may be proved by starting with the arithmetic mean–geometric mean inequality, and (16.81) is a particular case of the Schwarz inequality. Both of these may also be independently derived by using properties of convex functions [252, 215].

Two particular cases are worthy of note.

(i) Writing $\delta = b - a =$ range of integration, and taking $m = 1,\ n = -1$, we have

$$\int f + \int \frac{1}{f} \geq 2\delta, \tag{16.81a}$$

$$\left(\int f \right) \left(\int \frac{1}{f} \right) \geq \delta^2. \tag{16.81b}$$

(ii) When $m = 2,\ n = 0$, (16.81) gives

$$\int f^2 \int 1 \geq \left(\int f \right)^2. \tag{16.81c}$$

A geometrical representation of this may be obtained as follows. Let PQ be the arc of the curve $y = f(x)$ between $x = a$ and $x = b$. Let s be the area bounded by the arc PQ, the ordinates at P and Q, and the x-axis; and let V be the volume of the solid of revolution formed by rotating the area A about the x-axis. Then inequality (16.81c) is equivalent to

$$V\delta \geq \pi s^2.$$

By taking different forms of the function f, we should be able to obtain inequalities of interest.

For example:

(i) Let $f(t) = e^t,\ a = -\infty,\ b = 0,\ m$, and n positive numbers. From (16.80),

$$\frac{1}{m} + \frac{1}{n} \geq \frac{2}{m+n}.$$

(ii) Let $f(t) = e^t,\ a = -1,\ b = 1$, and m, n any real numbers. Inequality (16.80) gives

$$\frac{\sinh m}{m} + \frac{\sinh n}{n} \geq \frac{2\sinh(m+n)/2}{m+2}.$$

(iii) Let $f(t) = te^{-t},\ a = 0,\ b = \infty$. From (16.81),

$$\frac{\Gamma(m+1)\Gamma(n+1)}{\left\{\Gamma\left(\frac{m+n}{2}+1\right)\right\}^2} \geq \frac{m^{n+1}n^{n+1}}{\left(\frac{m+n}{2}\right)^{m+n+2}}. \tag{16.82}$$

(iv) Let $f(t) = \sqrt{(1-k^2\sin^2 t)},\ a = 0,\ b = x$. Then (16.81a) and (16.81b) give

$$\ell(x,k) + F(x,k) \geq 2x,$$
$$\ell(x,k)F(x,k) \geq x^2,$$

where ℓ and F are elliptic integrals of the first and second kinds.

As a slightly more general form of (16.80) and (16.81), we have

$$\int gf^m + \int gf^n \geq 2\int gf^{(m+2)/2},$$
$$\left(\int gf^m\right)\left(\int gf^n\right) \geq \left(\int gf^{(m+2)/2}\right)^2,$$

where f is as before and $g(x)$ is a nonnegative function of x. If $f(t) = t,\ a = 0,\ b = \infty$, the latter gives

$$\Gamma(m+1)\Gamma(n+1) \geq \left\{\Gamma\left(\frac{m+n}{2}+1\right)\right\}^2.$$

This may be compared with (16.82) above.

A variety of inequalities may be obtained from further generalizations.

16.16.3 Cauchy-Schwarz-Hölder Inequalities

Given a positive integer n and real numbers $a_k,\ b_k\ (k = 1, \ldots, n)$, Cauchy's inequality (Cauchy [147, pp. 373–374, Theorem XVI]) states that

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right). \tag{16.83}$$

Milne made the interesting discovery that if no $a_k^2 + b_k^2$ is 0, the expression

$$\left(\sum_{1}^{n}(a_k^2 + b_k^2)\right)\sum_{1}^{n} a_k^2 b_k^2/(a_k^2 + b_k^2)$$

lies between the left- and right-hand sides of (16.83). Callebaut [140] has shown that the function

$$R(x) = \left(\sum_{1}^{n} a_k^{r+x} b_k^{r-x}\right)\left(\sum_{1}^{n} a_k^{r-x} b_k^{r+x}\right) \tag{16.84}$$

with all $a_k,\ b_k$ positive is increasing with $|x|$. This function with $r = 1$ yields the left-hand side of (16.83) when $x = 0$ and the right-hand side when $x = 1$. In [254] it was observed that $R(x)$ is convex (16.10) with a minimum at $x = 0$. Callebaut's result follows immediately from this observation. Similarly, constructing appropriate convex functions, [255, 215, 216, 217] gave new proofs of a number of classical inequalities and numerous generalizations of the Cauchy and Callebaut inequalities. Among these generalizations was the following [217]. Let $a_1, a_2, \ldots, a_n,\ b_1, b_2, \ldots, b_n,\ p,\ q$ be positive numbers with $(1/p) + (1/q) = 1$. Then the function

$$H(x) = \left(\sum_1^n a_k^{1+(p-1)x} b_k^{1-x}\right)^{1/p} \left(\sum_1^n a_k^{1-x} b_k^{1+(q-1)x}\right)^{1/q}$$

is convex (16.10) with a minimum at $x = 0$. Consequently, $H(0) \le H(1)$, which yields

$$\sum_1^n a_k b_k \le \left(\sum_1^n a_k^p\right)^{1/p} \left(\sum_1^n b_k^q\right)^{1/q},$$

Hölder's inequality.

Generalizations of Cauchy's Inequality

We make use of the simple observation that for $K > 0$ the function K^x (and hence also K^{-x}) is convex (16.10). We also note that the sum of convex functions is convex and the product of a convex function by a positive constant is convex.

Let $\lambda_k,\ \mu_k,\ p_k$, and q_k $(k = 1, \ldots, n)$ be positive numbers. Then the function

$$T(x) = \left(\sum_1^n p_k \lambda_k^x\right)\left(\sum_1^n q_k \mu_k^x\right) + \left(\sum_1^n p_k \lambda_k^{-x}\right)\left(\sum_1^n q_k \mu_k^{-x}\right)$$

is convex with a minimum at $x = 0$. It is a constant if, and only if, $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \mu_1^{-1} = \mu_2^{-1} = \cdots = \mu_n^{-1}$.

Result 16.33. (Eliezer and Mond [255]). *Let $a_k,\ b_k,\ c_k$, and d_k $(k = 1, \ldots, n)$ be positive numbers and let $r_i,\ s_i$ $(i = 1, 2, 3, 4)$ be real. Then the function*

$$T_1(x) = \left(\sum_1^n a_k^{r_1+s_1x} b_k^{r_2+s_2x}\right)\left(\sum_1^n c_k^{r_3+s_3x} d_k^{r_4+s_4x}\right) + \left(\sum_1^n a_k^{r_1-s_1x} b_k^{r_2-s_2x}\right)\left(\sum_1^n c_k^{r_3-s_3x} d_k^{r_4-s_4x}\right)$$

is convex with a minimum at $x = 0$. It is a constant if and only if, for $k = 1, 2, \ldots, n,\ a_k^{s_1} b_k^{s_2} = c_k^{-s_3} d_k^{-s_4} = \alpha$, where α is independent of k.

Generalization of Hölder's Inequality

Let $\lambda_k,\ \mu_k,\ p_k,\ q_k\ (k = 1, \ldots, n),\ p$, and q be positive numbers. Then the function

$$S(x) = \left(\sum_1^n q_k \lambda_k^{px}\right)^{1/p} \left(\sum_1^n p_k \mu_k^{qx}\right)^{1/q} + \left(\sum_1^n q_k \lambda_k^{-qx}\right)^{1/q} \left(\sum_1^n p_k \mu_k^{-px}\right)^{1/p}$$

is convex (16.10). If

$$\left(\sum_1^n q_k\right)^{1/p} \left(\sum_1^n p_k\right)^{1/q} = \left(\sum_1^n q_k\right)^{1/q} \left(\sum_1^n p_k\right)^{1/p}$$

$\Big($e.g., if

$$\sum_1^n q_k = \sum_1^n p_k = 1\Big),$$

then $S(x)$ has a minimum at $x = 0$.

Result 16.34. [255]. *Let $a_k,\ b_k,\ c_k,\ d_k\ (k = 1, \ldots, n),\ p$, and q be positive numbers. Then the function*

$$S_1(x) = \left\{\frac{\sum a_k^{1+(p/q)x} b_k^{1-x}}{\sum a_k b_k}\right\}^{1/p} \left\{\frac{\sum c_k^{1+(q/p)} d_k^{1-x}}{\sum c_k d_k}\right\}^{1/q} + \left\{\frac{\sum a_k^{1-x} b_k^{1+(q/p)x}}{\sum a_k b_k}\right\}^{1/q} \left\{\frac{\sum c_k^{1-x} d_k^{1+(p/q)x}}{\sum c_k d_k}\right\}^{1/p}$$

is convex with a minimum at $x = 0$.

17

Applications

Applications in binomial expansion, scalar products, economics, the Cobb-Douglas production function and quasilinearity, interest formulas, physics, Gaussian functions, Chebyshev polynomials, determinants, geometry, statistics, information theory, and sums of powers are considered in this chapter.

Functional equations is a growing and fascinating part of mathematics because of its intrinsic mathematical beauty as well as its applications. Applications of functional equations abound in mathematics and elsewhere in diverse fields. This chapter is devoted to applications of functional equations in various fields. Applications have contributed to the growth and development of functional equations. Applications can be found in many fields—physics, economics, behavioural and social sciences, cognitive science, group theory, information theory, fuzzy set theory, artificial intelligence, iteration, dynamical systems, psychometry, nondifferentiable functions, etc. The list is getting longer by the day. It is nearly impossible to provide an informative survey of all the works on the various applications of functional equations. We have seen some in Chapter 1, primarily devoted to Cauchy and Pexider equations, and more in various other chapters. In what follows, I present some results or examples chosen from many fields in which functional equations play an important role. We will illustrate this with many interesting examples arising from geometry, physics, information theory, economics, statistics, topology, and taxation.

Functional equations have contributed greatly to the development of appropriate methods to handle and solve many problems (see the general references by Aczél [59, (i),(ii)], [12], Aczél and Dhombres [44], and Castillo and Ruiz-Cobo [144]).

17.1 Binomial Expansion

In Lacroix [592], Newton's binomial theorem

$$(1+x)^u = 1 + f_1(u)x + f_2(u)x^2 + \cdots = \sum_{n=0}^{\infty} f_n(u)x^n \tag{17.1}$$

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with $f_0(u) = 1,\ x, u \in \mathbb{R},\ f_n : \mathbb{R} \to \mathbb{R}$ was treated, and it was pointed out that $f_1(u)$ determines the other functions $f_k(u),\ k \geq 2$. Cauchy [147] used functional equations to obtain Newton's binomial theorem

$$(17.2)\quad (1+x)^u = 1+ux+\frac{u(u-1)}{2!}x^2+\cdots+\frac{u(u-1)\cdots(u-n-1)}{n!}x^n+\cdots.$$

Without going into the intricacies of convergence, etc. (which are fundamental for (17.2)), we obtain (17.2) from (17.1) by applying Cauchy's additive equation (A). Replacing u by $(u+v)$ in (17.1) and $(1+x)^{u+v} = (1+x)^u \cdot (1+x)^v,\ u, v \in \mathbb{R}$, we get the following:

(i) By comparing the coefficient of x, we have

$$f_1(u+v) = f_1(u) + f_1(u), \quad \text{for } u, v \in \mathbb{R};$$

say $f_1(u) = A_1(u)$, where A_1 is a solution of (A).

(ii) By comparing the coefficients of x^2, we obtain

$$\begin{aligned} f_2(u+v) &= f_2(u) + f_2(v) + A_1(u)A_1(v) \\ &= f_2(u) + f_2(v) + \frac{1}{2}[A_1(u+v)^2 - A_1(u)^2 - A_1(v)^2]; \end{aligned}$$

that is,

$$A_2(u+v) = A_2(u) + A_2(v),$$

where

$$A_2(u) = f_2(u) - \frac{1}{2}A_1(u)^2$$

or

$$f_2(u) = A_2(u) + \frac{1}{2}A_1(u)^2, \quad \text{for } u \in \mathbb{R}.$$

(iii) One more comparison of the coefficients of x^3 results in

$$\begin{aligned} f_3(u+v) &= f_3(u) + f_3(v) + f_2(u)\,f_1(v) + f_2(v)\,f_1(u) \\ &= f_3(u) + f_3(v) + \left[A_2(u) + \frac{1}{2}A_1(u)^2\right]A_1(v) \\ &\quad + \left[A_2(v) + \frac{1}{2}A_1(v)^2\right]A_1(u) \\ &= f_3(u) + f_3(v) + \frac{1}{2}[A_1(u)^2A_1(v) + A_1(v)^2A_1(u)] \\ &\quad + A_2(u)A_1(v) + A_2(v)A_1(u) \\ &= f_3(u) + f_3(v) + \frac{1}{6}[A_1(u+v)]^3 - \frac{1}{6}A_1(u)^3 - \frac{1}{6}A_1(v)^3 \\ &\quad + A_1(u+v)A_2(u+v) - A_1(u)A_2(u) - A_1(v)A_2(0); \end{aligned}$$

that is,

$$A_3(u+v) = A_3(u) + A_3(v), \quad \text{for } u, v \in \mathbb{R},$$

where

$$A_3(u) = f_3(u) - \frac{1}{6}A_1(u)^3 - A_1(u)A_2(u)$$

or

$$f_3(u) = A_3(u) + \frac{1}{6}A_1(u)^3 + A_1(u)A_2(u).$$

Assuming the regularity conditions on f_1, f_2, f_3 and noting that $f_1(1) = 1$, $f_2(1) = 0$, $f_3(1) = 0$ and $A_1(1) = 1$, $A_2(1) = -\frac{1}{2}$, $A_3(1) = -\frac{1}{6} + \frac{1}{2} = -\frac{1}{3}$, we have

$$\begin{aligned} f_1(u) &= u, \\ f_2(u) &= -\frac{1}{2}u + \frac{1}{2}u^2 = \frac{1}{2}u(u-1), \\ f_3(u) &= \frac{1}{3}u + \frac{1}{6}u^3 - \frac{1}{2}u^2 = \frac{1}{6}u(u-1)(u-2), \end{aligned}$$

resulting in (17.2) (compare Chapter 1—Sums of Powers). Here the additive equation is used. □

17.2 Scalar or Dot Product

Characterization of the scalar product is by means of the following properties [144]:

(i) The scalar product is invariant when the two vectors undergo the same rotation,

(ii)
$$(cx) \cdot y = c(x \cdot y) = x \cdot (cy),$$

where x and y are vectors and c is a scalar,

(iii)
$$(e_1 + e_2) \cdot e_3 = e_1 \cdot e_3 + e_2 \cdot e_3,$$

where e_1, e_2, and e_3 are unit and coplanar vectors.

Because of the property (i), the scalar product can be written as

$$x \cdot y = h(\theta, |x|, |y|),$$

where h is a function to be determined; that is, the scalar product of two vectors x and y depends on the moduli $|x|$ and $|y|$ of the vectors and their angle θ.

Condition (ii) gives

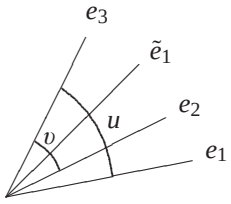
$$h(\theta, |x|, |y|) = |x||y|h(\theta, 1, 1) = |x||y| f(\theta),$$

where $f(\theta)$ is a function to be determined.

From condition (iii), taking into account that the vector sum of two unit and coplanar vectors making an angle θ is a vector with modulus $2\cos(\theta/2)$, we can write

$$2\cos\left(\frac{u-v}{2}\right) f\left(\frac{u+v}{2}\right) = f(u) + f(v),$$

where, as indicated in the figure below, u and v are angles associated with the pairs (e_1, e_3) and (e_2, e_3), respectively.



Making the change of variable

$$u = t + s, \quad v = t - s,$$

leads to

$$2\cos s f(t) = f(t+s) + f(t-s),$$

which is the same equation as (3.14). Thus its general solution is

$$f(t) = a\cos t + b\sin t, \quad t \in \mathbb{R}.$$

If we perform a rotation of an angle π around the axis e_2, we transform the pair (e_1, e_2) into $(\tilde{e}_1, e_2)$ and, because of property (i), we have

$$\begin{aligned} & e_1 \cdot e_2 = f(\theta) = \tilde{e}_1 \cdot e_2 = f(-\theta) \\ \Rightarrow\ & a\cos t + b\sin t = a\cos(-t) + b\sin(-t) \\ \Rightarrow\ & b = 0. \end{aligned}$$

Consequently, $x \cdot y = |x||y|a\cos\theta$.

Here a trigonometric equation is used. The vector or cross product can be characterized similarly (see [144]). □

17.3 Economics

17.3.1 Duopoly Model

This example deals with two firms, F_1 and F_2, that compete in the market. Naturally the sale function is a function of the prices p and advertising expenditures λ. Suppose $S(p_1, p_2, \lambda_1, \lambda_2)$, $S : \mathbb{R}_+^4 \to \mathbb{R}$, gives the sales of firm F_1 as a function of p_1 and p_2 and λ_1 and λ_2, where p_1 and p_2 are the prices and λ_1 and λ_2 are advertising expenditures of the firms F_1 and F_2. To determine S, some natural (practical) assumptions are made. We deal with two models based on different assumptions.

Assumption 1. $S(p_1, p_2, \lambda_1, \lambda_2)$ is increasing in p_2 and λ_1. It is natural to assume that an increase in the unit price p_2 of F_2 increases the sales of F_1. It is also natural to assume that an increase in the advertising expenditures λ_1 of F_1 increases its sales.

Assumption 2. $S(p_1, p_2, \lambda_1, \lambda_2)$ is decreasing in p_1. An increase in the price p_1 of F_1 causes a decrease in its sales.

Assumption 3. The $S(p_1, p_2, \lambda_1, \lambda_2)$ function is continuous, at least at a point, in all its arguments.

Remark. Note that we do not assume that $S(p_1, p_2, \lambda_1, \lambda_2)$ is decreasing with respect to λ_2 since the advertising effort of one company can also benefit its opponent.

17.3.1.1 Duopoly Model I

Assumption 4. An additive change in prices and advertising expenditures leads to an increment in sales that is the sum of the independent contributions of prices and advertising expenditures. More precisely,

$$\begin{aligned} &S(p_1 + q_1, p_2 + q_2, \lambda_1 + \mu_1, \lambda_2 + \mu_2) \\ &\qquad = S(p_1, p_2, \lambda_1, \lambda_2) + U(q_1, q_2, p_1, p_2) + V(\mu_1, \mu_2, \lambda_1, \lambda_2) \end{aligned} \tag{17.3}$$

for $p_1, q_1, p_2, q_2, \lambda_1, \mu_1, \lambda_2, \mu_2 \in \mathbb{R}_+$.

Equation (17.3) expresses that when changing prices and advertising expenditures, the resulting sales $S(p_1 + q_1, p_2 + q_2, \lambda_1 + \mu_1, \lambda_2 + \mu_2)$ are the old sales $S(p_1, p_2, \lambda_1, \lambda_2)$ plus two increments: $U(q_1, q_2, p_1, p_2)$, which depends only on prices, and $V(\mu_1, \mu_2, \lambda_1, \lambda_2)$, which depends only on advertising expenditures. This means that prices and advertising act independently (i.e., without interactions).

The general solution of functional equation (17.3) is

$$\begin{aligned} S(p_1, p_2, \lambda_1, \lambda_2) &= \alpha(p_1, p_2) + \beta(\lambda_1, \lambda_2), \\ U(q_1, q_2, p_1, p_2) &= \alpha(p_1 + q_1, p_2 + q_2) - \alpha(p_1, p_2) - a, \\ V(\mu_1, \mu_2, \lambda_1, \lambda_2) &= a + \beta(\lambda_1 + \mu_1, \lambda_2 + \mu_2) - \beta(\lambda_1, \lambda_2), \end{aligned} \tag{17.4}$$

where $\alpha(p_1, p_2)$ and $\beta(\lambda_1, \lambda_2)$ are arbitrary functions and a is an arbitrary constant.

Interchanging λ_1 and μ_1 and λ_2 and μ_2 in (17.3) and using (17.3), we obtain

$$\begin{aligned} &S(p_1, p_2, \mu_1, \mu_2) + U(q_1, q_2, p_1, p_2) + V(\lambda_1, \lambda_2, \mu_1, \mu_2) \\ &\quad = S(p_1, p_2, \lambda_1, \lambda_2) + U(q_1, q_2, p_1, p_2) + V(\mu_1, \mu_2, \lambda_1, \lambda_2). \end{aligned}$$

With $\mu_1 = 0 = \mu_2$, this equation yields S in (17.4). Substitution of this value of S in (17.3) gives

$$\begin{aligned} &\alpha(p_1 + q_1, p_2 + q_2) + \beta(\lambda_1 + \mu_1, \lambda_2 + \mu_2) \\ &\quad = \alpha(p_1, p_2) + \beta(\lambda_1, \lambda_2) + U(q_1, q_2, p_1, p_2) + V(\mu_1, \mu_2, \lambda_1, \lambda_2). \end{aligned}$$

Separating p, q's and λ, μ's, we obtain the U and V in (17.4).

Consequently, all functions $S(p_1, p_2, \lambda_1, \lambda_2)$ satisfying Assumption 4 have the structure in (17.3) (i.e., prices and advertising expenditures act independently), not only for the increments of sales but for the sales themselves. The fact that we get two arbitrary functions means that Assumption 4 is insufficient to determine the full structure of S. Thus we can make new assumptions, such as, for example, assuming that the U function is independent of p_1 and p_2 and the V function is independent of λ_1 and λ_2. With this, functional equation (17.3) transforms to the following assumption.

Assumption 4a.

$$\begin{aligned} &S(p_1 + q_1, p_2 + q_2, \lambda_1 + \mu_1, \lambda_2 + \mu_2) \\ &\quad = S(p_1, p_2, \lambda_1, \lambda_2) + U(q_1, q_2) + V(\mu_1, \mu_2). \end{aligned} \tag{17.3a}$$

Its general solution is

$$\begin{aligned} S(p_1, p_2, \lambda_1, \lambda_2) &= a + bp_1 + cp_2 + d_1\lambda_1 + d_2\lambda_2, \\ U(q_1, q_2) &= -a + bq_1 + cq_2, \\ V(\mu_1, \mu_2) &= a + d_1\mu_1 + d_2\mu_2, \end{aligned} \tag{17.4a}$$

where a, b, c, d_1, and d_2 are arbitrary constants, which, according to Assumptions 1, 2, and 3, must satisfy the inequality constraints $c, d > 0$ and $b < 0$.

Interchanging p_1 and q_1 and p_2 and q_2 in (17.3a), we get

$$\begin{aligned} &S(q_1, q_2, \lambda_1, \lambda_2) + U(p_1, p_2) + V(\mu_1, \mu_2) \\ &\quad = S(p_1, p_2, \lambda_1, \lambda_2) + U(q_1, q_2) + V(\mu_1, \mu_2); \end{aligned}$$

that is,

$$\begin{aligned} S(p_1, p_2, \lambda_1, \lambda_2) - U(p_1, p_2) &= S(q_1, q_2, \lambda_1, \lambda_2) - U(q_1, q_2) \\ &= r(\lambda_1, \lambda_2) \quad \text{(say)}. \end{aligned}$$

Substituting this value of S in (17.3a) yields

$$\begin{aligned} &U(p_1 + q_1, p_2 + q_2) + r(\lambda + \mu_1, \lambda_2 + \mu_2) \\ &\quad = U(p_1, p_2) + r(\lambda_1, \lambda_2) + U(q_1, q_2) + V(\mu_1, \mu_2); \end{aligned}$$

that is,

$$U(p_1 + q_1, p_2 + q_2) = U(p_1, p_2) + U(q_1, q_2) + a,$$
$$r(\lambda_1 + \mu_1, \lambda_2 + \mu_2) = r(\lambda_1, \lambda_2) + V(\mu_1, \mu_2) - a,$$

where a is a constant. Note that V and r differ by a constant.

Hence we get (see Chapter 1)

$$U(p_1, p_2) = -a + A_1(p_1) + A_2(p_2),$$
$$r(\lambda_1, \lambda_2) = A_3(\lambda_1) + A_4(\lambda_2),$$

and

$$V(\mu_1, \mu_2) = +a + A_3(\mu_1) + A_4(\mu_2),$$

and so

$$S(p_1, p_2, \lambda_1, \lambda_2) = a + A_1(p_1) + A_2(p_2) + A_3(\mu_1) + A_4(\mu_2),$$

where A_i $(i = 1 \text{ to } 4)$ satisfy (A).

By Assumption 1, we get $A_2(p) = c_2 p$, $A_3(\mu) = c_3\mu$; by Assumption 2, $A_1(p) = c, p$; and by Assumption 3, $A_4(\mu) = c_4\mu$. Thus we obtain (17.4a).

It is worth mentioning that each of the variables p_1, p_2, λ_1, and λ_2 act independently, which is obvious from (17.4a) (one summand for each) but was not from its equivalent condition (17.3), where they were grouped in couples.

Note that the linear structure of $S(p_1, p_2, v_1, v_2)$ is a direct consequence of (17.3a). The arbitrary constants a, b, c, d_1, and d_2 are called *parameters*. Thus we obtain a parametric family of solutions that is an extension of the model proposed in [302].

17.3.1.2 Duopoly Model II

In this model, we make a slight modification of the previous model, assuming multiplicative changes in advertising.

Assumption 5. An additive change in prices and a multiplicative change in advertising expenditures leads to an increment in sales that is the sum of the independent contributions of prices and advertising expenditures. More precisely,

$$\begin{aligned} &S(p_1 + q_1, p_2 + q_2, \lambda_1\mu_1, \lambda_2\mu_2) \\ &\quad = S(p_1, p_2, \lambda_1, \lambda_2) + U(q_1, q_2, p_1, p_2) + V(\lambda_1, \lambda_2, \mu_1, \mu_2). \end{aligned} \tag{17.5}$$

The general solution of (17.5), which can be obtained in a similar way as the solution of (17.3), is

$$
\begin{aligned}
S(p_1, p_2, \lambda_1, \lambda_2) &= \alpha(p_1, p_2) + \beta(\lambda_1, \lambda_2), \\
U(q_1, q_2, p_1, p_2) &= \alpha(p_1 + q_1, p_2 + q_2) - \alpha(p_1, p_2) - a, \\
V(\lambda_1, \lambda_2, \mu_1, \mu_2) &= a + \beta(\lambda_1\mu_1, \lambda_2\mu_2) - \beta(\mu_1, \mu_2),
\end{aligned} \tag{17.6}
$$

where $\alpha(p_1, p_2)$ and $\beta(v_1, v_2)$ are arbitrary functions and a is an arbitrary constant.

As before, interchanging p_1 and q_1 and p_2 and q_2, we have S in (17.6). Substitution of this S in (17.5) yields U and V in (17.6). It is interesting to see that the S functions in (17.3) and (17.5) coincide, showing that Assumptions 4 and 5 are equivalent.

If now we assume that the U function is independent of p_1 and p_2 and the V function is independent of λ_1 and λ_2, as before, functional equation (17.5) transforms to the following assumption.

Assumption 5a.

(17.5a)

$$S(p_1 + q_1, p_2 + q_2, \lambda_1\mu_1, \lambda_2\mu_2) = S(p_1, p_2, \mu_1, \mu_2) + U(q_1, q_2) + V(\lambda_1, \lambda_2),$$

whose general solution is

$$
\begin{aligned}
S(p_1, p_2, \lambda_1, \lambda_2) &= a + bp_1 + cp_2 + d\log\lambda_1 + d_2\log\lambda_2, \\
U(p_1, p_2) &= -a + bp_1 + cp_2, \\
V(\lambda_1, \lambda_2) &= a + d_1\log\lambda_1 + d_2\log\lambda_2,
\end{aligned} \tag{17.6a}
$$

where a, b, c, d_1, and d_2 are arbitrary constants, which, according to Assumptions 1, 2, and 3, must satisfy the inequality constraints $c, d > 0$ and $b < 0$. Once again, p_1, p_2, v_1, and v_2 act independently.

Interchanging p_i and q_i and λ_i and μ_i $(i = 1, 2)$ separately in (17.5a) results in

$$S(p_1, p_2, \mu_1, \mu_2) = U(p_1, p_2) + r(\mu_1, \mu_2) \quad \text{(say)}$$

and

$$r(\mu_1, \mu_2) - V(\mu_1, \mu_2) = \text{constant} = c_1.$$

This value of S in (17.5a) gives

$$U(p_1 + p_2, q_1 + q_2) = U(p_1, p_2) + U(q_1, q_2) + c_2;$$

that is,

$$U(p_1, p_2) = c_3 + A_1(p_1) + A_2(p_2)$$

and

$$r(\lambda_1\mu_1, \lambda_2\mu_2) = r(\mu_1, \mu_2) + V(\lambda_1, \lambda_2) + \text{constant},$$

meaning

$$V(\lambda_1, \lambda_2) - r(\lambda_1, \lambda_2) = \text{constant},$$
$$V(\lambda_1\mu_1, \lambda_2\mu_2) = V(\lambda_1, \lambda_2) + V(\mu_1, \mu_2) + c_4,$$
$$V(\lambda_1, \lambda_2) = c_5 + L_1(\lambda_1) + L_2(\lambda_2),$$

where A_i $(i = 1, 2)$ are additive and L_i $(i = 1, 2)$ are logarithmic. Assumptions 1 to 3 give

$$A_i(p) = b_i p, \quad L_i(\lambda) = d_i \log \lambda, \quad i = 1, 2,$$

yielding (17.6a).

Equation (17.6a) shows that S is more sensitive to prices than to advertising expenditures. However, this is not a surprise since in (17.5a) we required a larger effort in advertising (multiplicative) than in price (additive) for the same increment (additive) in sales. This assumption led to (17.6a) as the only solution and to the appearance of the logarithm function, which has not been arbitrarly selected but arises as a consequence of the assumptions.

Now we have to explain why (17.3a) and (17.5a) have the same solution for S. The reason is what makes the difference in the independence on p_1 and p_2 of the U function and the independence of λ_1 and λ_2 of the V function, which was assumed only for equations (17.3a) and (17.5a). In other words, the different forms of the U and V functions in solutions (17.4) and (17.6) cover for the differences in functional equations (17.4) and (17.5) without these differences being apparent through the $S(p_1, p_2, v_1, v_2)$ function.

17.3.2 Cobb-Douglas (CD) Production Function and Quasilinearity

17.3.2.1 Quasilinearity

Let $D_1 \subset \mathbb{R}^n$ be a domain. A function $\Phi : D_1 \to \mathbb{R}$ is called *quasilinear* (Aczél [12, p. 154]) (see also Chapter 15) if there exist real constants $a_1, a_2, \ldots, a_n,\ b$ with $a_1 a_2 \cdots a_n \neq 0$ and a continuous and strictly monotonic function f with inverse f^{-1}, both with suitable domains, such that

$$\text{(QL)} \qquad \Phi(x_1, x_2, \ldots, x_n) = f^{-1}[a_1 f(x_1) + a_2 f(x_2) + \cdots + a_n f(x_n) + b].$$

If $b = 0,\ a_1 = a_2 = \cdots = a_n = 1$, (QL) becomes the so-called *quasiaddition*

$$\Phi(x_1, x_2, \ldots, x_n) = f^{-1}[f(x_1) + f(x_2) + \cdots + f(x_n)].$$

If $b = 0,\ a_1 > 0, a_2 > 0, \ldots, a_n > 0,\ \sum a_\nu = 1$, (QL) becomes the so-called *quasilinear mean*. If $f(x) = x$, the word quasi can be deleted in the preceding definition.

It is well known that a linearly homogeneous production function $\Phi : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+$ with CES (constant elasticity of substitution), σ, is a CD (Cobb and Douglas [167]) production function

(CD) $$\Phi(x_1, x_2) = cx_1^{\alpha_1} x_2^{\alpha_2},$$

$c > 0,\ a_1, a_2 \in \mathbb{R}_+^*$, with $a_1 + a_2 = 1$, for $\sigma = 1$, or an ACMS (Arrow, Chenery, Minhas, and Solow [79]) production function

(ACMS) $$\Phi(x_1, x_2) = (\beta_1 x_1^{-\rho} + \beta_2 x_2^{-\rho})^{-\frac{1}{\rho}},$$

$\rho > -1,\ \rho \neq 0,\ \beta_1, \beta_2 > 0$, for $\sigma = 1/(1+\rho)$.

Note that, if $c = 1$, the function value (CD) is a mean value of order 1, and if $\beta_1 + \beta_2 = 1$, the function value (ACMS) is a mean value of order $-\rho = 1 - \frac{1}{\sigma}$.

In what follows, let us call a function $\Phi : \mathbb{R}_+^* \times \cdots \times \mathbb{R}_+^*$ (n times ($n \geq 2$)) $\to \mathbb{R}_+$ a *CD-type function* if

(17.7) $$\Phi(x_1, x_2, \ldots, x_n) = cx_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

$c > 0$, and $\alpha_1 + \cdots + \alpha_n = 1$, and an *ACMS-type function* if

(17.7a) $$\Phi(x_1, x_2, \ldots, x_n) = (\beta_1 x_1^{-\rho} + \beta_2 x_2^{-\rho} + \cdots + \beta_n x_n^{-\rho})^{-\frac{1}{\rho}}$$

$\beta_i > 0,\ c \neq 0$.

Several characteristics of the CD-type functions, by means of functional equations, can be found in Eichhorn [250].

Production functions of type (17.7a) with $\rho > -1,\ \beta_1 > 0,\ \beta_2 > 0, \ldots, \beta_n > 0$ have been characterized via partial elasticities of substitution.

Now a *joint* characterization of the CD-type and the ACMS-type functions by means of a property that differs completely from CES is given. This property is the so-called quasilinearity of (17.7) and (17.7a). It will be shown that these are the only systems of functions that are at the same time linearly homogeneous and quasilinear. In passing, it may be noted that differentiability assumptions are not needed.

Note that both the CD-type functions and the ACMS-type functions are quasilinear:

$$f(x) = \log x, \quad f^{-1}(x) = e^x,$$

is CD (17.7); that is,

$$\Phi(x_1, x_2, \ldots, x_n) = \exp(\alpha_1 \log x_1 + \alpha_2 \log x_1 + \cdots + \alpha_n \log x_n + \log C),$$

$$f(x) = x^{-\rho}, \quad f^{-1}(x) = x^{-\frac{1}{\rho}},$$

$b = 0$ and gives ACMS (17.7a).

Moreover, both the CD-type functions and the ACMS-type functions are linearly homogeneous.

17.3.2.2 Determination of all Quasilinear, Linearly Homogeneous Functions

Theorem 17.1. (Eichhorn [249], Eichhorn and Kolm [251]). *Let the function* $\Phi : \mathbb{R}_+^n \to \mathbb{R}$ *be linearly homogeneous. It is a CD-type function* (*see* (17.7)) *or an ACMS-type function* (*see* (17.7a)) *if and only if it is quasilinear* (QL) (*see also* Chapter 15).

Proof. The proof involves Cauchy equations, extensions, and the solution of (1.94b$_1$). Let Φ be a solution of (QL). Then, by linear homogeneity, we get

$$\lambda f^{-1}[a_1 f(x_1) + \cdots + a_n f(x_n) + b] = f^{-1}[a_1 f(\lambda x_1) + \cdots + a_n f(\lambda x_n) + b],$$

where $\lambda > 0$ and $a_1, a_2, \ldots, a_n,\ b$ real constants with $a_1 a_2 \cdots a_n \neq 0$.

Set

$$\begin{aligned} g(x) &:= f^{-1}(x), & f_\lambda(x) &:= f(\lambda x), \\ f(x_\nu) &=: u_\nu \quad \nu = 1, 2, \ldots, n, & h_\lambda(x) &= f_\lambda g(x), \end{aligned}$$

in the equation above to have

$$h_\lambda(a_1 u_1 + \cdots + a_n u_n + b) = a_1 h_\lambda(u_1) + \cdots + a_n h_\lambda(u_n) + b, \tag{17.8}$$

$\lambda > 0,\ a_1, a_2, \ldots, a_n,\ b$ real constants with $a_1 a_2 \cdots a_n \neq 0$, the solution of which is given in Lemma 17.2. h_λ is not necessarily defined on all of $\mathbb{R}$. It is defined on the open set $S = I_1 \cup I_1$, where $I_1 = \{u : u = f(x),\ x \in \mathbb{R}_+^*\}$ is open in $\mathbb{R}$ (f is a continuous, strictly monotonic function, and $I_1' = \{a_1 I_1 + \cdots + a_n I_1 + b\} = \{v : v = a_1 u_1 + \cdots + a_n u_n + b,\ u_i \in I_1\}$).

Lemma 17.2. [251]. *For every fixed $\lambda > 0$, each nonconstant continuous solution $h_\lambda : S \subseteq \mathbb{R} \to \mathbb{R}$ of equation* (17.8) *can be written as*

$$h_\lambda(u) = \alpha_\lambda u + \begin{cases} \beta_\lambda & \text{for } u \in I_1, \\ \beta_\lambda' & \text{for } u \in I_1', \end{cases} \tag{17.8a}$$

where $\alpha_\lambda \neq 0,\ \beta_\lambda$, and β_λ' are suitably chosen real constants.

From (17.8) with $u = f(x),\ a_\lambda = a(x),\ \beta_x = \beta(\lambda),\ \beta_\lambda' = \beta'(\lambda)$, and the definitions of $g,\ f_\lambda,\ h_\lambda$, we see an equation like (1.94b$_1$),

$$f(\lambda x) = \alpha(\lambda) f(x) + \beta(\lambda), \quad \alpha(\lambda) \neq 0 \text{ for all } \lambda.$$

From Result 1.61, we have from (1.94b$_1$) either

$$\begin{aligned} f(x) &= c_3 L(x) + c_1 = g(x) = L(x) + c_2, \\ h(x) &= \alpha(x) = c_3, \end{aligned}$$

or

$$k(x) = \beta(x) = c_3 L(x) + c_1 - c_2 c_3;$$

that is, $f(x) = L(x) + b,\ \alpha(x) = 1,\ \beta(x) = L(x),\ L$ logarithmic (from continuity and strict monotonicity), and $f(x) = a \log x + b$. This results in CD-type function (17.7) or

$$f(x) = cc_3(M(x) - 1) + c_1 = g(x) = c(M(x) - 1) + c_2,$$
$$h(x) = c_3 M(x) = \alpha(x),$$
$$k(x) = (cc_3 - c_2c_3)(M(x) - 1) + c_1 - c_2c_3 = \beta(x);$$

that is,

$$f(x) = c(M(x) - 1) + c_1,$$
$$\alpha(x) = M(x),$$
$$\beta(x) = (c - c_1)(M(x) - 1),$$

where M is multiplicative (M) (and from the continuity and monotonicity), and we have that $f(x) = rx^{-\rho} + \delta,\ r, \rho \neq 0$ results in

$$\Phi(x_1, \ldots, x_n) = \left[\sum_{i=1}^{n} a_i x_i^{-\rho} + \frac{1}{r}(b + a_1\delta + \cdots + a_n\delta - \delta)\right]^{-\frac{1}{r}},$$

which is ACMS-type function (17.7a). The converse is easy to check. This proves the theorem. □

17.4 Business Mathematics—Interest

17.4.1 Interest Formula

See Castillo and Ruiz-Cobo [144] and Eichhorn [250], and let an amount K be deposited for a period of length t and $F(K, t)$ be the amount accrued by interest compounding. $F(K, t)$ is subject to two conditions:

(i) F does not change by splitting K into two parts, $x + y$, and, left for the same period, this yields

$$F(x + y, t) = F(x, t) + F(y, t), \quad x, y \in \mathbb{R}_+,\ t \in \mathbb{R}_+^*.$$

That is, the final amount depends only on the total initial investment but not on the number of investments it can be divided into.

(ii) The amount to which $F(K, t_1)$ during a time interval of length t_2 is equal to the amount to which K increases during the period $t_1 + t_2$ gives

$$F(K, t_1 + t_2) = F(F(K, t_1), t_2), \quad K, t_1, t_2 \in \mathbb{R}_+^*;$$

that is, the final amount only depends on the total time, $t_1 + t_2$, the capital is deposited, but not on the tentative period of deposit.

For fixed t, the first equation is an additive Cauchy equation giving (by Theorem 1.2)

$$F(x, t) = c(t)x \quad \text{for } x \in \mathbb{R}_+,$$

where $c(t) > 0$. This F in the second equation gives

$$c(t_1 + t_2) = c(t_1)c(t_2) \quad \text{for } t_1, t_2 > 0.$$

Since $c(t)$ is a monotonically increasing function satisfying (E) by Corollary 1.36a, $c(t) = e^{dt}$, where $d > 0$. Thus $F(x, t) = xq^t$, where $q = e^d > 1$.

It is interesting to point out the following three important facts:

(i) If $f(x, t_1 + t_2) < f[f(x, t_1), t_2]$, we get more if we cancel the actual deposit and initiate a new one by reinvestment.
(ii) If $f(x + y, t) < f(x, t) + f(y, t)$, we are invited to make many small investments instead of a single one for the whole amount.
(iii) If the inequalities in (i) and (ii) are reversed, the bank is offering more than necessary.

Thus, under this point of view, the optimal bank offer corresponds to the equalities of the expressions above. □

17.4.2 Simple Interest

See [144]. Let $F(k, t)$ be the accrued amount of capital k invested for a period of length t. Then the simple interest $F(k, t)$ is subject to the following two conditions:

(i) As in (i), above

$$F(x + y, t) = F(x, t) + F(y, t), \quad x, y \in \mathbb{R}_+, \ t \in \mathbb{R}_+^*.$$

(ii) The final amount depends only on the investment and the total time and not on the periods of time it can be divided into; that is,

$$F(k, t_1 + t_2) = F(k, t_1) + F(k, t_2), \quad x \in \mathbb{R}_+, \ t_1, t_2 \in \mathbb{R}_+^*.$$

As in the example above, (i) gives $F(x, t) = c(t)x$ and this F in (ii) yields $c(t_1 + t_2) = c(t_1) + c(t_2)$, which, again by Theorem 1.2, yields $c(t) = bt$ and then $F(x, t) = bxt$ (cf. the application in Chapter 1).

17.4.3 Interest Rates

See [144]. In the examples above, we characterized the compound and simple interest formulas from some properties that were written as functional equations. However, some actual bank policies differ from those previously stated. As a matter of fact, the interest rate is usually dependent on the period and/or the amount of the deposit. This motivates the search for new interest formulas to be applied in the new situation of the actual economy.

As in the second example, let $F(x, t)$ be the future value of the capital x invested for a period of time of duration t, and let $h(x, t) = F(x, t)/x$ be the accumulated investment rate. These two properties suggest the functional equation

$$h(x, t_1 + t_2) = h(x, t_1)h(x, t_2), \quad x, t \in \mathbb{R}_+,$$

where we assume $h(x, 0) = 1$ and h is continuous and increasing in both arguments. This equation for fixed x is a Cauchy equation (E), so its general continuous solution is given by

$$h(x, t) = \exp[C(x)t] = p(x)^t \Rightarrow F(x, t) = xp(x)^t, \tag{17.9}$$

where $p(x)$ is a nonnegative and increasing arbitrary function. Note that the cases of compound and continuous interest are particular cases with $p(x) = 1 + x$ and $p(x) = \exp(x)$, respectively. Note also that the case $F(x, t) = x[1 + r(x)]^t$ (that is, with the interest rate being dependent on the initial investment) is included.

Another possibility is to assume that the accumulated investment rate for an investment x is the accumulated investment rate for a period of duration y raised to a power that is a function of y and x; that is,

$$h(x, t) = h(y, t)^{k(y,x)}, \quad t, x, y \in \mathbb{R}_+^*,$$

with general (with the indicated conditions) solution

$$h(x, t) = p(t)^{q(x)} = m(x)^{n(t)}; \quad k(y, x) = \frac{q(x)}{q(y)} = \frac{\log m(x)}{\log m(y)}.$$

Hence, we get

$$f(x, t) = xm(x)^{n(t)},$$

which generalizes (17.9).

Finally, we can assume that the accumulated investment rate for a period $t_1 + t_2$ can be obtained from the accumulated investment rates for periods t_1 and t_2; that is,

$$h(x, t_1 + t_2) = U[h(x, t_1), h(x, t_2)].$$

Thus, its general solution is

$$h(x, t) = w[tp(x)], \quad U(x, y) = w[w^{-1}(x) + w^{-1}(y)],$$

where $w(x)$ and $p(x)$ are arbitrary functions and $w(x)$ is invertible. This leads to

$$F(x, t) = xw[tp(x)],$$

which also generalizes the interest formula (17.9). One important particular case is given by $p(x) = x$; that is, $F(x, t) = xw(tx)$. The product measures the power of investment. □

17.5 Physics

17.5.1 Quantum Physics

D'Alembert's equation is one of the most important functional equations. This equation was originally introduced as a tool for studying the vibrating string problem

(see Chapter 3 (VS)) and for axiomatic investigations of the parallelogram law for addition of vectors (see Chapter 1). It has been applied in non-Euclidean mechanics [44] and harmonic analysis (Hewitt and Zuckerman [370]).

Investigations on the mathematical foundation of quantum mechanics have shown that a quantum mechanical influence function satisfies a generalized d'Alembert functional equation. In his studies of discrete models for classical and quantum physics, Hemion [365, 366] introduced the concept of an influence $f : \mathbb{R} \to \mathbb{R}$. The function f provides a measure of the influence between physical states (or configurations). By applying the principle of strong causality (the future cannot influence the past), Hemion has argued that f must satisfy the condition

$$\sum_{i=1}^{n} f(y_i) = 0 \Rightarrow \sum_{i=1}^{n} [f(x + y_i) + f(x - y_i)] = 0 \tag{17.10}$$

for all $x, y_i \in \mathbb{R}$. It is clear that (C) implies (17.10), so (17.10) provides an interesting generalization of (C). A physical interpretation states that if a present total influence vanishes, then it still vanishes when future effects are included.

Closely related to (17.10) is the condition

$$\sum_{i=1}^{n} f(x_i) = \sum_{j=1}^{m} f(y_j) \Rightarrow \sum_{i=1}^{n} [f(x + x_i) + f(x - x_i)]$$

$$= \sum_{j=1}^{n} [f(x + y_j) + f(x - y_j)] \tag{17.10a}$$

for all $x_i, y_i \in \mathbb{R}$.

Conditions (17.10) and (17.10a) are equivalent if f is continuous and has a zero. If in addition $f(0) = 1$, then (C), (17.10), and (17.10a) are all equivalent (Gudder [330]).

Definition. Let V be a real topological linear space. A subset C is balanced if $tC \subseteq C$ for all $|t| \leq 1$. A subset C is *absorbing* if for every $x \in V$ there exists an $\varepsilon > 0$ such that $tx \in C$ for $|t| < \varepsilon$.

Result 17.3. [330]. *If $f : V \to \mathbb{R}$ is continuous and satisfies* (17.10a)*, then there exists a continuous linear functional $g : V \to \mathbb{R}$ such that either $f(x) = f(0) \cos g(x)$ or $f(x) = f(0) \cosh g(x)$ for all $x \in V$. Further, g is unique in the sense that if $f \neq 0$ and $f(x) = f(0) \cos k(x)$ or $f(x) = f(0) \cosh k(x)$, for a continuous linear functional k, then $k = \pm g$.*

Corollary 17.4. [330]. *If $f : V \to \mathbb{R}$ is continuous, satisfies* (17.10)*, and has a zero, then there is a continuous linear functional $g : V \to \mathbb{R}$ such that $f(x) = f(0) \cos g(x)$.*

Corollary 17.5. [330]. *Let V be a real Hilbert space. If $f : V \to \mathbb{R}$ is a continuous solution of* (17.10a)*, then there is a unique $y \in V$ such that $f(x) = f(0) \cos\langle x, y\rangle$ or $f(x) = f(0) \cosh\langle x, y\rangle$ for all $x \in V$ (uniqueness means that if there is a z satisfying the representation above, then $z = \pm y$).*

Corollary 17.6. [330]. *Let V be a real Hilbert space. Suppose $f : V \to \mathbb{R}$ is a continuous solution of* (17.10) *and has a zero. Then there exists a $y \in V$ such that $f(x) = f(0)\cos\langle x, y\rangle$.*

17.5.2 Gaussian Function

In digital image analysis, edge detection, line detection, and texture classification are basic to image understanding and interpretation. Since a typical image is devoid of predetermined directions of edge, line, or texture, rotation-invariant filters are important for their detection. In designing such filters [342], the concept of a rotation-invariant separable function is often used. Also, in many problems in optics and quantum mechanics, we encounter such functions. For instance, two independent particles in quantum mechanics whose wave function is a product of the functions of individual coordinates are also a product when expressed in the centre of mass and relative coordinates. We show that every rotation-invariant, separable, real-valued function of two variables that satisfies the functional equation (17.18) is either Gaussian or identically zero. In Mann, Revzen, Khanna, and Takahashi [622], the authors study bifactorizable quantum wave functions. While proving that if two quantum systems are prepared independently and if their centre of mass is found to be in a pure state, then each of the component systems is also in a pure state, which in the coordinate representation is a Gaussian wave function, the authors [622] came across the functional equation (17.18). This result is established without assuming any regularity conditions on the unknown functions. Our approach is from the functional equation point of view and is simple, direct, and interesting.

Let $\Omega = \left\{\theta \in \mathbb{R} \mid \theta \neq \frac{n\pi}{2},\ n = 0, 1, 2, 3, \ldots\right\}$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *Gaussian* if and only if

$$f(x_1, x_2, \ldots, x_n) = ke^{A(x_1^2 + x_2^2 + \cdots + x_n^2)}, \quad (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \tag{17.11}$$

where k is a real constant and $A : \mathbb{R} \to \mathbb{R}$ satisfies (A). A function $G : \mathbb{R}^2 \to \mathbb{R}$ is said to be *separable* if there exist functions $g, h : \mathbb{R} \to \mathbb{R}$ such that

$$G(x, y) = g(x)h(y) \quad \text{for } x, y \in \mathbb{R}. \tag{17.12}$$

A function $G : \mathbb{R}^2 \to \mathbb{R}$ is said to be *rotation invariant* provided

$$G(x, y) = G(x_\theta, y_\theta) \quad \text{for all } \theta \in \Omega, \tag{17.12a}$$

where

$$\begin{pmatrix} x_\theta \\ y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

If a function $G : \mathbb{R}^2 \to \mathbb{R}$ is rotation invariant and separable, then

$$g(x)h(y) = k(x_\theta)\ell(y_\theta), \tag{17.13}$$

for $x, y \in \mathbb{R},\ \theta \in \Omega$, where $g, h, k, \ell : \mathbb{R} \to \mathbb{R}$.

Main Theorem 17.7. (Kannappan and Sahoo [508]). *If $G : \mathbb{R}^2 \to \mathbb{R}$ is a rotation-invariant separable function, then G is either identically zero or Gaussian.*

To establish this result, we need the following results.

Lemma 17.8. [508]. *Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation*

$$f(x+y) = f(x)\,f(y)g(xy) \quad \text{for } x, y \in \mathbb{R}, \tag{17.14}$$

where f and g are not identically zero functions. Then the general solution of (17.14) *is given by*

$$f(x) = ce^{\frac{1}{2}A_1(x^2)+A_2(x)}, \quad g(x) = \frac{1}{c}e^{A_2(x)}, \quad \text{for } x \in \mathbb{R}, \tag{17.15}$$

where $A_1, A_2 : \mathbb{R} \to \mathbb{R}$ satisfy (A) *and c is an arbitrary nonzero real constant.*

Remark. First, the solution of (17.14) is obtained without assuming any regularity conditions whatsoever on f or g. Second, if we assume some regularity condition on f, say f is measurable or continuous or continuous at a point, then we obtain, as in Sahoo [721], $f(x) = ce^{\frac{1}{2}ax^2+bx}$ and $g(x) = \frac{1}{c}e^{ax}$. The measurability or continuity on f implies the same on g.

Corollary 17.8a. *The general nontrivial solution of*

$$f_1(x+y) = f_2(x)\,f_3(y)g(xy) \quad \text{for } x, y \in \mathbb{R}, \tag{17.16}$$

where $f_1,\ f_2,\ f_3, g : \mathbb{R} \to \mathbb{R}$, is given by

$$f_1(x) = abce^{\frac{1}{2}A_1(x^2)+A_2(x)},$$
$$f_2(x) = \frac{1}{a}f_1(x), \quad f_3(x) = \frac{1}{b}f_1(x), \quad g(x) = \frac{1}{c}e^{A_1(x)}, \quad \text{for } x \in \mathbb{R},$$

where $A_1, A_2 : \mathbb{R} \to \mathbb{R}$ satisfy (A) *and $a,\ b$, and c are arbitrary nonzero real constants.*

Corollary 17.8b. *The general nontrivial solution of*

$$f_1(x-y) = f_2(x)\,f_3(y)g(xy) \quad \text{for } x, y \in \mathbb{R}, \tag{17.17}$$

where $f_1,\ f_2,\ f_3, g : \mathbb{R} \to \mathbb{R}$, is given by

$$f_1(x) = abce^{\frac{1}{2}A_1(x^2)+A_2(x)},$$
$$f_2(x) = \frac{1}{a}f_1(x), \quad f_3(x) = \frac{1}{b}f_1(-x), \quad g(x) = \frac{1}{c}e^{A_1(-x)}, \quad \text{for } x \in \mathbb{R},$$

where $A_1, A_2 : \mathbb{R} \to \mathbb{R}$ are additive and $a,\ b,\ c$ are nonzero reals.

Proof of the Main Theorem. (Kannappan and Sahoo [508]). If G is the zero function, it is obviously rotation invariant and separable. So, we assume G is not

identically zero. Suppose $G(x_0, y_0) = 0$. Then, by using separability and rotation invariance, we can show that $G(x, y) = 0$ for all $x, y \in \mathbb{R}$. The separability condition (17.12) gives $g(x_0)k(y_0) = 0$. Assume $g(x_0) = 0$. Then again (17.12) gives $G(x_0, y) = 0$ for all y. Given any (x_1, y_1), it is possible to find a suitable (x_0, y) or a rotation of it, so that (x_1, y_1) is obtained as the image of (x_0, y) by a suitable rotation. Hence, $G(x_1, y_1) = 0$, so G is nowhere zero. Then, from (17.13), we obtain

$$k(\mu x + \nu y)\ell(-\nu x + \mu y) = g(x)h(y), \quad \text{for all } x, y \in \mathbb{R}, \tag{17.18}$$

for all (μ, ν) not equal to $(0, 1)$, $(1, 0)$, $(0, -1)$, $(-1, 0)$, where $\mu = \cos\theta$ and $\nu = \sin\theta$. Since G is nowhere zero, we conclude that k, ℓ, g, and h are also nowhere zero. Letting $x = 0$ in (17.18), we get

$$h(y) = ak(\nu y)\ell(\mu y), \tag{17.19}$$

where a is a nonzero constant. Similarly, letting $y = 0$ in (17.18), we get

$$g(x) = bk(\mu x)\ell(-\nu x), \tag{17.19a}$$

where b is a nonzero constant. Using (17.19) and (17.19a) in (17.18), we obtain

$$k(\mu x + \nu y)\ell(-\nu x + \mu y) = abk(\mu x)\ell(-\nu x)k(\nu y)\ell(\mu y),$$

which is

$$\frac{k(\mu x + \nu y)}{k(\mu x)k(\nu y)} = ab\frac{\ell(-\nu x)\ell(\mu y)}{\ell(-\nu x + \mu y)}.$$

Replacing x by $\frac{1}{\mu}x$ and y by $\frac{1}{\nu}y$ in the equation above, we get

$$\frac{k(x + y)}{k(x)k(y)} = ab\frac{\ell\left(-\frac{\nu}{\mu}x\right)\ell\left(\frac{\mu}{\nu}y\right)}{\ell\left(-\frac{\nu}{\mu}x + \frac{\mu}{\nu}y\right)}. \tag{17.20}$$

Letting $\frac{\mu}{\nu} = x$ any nonzero real (recall $\frac{\mu}{\nu} = \cot\theta$, and it is possible to choose $\frac{\mu}{\nu} = x$ any nonzero real) in (17.20), we see that the right side of (17.20) is a function of xy. Hence we have

$$k(x + y) = k(x)k(y)\phi(xy) \tag{17.14}$$

for all $x, y \in \mathbb{R}$ with $x \neq 0$. By Lemma 17.8 and the remark following it, we obtain

$$k(x) = ce^{\frac{1}{2}A_1(x^2)+A_2(x)}, \quad \phi(x) = \frac{1}{c}e^{A_1(x)}. \tag{17.15}$$

Letting $(\mu, \nu) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in (17.20), we get

$$\frac{k(x + y)}{k(x)k(y)} = ab\frac{\ell(-x)\ell(y)}{\ell(-x + y)}. \tag{17.16}$$

Using (17.15) in (17.16), we see that

$$\ell(-x+y) = abc\ell(-x)\ell(y)e^{-A_1(xy)}.$$

Thus, by Corollary 17.8b (which could use Corollary 17.8a), we have

$$\ell(x) = \frac{1}{abc}e^{\frac{1}{2}A_1(x^2)+A_3(x)}. \tag{17.21}$$

Using (17.15), (17.21), (17.12), (17.12a), and (17.18), we obtain

$$\begin{aligned} G(x,y) &= k(\mu x+\nu y)\ell(-\nu x+\mu y) \\ &= \frac{1}{ab}e^{\frac{1}{2}A_1(x^2+y^2)+A_2(\mu x+\nu y)+A_3(-\nu x+\mu y)}. \end{aligned} \tag{17.22}$$

Since G is rotation invariant for all $\theta \in \Omega$, G in (17.22) remains the same for different choices of (μ,ν). By choosing $(\mu,\nu) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, we get

$$G(x,y) = \frac{1}{ab}\exp\left(\frac{1}{2}A_1(x^2+y^2) + A_2\left(\frac{x+y}{\sqrt{2}}\right) + A_3\left(\frac{-x+y}{\sqrt{2}}\right)\right). \tag{17.23}$$

Similarly, choosing $(\mu,\nu) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, we see that (17.22) becomes

$$G(x,y) = \frac{1}{ab}\exp\left(\frac{1}{2}A_1(x^2+y^2) + A_2\left(\frac{-x-y}{\sqrt{2}}\right) + A_3\left(\frac{x-y}{\sqrt{2}}\right)\right). \tag{17.23a}$$

Hence, from (17.23) and (17.23a), we obtain

$$A_2\left(\frac{x+y}{\sqrt{2}}\right) = A_3\left(\frac{x-y}{\sqrt{2}}\right)$$

for all $x, y \in \mathbb{R}$. Now, from (17.23), we have

$$G(x,y) = \frac{1}{ab}e^{\frac{1}{2}A_1(x^2+y^2)}.$$

This completes the proof of the theorem. □

We conclude this result with the following remark. A function $G : \mathbb{R}^2 \to \mathbb{R}$ is circularly symmetric if $G(x,y) = f(\sqrt{x^2+y^2})$ for some real-valued function f. It was shown in [721] that circularly symmetric separable functions are Gaussian. Circularly symmetric functions are precisely rotation invariant, and not the converse. In harmonic analysis, such functions are generally called radial functions [369]. Thus this result can also be deduced from [721]. However, here we accomplished our task by solving a quite different and interesting functional equation (17.14) and the related ones considered by others in different contexts [622]. Furthermore, this approach allows us to study linear transformation-invariant separable functions.

17.5.3 Chebyshev Polynomials

17.5.3.1 Introduction

We now discuss yet another example of the application of the cosine equation. In 1750, d'Alembert [189] introduced the cosine functional equation (C), which arose in modelling the motion of a stretched string [808] and in the foundations of mechanics.

To see how one can "find" the cosine in (C), assume that the not identically zero solution f is twice continuously differentiable. Then, from

$$\frac{f(x+y)+f(x-y)-2f(x)}{y^2} = f(x)\frac{f(y)+f(-y)-2f(0)}{y^2},$$

using $f(-y) = f(y)$ and $f(0) = 1$, which follow from (C), and taking the limit as y tends to 0, one obtains

$$f''(x) = f(x)f''(0).$$

Hence

$$f(x) = \begin{cases} \cos(cx) & \text{if } f''(0) \le 0, \\ \cosh(cx) & \text{if } f''(0) > 0, \end{cases}$$

where $c := \sqrt{|f''(0)|}$ (see Theorem 3.9, Chapter 3).

It is worth remarking that d'Alembert was among those calling for a theory of limits that would justify the argument just given. It is also worth remarking that the technique of reducing a functional equation (such as (C)) to a differential equation has been a mainstay for the past 250 years (see [44]).

Indeed, in Kannappan [424, Theorem 3.21], (C) was solved in great generality, in particular where x, y are elements of an additive Abelian group and $f(x)$ is a complex number without assuming any regularity (e.g., continuity) in the function. In the theorem it is proved that, given a solution of d'Alembert's functional equation with $f(0) = 1$, there is a function $e : \mathrm{dom}(f) \to \mathbb{C}$ such that $e(0) = 1$ and $e(x+y) = e(x)e(y)$ and $2f(x) = e(x)+e(-x)$ for all $x \in \mathrm{dom}(f)$. In the classical cases, $e(x) = e^{icx}$ for $\cos(cx)$ and $e(x) = e^{cx}$ for $\cosh(cx)$.

Now d'Alembert's equation is solved when the domain of f is the additive group of the integers and the codomain of f is a commutative ring F. It is here that, perhaps surprisingly, the Chebyshev polynomials show up.

Definition. Chebyshev polynomial $T_m \in \mathbb{Z}[X]$ is given by, for $m \in \mathbb{Z}_+^*$,

$$T_m(X) = \sum_{k=0}^{q} \binom{m}{2k} X^{m-2k}(X^2-1)^k, \tag{17.24}$$

where q is the largest integer with $2q \le m$ (see [197]).

If $p \in \mathbb{Z}[X]$, say

$$p(X) = p_0 + p_1X + \cdots + p_dX^d,$$

and if $r \in F$, then, as usual,

$$p(r) := p_0 + p_1 r + \cdots + p_d r^d.$$

The general reference for Chebyshev polynomials (Tchebycheff—hence T) is Rivlin [704]. The occurrence of T_n here is really as a polynomial, not a polynomial function as in Rivlin generally.

Result 17.9. [215]. *Let* $f : \mathbb{Z} \to F$ *with* $f(0) = 1$. *Then*

$$f(m+n) + f(m-n) = 2f(m)f(n), \quad \text{for } m, n \in \mathbb{Z}, \tag{17.25}$$

if, and only if,

$$f(n) = T_{|n|}(f(1)), \quad n \in \mathbb{Z}.$$

The identically zero function satisfies (17.25) but is not of the form $T_{(n)}(f(1))$; this is why $f(0) = 1$ is a constant assumption.

17.5.3.2 Reduction to a Difference Equation

The result below shows that the two variables m, n in (17.25) can, over $\mathbb{Z}$, be replaced by a single variable equation.

Proposition 17.10. (Davison [211]). *Let* $f : \mathbb{Z} \to F$ *with* $f(0) = 1$. *Then* f *satisfies equation* (17.25) *if, and only if,*

$$f(n+2) + f(n) = 2f(1)f(n+1), \quad n \in \mathbb{Z}. \tag{17.26}$$

Proof. Assume f satisfies equation (17.25). With $m = n+1$, $n = 1$, we get

$$f(n+1+1) + f(n+1-1) = 2f(n+1)f(1);$$

this is equation (17.26).

Assume conversely that f satisfies equation (17.26). Then f is even; that is,

$$f(-n) = f(n), \quad n \in \mathbb{Z}.$$

It clearly suffices to prove evenness for all $n \in \mathbb{Z}_+$. This is proved by induction on $n \in \mathbb{Z}_+$. It is trivially true for $n = 0$. Also $f(-1) = f(1)$ since $f(0) = 1$. Assume it is true for all $n \in \mathbb{Z}_+$, with $0 \le n \le N$, where $N \ge 1$. Then it is true for $n = N+1$; using (17.26),

$$f(N+1) + f(N-1) = 2f(1)f(N).$$

Since $f(N-1) = f(-(N-1))$ and $f(N) = f(-N)$, by the induction hypothesis

$$f(N+1) + f(1-N) = 2f(1)f(-N). \tag{17.27}$$

But again using equation (17.26) with $n = -N - 1$,

$$f(-N+1) + f(-N-1) = 2f(1)f(-N). \tag{17.27a}$$

Comparing the two equations above yields $f(-N-1) = f(N+1)$; that is, f is even for all $n \in \mathbb{Z}_+$ by induction.

To show that f satisfies equation (17.25), the function $g : \mathbb{Z}^2 \to F$ defined next must be identically zero:

$$g(k,\ell) := f(k+\ell) + f(k-\ell) - 2f(k)f(\ell), \quad (k,\ell) \in \mathbb{Z}^2.$$

Now, since f, assumed to satisfy equation (17.26), has been shown to be even,

$$g(k,\ell) = g(\ell,k) = g(-k,\ell), \quad (k,\ell) \in \mathbb{Z}^2.$$

So iterating the involutions $(k,\ell) \to (\ell,k)$ and $(k,\ell) \to (-k,\ell)$, it follows that F is identically zero on $\mathbb{Z}^2$ if, and only if, g is zero on

$$Y := \{(k,\ell) \in \mathbb{Z}^2 : 0 \le \ell \le k\}.$$

Using equation (17.26) to express $f(k+\ell)$ in terms of $f(k+\ell-1)$ and $f(k+\ell-2)$ and similarly $f(k-\ell)$ in terms of $f(k-\ell-1)$ and $f(k-\ell-2)$, it follows that

$$g(k,\ell) = 2f(1)F(k-1,\ell) - g(k-2,\ell), \quad (k,\ell) \in \mathbb{Z}^2. \tag{17.28}$$

The "size" of $(k,\ell) \in Y$ is $k+\ell$—the taxicab distance from $(0,0)$ to (k,ℓ) in Y. By induction on the "size" of $(k,\ell) \in Y$, it is easy to show, using equation (17.28), that g is zero on Y. ($g(0,0) = 0$ is true since $f(0) = 1$, as is $g(1,0) = 0$. $g(1,1) = f(2) + f(0) - 2f(1')f(1) = 0$ since f satisfies (17.26).) This completes the proof that (17.26) implies (17.25). □

Corollary 17.11. *Let $f : \mathbb{Z} \to F$ with $f(0) = 1$. Then f satisfies equation* (17.26) *if, and only if, it is even and*

$$f(n+2) + f(n) = 2f(1)f(n+1), \quad n \in \mathbb{Z}_+.$$

Proof. Assume f satisfies (17.26). Then, as above, f must be even. Clearly, f satisfies (17.28), as the domain of equation (17.26) includes the domain of equation (17.28). So this direction is proved.

Assume, conversely, that f satisfies (17.28) and is even. Then

$$\begin{aligned} f(-1) + f(1) &= f(1) + f(1) \quad (f(-1) = f(1)) \\ &= 2f(1)f(0) \quad (f(0) = 1). \end{aligned}$$

So f satisfies equation (17.28) for $n = -1$. Now let $n \in \mathbb{Z}$ with $n \le -2$. Then

$$\begin{aligned} f(n+2) + f(n) &= f(n) + f(n+2) \\ &= f(-n) + f(-n-2) \quad (-n-2,\ -n \in \mathbb{Z}) \\ &= 2f(1)f(-n-1) \quad (n-1 \in \mathbb{Z}) \\ &= 2f(1)f(n+1). \end{aligned}$$

So f satisfies equation (17.28) for all integers $n \leq -1$. Thus

$$f(n+2) + f(n) = 2f(1)f(n+1), \quad n \in \mathbb{Z},$$

as claimed.

Equation (17.26) is a linear difference equation of the second order: Since $f(0) = 1$ if $f(1)$ is given, then $f(2)$, and recursively $f(3)$, $f(4), \ldots$ are determined. □

17.5.3.3 The Universal Solution

Definition. Let $T : \mathbb{Z} \to \mathbb{Z}[X]$ be given by $T(0) = 1$, $T(1) = X$, $T(-n) = T(n)$, $n \in \mathbb{Z}_+$, and

$$T(n+2) + T(n) = 2XT(n+1), \quad n \in \mathbb{Z}_+. \tag{17.29}$$

It is customary to write $T(n)$ as $T_n(X)$. Thus $T_2(X) = 2X^2 - 1$, $T_3(X) = 4X^3 - 3X$, $T_4(X) = 8X^4 - 8X^2 + 1$ follow immediately from equation (17.29).

By Proposition 17.10, $T : \mathbb{Z} \to \mathbb{Z}(X)$ is a solution of d'Alembert's functional equation (17.25). Indeed more is true!

Proposition 17.12. (Davison [213]). *Let $f : \mathbb{Z} \to R$ with $f(0) = 1$. If f satisfies equation* (17.25)*, then*

$$f(n) = T_n(f(1)), \quad n \in \mathbb{Z}, \tag{17.30}$$

where $n \mapsto T_n(X)$ is the family of polynomials from the definition above.

Proof. Since both f and T are even (one by virtue of satisfying equation (17.25), the other by definition), it suffices to prove (17.30) for all $n \in \mathbb{Z}_+$. Now $f(0) = 1 = T_0(f(1))$, and $f(1) = T_1(f(1))$. So assume it has been shown that $f(n) = T_n(f(1))$ for all $n \in \mathbb{Z}_+$ with $n \leq N$, where $N \in \mathbb{Z}_+$ and $N \geq 1$. Then

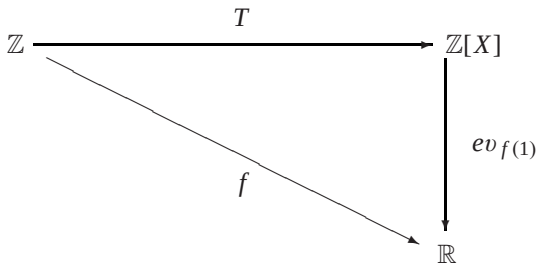
$$\begin{aligned} f(N+1) &= 2f(1)f(N) - f(N-1) && (f \text{ satisfies (17.25)}) \\ &= 2T_1(f(1))T_N(f(1)) - T_{N-1}(f(1)) && (\text{induction hypothesis}) \\ &= T_{N+1}(f(1)) && (T \text{ satisfies (17.29)}). \end{aligned}$$

Thus the result is true for $n \leq N+1$, so the result follows for all $n \in \mathbb{Z}_+$.

Note that what makes the preceding proof work is that, for each $r \in F$, the evaluation ev_r of $p \in \mathbb{Z}(X)$ at r is a homomorphism from $\mathbb{Z}(X)$ to F:

$$\begin{aligned} ev_r(p+q) &= ev_r(p) + ev_r(q) \quad [(p+q)(r) = p(r) + q(r)], \\ ev_r(p \cdot q) &= ev_r(p) + ev_r(q) \quad [(pq)(r) = p(r)q(r)]. \end{aligned}$$

So Proposition 17.12 can be restated in diagram form as



Given f satisfying equation (17.25), there is a unique homomorphism (of commutative rings) $ev_{f(1)}$ such that $f = ev_{f(1)} \circ T$. Also $ev_{f(1)}$ is completely specified by

$$ev_{f(1)}(1) = 1 \quad \text{and} \quad ev_{f(1)}(X) = f(1):$$

There is one, and only one, ring homomorphism that sends 1 (of $\mathbb{Z}$) to 1 (of F) and X of $\mathbb{Z}[X]$ to $r \in F$.

17.5.3.4 Identification of the Universal Solution

The difference equation for T is

$$T(n+2) - 2XT(n+1) + T(n) = 0, \quad n \in \mathbb{Z}_+. \tag{17.31}$$

The (quadratic) indicial equation for this is

$$\lambda^2 - 2X\lambda + 1 = 0.$$

This has roots

$$\lambda_1 = X + \sqrt{X^2 - 1}, \quad \lambda_2 = X - \sqrt{X^2 - 1}.$$

These roots lie in the quadratic extension K of $\mathbb{Z}[X]$, where

$$K = \left\{ \begin{bmatrix} p & q \\ (X^2-1)q & p \end{bmatrix} : p, q \in \mathbb{Z}[X] \right\},$$

so that, since

$$\begin{bmatrix} 0 & 1 \\ (X^2-1) & 0 \end{bmatrix}^2 = \begin{bmatrix} X^2-1 & 0 \\ 0 & X^2-1 \end{bmatrix},$$

$$\lambda_1 = \begin{bmatrix} X & 1 \\ X^2-1 & X \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} X & -1 \\ 1-X^2 & X \end{bmatrix}.$$

(Note that the characteristic polynomial of λ_1 is $t^2 - 2Xt + 1$.)

Hence, for some α_1 and $\alpha_2 \in K$,

$$2T(n) = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n, \quad n \in \mathbb{Z}_+.$$

From the initial conditions, $T(0) = 1$, and $T(1) = X$, so $\alpha_1 = 1$ and $\alpha_2 = 1$. Thus

$$2T_n(X) = \sum_{j=0}^{n} \binom{n}{j} X^{n-j} \left[(\sqrt{X^2-1})^j + (-1)^j (\sqrt{X^2-1})^j \right]$$
$$= \sum_{k=0}^{q} \binom{n}{2k} X^{n-2k} 2(X^2-1)^k,$$

where $j = 2k$ since for j odd $1 + (-1)^j = 0$. Hence the following result has been proved.

Proposition 17.13. [211]. *Suppose* $T : \mathbb{Z} \to \mathbb{Z}[X]$ *is given by* (17.29)*. Then* $T(n)$ $(= T_n(X))$ *is given by* (17.24)*. In other words, the univeral solution to the d'Alembert equation over* $\mathbb{Z}$ *is given by the family of Chebyshev polynomials.*

Since $\lambda_2 = \lambda_1^{-1}$,

$$\left(\begin{bmatrix} X & 1 \\ X^2 - 1 & X \end{bmatrix}^{-1} = \begin{bmatrix} X & -1 \\ 1 - X^2 & X \end{bmatrix} \right),$$

and the solution is seen to agree with Kannappan's general description. Define $E(n) := \lambda_1^n$. Then $E(m+n) = E(m)E(n)$, and $E(0) = 1$ and

$$2T(n) = E(n) + E(-n).$$

17.5.3.5 General Remarks

One direction of the theorem says: If $f : \mathbb{Z} \to F$ satisfies d'Alembert's equation, then $f(n) = T_{(n)}(f(1))$ for all integers n. This has been proved. Proposition 17.12 gives this for the universal T, but Proposition 17.13 identifies the universal T as the Chebyshev family.

The other direction of the theorem is just as easy now: The Chebyshev family is given by $(E(n)+E(-n))/2$ and so satisfies d'Alembert's equation, and consequently so does any homomorphic image via evaluation maps $T_n(X) \to T_n(f(1))$. Thus the theorem has been proved.

Finally, the well-known definition of the Chebyshev polynomial (see Rivlin [704, equation (1.2)]) is a consequence of the theorem: Define for $\theta \in \mathbb{R}$ the function $f : \mathbb{Z} \to \mathbb{R}$ by $n \mapsto \cos(n\theta)$. Then f satisfies d'Alembert's functional equation, as was noted previously. Hence, by the theorem,

$$\cos(n\theta) = f(n) = T_n(f(1)) = T_n(\cos\theta). \tag{17.32}$$

In a similar way, it can be shown that

$$\cosh(nt) = T_n(\cosh t), \quad n \in \mathbb{Z},\ t \in \mathbb{R}. \tag{17.32a}$$

These equations can be subsumed under the general result

$$\frac{X^n + X^{-n}}{2} = T_n\left(\frac{X + X^{-1}}{2}\right), \quad n \in \mathbb{Z}, \tag{17.33}$$

where the theorem has been applied to the function $n \mapsto \frac{X^n+X^{-n}}{2} \in \mathbb{Q}[X, X^{-1}]$. So equation (17.32) follows from (17.33) by evaluating X at $e^{i\theta}$, as does (17.32a) with evaluation at e^t.

17.6 Topology

Topology can be introduced axiomatically in many ways through open sets, closed sets, the interior operator, the closure operator, etc. Here it is done through the interior operator (see Kannappan [425]). For characterization by a single functional equation through the closure operator, see Monteiro [638], Aczél [10], and Lin [608].

It is well known that a set X together with a function i (called the "interior" operator) on subsets of X satisfying the following conditions can be endowed with a topology (Kannappan [425], Vaidyanathaswamy [808]):

(K1) $\quad i(C) \subseteq C \quad$ for $C \subseteq X$;

(K2) $\quad i(C \cap H) = i(C) \cap i(H), \quad C, H \subseteq X$;

(K3) $\quad i(X) = X$;

(K4) $\quad i(i(C)) = i(C), \quad C \subseteq X$.

Evidently, $(X, \cap)$ is an Abelian semigroup. Also, the axioms (K1) to (K4) are equivalent to

$$i(X) = X, \tag{17.34}$$

$$i(C \cap H) = C \cap H \cap i^2(C) \cap i^2(H), \quad \text{for } C, H \subseteq X, \tag{17.35}$$

where $i^2(C) = i[i(C)]$.

Here we consider more generally the problem of defining a natural topology on an arbitrary Abelian semigroup and prove that, for an arbitrary Abelian semigroup G (written multiplicatively) with the identity element e, the conditions corresponding to (17.34) and (17.35),

$$f(e) = e \tag{17.34a}$$

and

$$f(xy) = xyf^2(x)f^2(y), \quad x, y \in G \; [f^2(x) = f(f(x))] \tag{17.35a}$$

(where f is a function from G into G), with further simple assumptions on the operation $\cdot$ in G, imply the analogues of (K2), (K4), and (K1) ((K3) corresponds to (17.34a)),

$$f(xy) = f(x)f(y), \quad x, y \in G, \tag{17.36}$$

$$f^2(x) = f(x), \tag{17.37}$$

and

$$f(x) \leq x \tag{17.38}$$

in a reasonably chosen ordering $<$.

Also, further analogues of known properties of the interior will be derived from (17.34a) and (17.35a), and it will also be examined how these modify if the above-mentioned further assumption about the operation $\cdot$ in G is not made. Here the interior operator is characterized by a single functional equation. A counterexample is provided at the end to show that the same is not true in every Abelian semigroup with unit element without any further assumptions on $(\cdot)$.

Result 17.14. (Kannappan [425]). *Let G be an arbitrary Abelian semigroup* (*multiplicative*) *with identity element e. Then, for a function $f : G \to G$ satisfying* (17.34a), *the two conditions* (17.35a) *and* (17.36) *with*

$$f(x) = xf^2(x), \quad x \in G, \tag{17.39}$$

are equivalent.

Proof. Let f satisfy (17.35a). Putting $y = e$ and using (17.34a), we get $f(x) = xf^2(x)$, which is (17.39). From (17.35a), (17.39), and the Abelian property of G, we get (17.36). Conversely, from (17.36) and (17.39), (17.35a) is immediate. This completes the proof of this result.

Remark. Equation (17.36) is a familiar property of the interior (K2). Naturally, one would expect (K4)

$$f(x) = f^2(x) \tag{17.37}$$

instead of (17.39). But a counterexample is provided at the end to illustrate that (17.34a) and (17.35a) do not imply (17.37) in every Abelian semigroup. But, as mentioned earlier, with the further assumption $x \cdot x = x$ (idempotence) in G for every $x \in G$, (17.37) is also true. As a matter of fact, we prove the following result.

Result 17.15. [425]. *Let G be a semigroup* (*Abelian or not*) *such that $x \cdot x = x$ for every $x \in G$. Then, if $f : G \to G$ satisfies* (17.39), *f also satisfies* (17.37).

Proof. From (17.39) and $x \cdot x = x$, for every $x \in G$, it follows that

$$f(x) = xf^2(x) = x \cdot xf^2(x) = xf(x) \quad \text{for all } x \in G. \tag{17.40}$$

Replacing x by $f(x)$ in (17.40) and using (17.39), we obtain

$$f^2(x) = f(x)f^2(x) = xf^2(x)f^2(x) = xf^2(x) = f(x),$$

which is (17.37). This completes the proof of this result.

In order to obtain the analogues (17.38) of (K1) and further inclusion, the relation $\leq$ is defined in G as follows.

Definition. For $x, y \in G$, define $x \geq y$ if, and only if, there exists a $z \in G$ such that $xz = y$.

Remark. This relation $\geq$ in G is reflexive and transitive, so it is a quasiordering.

In fact, for every $x \in G$, $xe = x$. So, by the definition, $x \geq x$. Let $x \geq y$ and $y \geq z$. Then there are $u, v \in G$ such that $xu = y$ and $yv = z$. Hence $xuv = yv = z$. Thus $x \geq z$.

Theorem 17.16. [425]. *Let G be an Abelian semigroup with the quasiordering defined by the definition above. If $f : G \to G$ satisfies* (17.34a) *and* (17.35a)*, then f also satisfies*

$$x \geq y \quad \textit{implies} \quad f(x) \geq f(y), \quad x, y \in G, \tag{17.41}$$

$$x \geq f(x), \quad x \in G, \tag{17.41a}$$

$$f^2(x) \geq f(x), \tag{17.41b}$$

and

$$f(x) \geq f^2(x). \tag{17.41c}$$

Proof. Let f satisfy (17.34a) and (17.35a). Let $x \geq y$. Then there exists a $z \in G$ such that $xz = y$. Since f satisfies (17.35a), by Result 17.14 we have $f(y) = f(xz) = f(x)f(z)$, so, by the definition, $f(x) \geq f(y)$. Since f satisfies (17.35a), f also satisfies (17.39), from which there results that $x \geq f(x)$ and $f^2(x) \geq f(x)$. From (17.41) and (17.38) follows (17.41c). Thus the proof of this result is complete.

Remark. Unless the quasiordering $\geq$ in G is a partial ordering (that is, $x \geq y$ and $y \geq x$ imply $x = y$), (17.41b) and (17.41c) in general do not imply (17.38), as is shown in the following example.

Counterexample. Let G be a group (Abelian) such that $x^2 = e$ for every $x \in G$. For example, let G be $\{e, a, b, c\}$ and the multiplication defined by

e	a	b	c
a	e	c	b
b	c	e	a
c	b	a	e

.

Let $f : G \to G$ be defined such that

$$f(e) = e, \quad f(a) = b, \quad f(b) = c, \quad \text{and} \quad f(c) = a. \tag{17.42}$$

Then, evidently, $f(xy) = f(x)f(y)$ for all $x, y \in G$, so (17.36) is satisfied. From the definition of f, we have

$$f^2(e) = e, \quad f^2(a) = c, \quad f^2(b) = a, \quad \text{and} \quad f^2(c) = b.$$

Evidently, $f^2(x) \neq f(x)$ for $x \in G\backslash\{e\}$. Hence (17.37) is false. Further,

$$ef^2(e) = e, \quad af^2(a) = b, \quad bf^2(b) = c, \quad \text{and} \quad cf^2(c) = a. \tag{17.43}$$

From (17.42) and (17.43), we get $x \cdot f^2(x) = f(x)$ for all $x \in G$. Hence (17.37) is true. Thus, in general, (17.34a), (17.36), and (17.39) (or (17.34a) and (17.35a)) do not imply (17.37).

Remark. (Lin [608]). The commutativity requirement can be removed when the right side of (17.35a) is suitably rearranged:

$$f(xy) = f^2(x)xyf^2(y). \tag{17.35b}$$

If $f : G \to G$ satisfies (17.34a) and (17.35b), then

(i) (17.36) and

$$f(x) = xf^2(x) = f^2(x) \cdot x \tag{17.39a}$$

hold. (The converse also holds.)

(ii) $f^2(x) = f(x) \Leftrightarrow f(x)$ is idempotent for every x.

17.6.1 Integral

17.6.1.1 Simpson's Rule

The general solution of the functional equation

$$f(x) - g(y) = (x - y)[h(sx + ty) + \phi(x) + \psi(y)] \tag{17.44}$$

for all $x, y \in \mathbb{R}$ with s and t being a priori known parameters is determined without any regularity assumptions (differentiability, continuity, measurability, etc.) imposed on the real functions f, g, h, ϕ, and ψ. The motivation for studying this equation came from Simpson's Rule for evaluating definite integrals (Kannappan, Riedel, and Sahoo [500]).

In connection with Simpson's rule for evaluating definite integrals, one comes across the functional equation

$$f(x) - f(y) = \frac{(x - y)}{6}\left[g(x) + 4g\left(\frac{x + y}{2}\right) + g(y)\right], \tag{17.45}$$

where f is an antiderivative of g. This equation is a special case of

$$f(x) - g(y) = (x - y)[h(x + y) + \psi(x) + \phi(y)], \tag{17.44a}$$

where $f, g, h, \psi, \phi : \mathbb{R} \to \mathbb{R}$ are unknown functions. First, we treat equation (17.44a) (see [583]). The middle term of equation (17.45), $4g\left(\frac{x+y}{2}\right)$, is due to the fact that in Simpon's Rule one partitions the interval into subintervals of equal lengths. However, there is no reason why one should be restricted to such an equal

partition. If we allow unequal partitions, then the middle term is no longer of the form $4g\left(\frac{x+y}{2}\right)$ but rather is of the form $\alpha g(sx+ty)$, where α, s, and t are constants. Taking this into account, we have a generalization of (17.44a) to (17.44b),

$$f(x) - f(y) - (x-y)[h(sx+ty) + g(x) + g(y)], \tag{17.44b}$$

for all $x, y \in \mathbb{R}$, with s and t being a priori chosen parameters.

Our objective now is to determine the general solution of the functional equation (17.44b) without any regularity assumptions (differentiability, continuity, measurability, etc.) imposed on the unknown functions. Further, utilizing the solution of this equation, we determine the general solution of the functional equation (17.44).

Before obtaining the solution of (17.44), we give the motivation—Simpson's Rule (Kannappan and Sahoo [514]).

For a function $f : \mathbb{R} \to \mathbb{R}$ differentiable on $\mathbb{R}$, the mean value theorem states that

$$\frac{f(x) - f(y)}{x - y} = f'(\eta),$$

which holds for η between x and y (see Chapter 7). Obviously, the mean value η depends on x and y, and one may ask for what f the mean value η depends on x and y in a given manner. For more information, see Aczél [31], Aczél and Kuczma [54], and Chapter 7.

Simpson's Rule is an elementary numerical method for evaluating a definite integral $\int_a^b f(t)dt$. The method consists of partitioning the interval $[a, b]$ into subintervals of equal lengths and then interpolating the graph of f over each subinterval with a quadratic function. If $a = x_0 < x_1 < x_2 < \cdots < x_{2n} = b$ is a partition of $[a, b]$ into $2n$ subintervals, each of length $\frac{b-a}{2n}$, then

$$\int_a^b f(t)dt \simeq \frac{b-a}{6n}[f(x_0) + 4f(x_1) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})].$$

This approximation formula is called *Simpson's Rule.* It is well known that the error bound for the Simpson's Rule approximation is

$$\left|\int_a^b f(t)dt - \frac{b-a}{6n}[f(x_0) + 4f(x_1) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})]\right| \leq \frac{k(b-a)^5}{180n^4},$$

where $k = \sup\{|f^{(4)}(x)| \mid x \in [a, b]\}$. It is easy to note from this inequality that if f is four times continuously differentiable and $f^{(4)}(x) = 0$, then

$$\int_a^b f(t)dt = \frac{b-a}{6n}[f(x_0) + 4f(x_1) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})].$$

This is obviously true if $n = 1$, and it reduces to

$$\int_a^b f(t)dt = \frac{b-a}{6}[f(x_0) + 4f(x_1) + f(x_2)].$$

Letting $a = x,\ b = y$, and $x_1 = \frac{x+y}{2}$ in the formula above, we obtain

$$\text{(SR)} \qquad \int_x^y f(t)dt = \frac{y-x}{6}\left[f(x) + 4f\left(\frac{x+y}{2}\right) + f(y)\right].$$

This integral equation (SR) holds for all $x, y \in \mathbb{R}$ if f is a polynomial of degree at most three. However, it is not obvious that if (SR) holds for all $x, y \in \mathbb{R}$, then the only solution f is the cubic polynomial. The integral equation (SR) leads to the functional equation

$$g(y) - g(x) = \frac{y-x}{6}\left[f(x) + 4f\left(\frac{x+y}{2}\right) + f(y)\right], \tag{17.45}$$

where g is an antiderivative of f. The equation above is a special case of the functional equation

$$f(x) - g(y) = (x-y)[h(x+y) + \psi(x) + \phi(y)] \tag{17.44a}$$

for all $x, y \in \mathbb{R}$. Now our goal is to determine the general solution of the functional equation (17.44a). The two functional equations

$$g(x) - g(y) = (x-y)f(x+y) + (x+y)f(x-y), \quad \text{for all } x, y \in \mathbb{R}, \tag{17.46}$$

and

$$xf(y) - yf(x) = (x-y)[g(x+y) - g(x) - g(y)], \quad \text{for all } x, y \in \mathbb{R}, \tag{17.47}$$

are instrumental in solving the functional equation (17.44a). □

17.6.1.2 Solution of the Functional Equation (17.46)

Result 17.17. *The functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation* (17.46) *if and only if*

$$f(x) = ax^2 + A(x), \quad g(x) = 2ax^4 + 2xA(x) + b, \tag{17.46a}$$

where A is additive and a and b are arbitrary constants.

17.6.1.3 Solution of the Functional Equation (17.47)

The general solution of the functional equation (17.47) without any regularity assumption on the unknown functions is given by the following.

Result 17.18. *The functions* $f, g : \mathbb{R} \to \mathbb{R}$ *satisfy the functional equation* (17.47) *if and only if*

$$f(x) = 3ax^3 + 2bx^2 + cx + d, \quad g(x) = -ax^3 - bx^2 - A(x) - d, \tag{17.47a}$$

where A *is additive and* $a,\ b,\ c$, *and* d *are arbitrary constants.*

Using Result 17.18, we solve the functional equation

$$f(x) - g(y) = (x - y)(h(x + y) + k(x) + k(y)), \quad x, y \in \mathbb{R}. \tag{17.48}$$

Theorem 17.19. (Kannappan [475]). *The functions* $f, g, h, k : \mathbb{R} \to \mathbb{R}$ *satisfy* (17.48) *if and only if*

$$\begin{cases} f(x) = 3ax^4 + 2bx^3 + cx^2 + dx + \alpha = g(x), \\ h(x) = ax^3 + bx^2 + A(x) + d - 2\beta, \\ k(x) = 2ax^3 + bx^2 + cx - A(x) + \beta, \end{cases} \tag{17.48a}$$

where A *is additive and* $a,\ b,\ c,\ d,\ \alpha$, *and* β *are arbitrary constants.*

Proof. Letting $x = y$ in (17.48), we see that $f(x) = g(x)$. Hence (17.48) reduces to

$$f(x) - f(y) = (x - y)[h(x + y) + k(x) + k(y)]. \tag{17.49}$$

Putting $y = 0$ in (17.49), we obtain

$$f(x) = f(0) + x[h(x) + k(x) + k(0)].$$

Putting this equation into (17.49) and rearranging, we obtain

$$y[h(x)+k(x)]-x[h(y)+k(y)] = (x-y)[h(x+y)-h(x)-h(y)-k(0)]. \tag{17.50}$$

Defining

$$\phi(x) := h(x) + k(x) \quad \text{and} \quad g(x) := -h(x) - k(0), \tag{17.51}$$

and using (17.51) and (17.50), we have

$$x\phi(y) - y\phi(x) = (x - y)[g(x + y) - g(x) - g(y)] \tag{17.47}$$

for all $x, y \in \mathbb{R}$. The general solution of (17.47) can be obtained from Result 17.18 as

$$\phi(x) = 3ax^3+2bx^2+cx+d_0 \quad \text{and} \quad g(x) = -ax^3-bx^2-A(x)-d_0, \tag{17.51a}$$

where $a,\ b,\ c$, and d_0 are constants. From (17.51) and (17.51a), we obtain

$$k(x) = \phi(x) + g(x) + \beta, \tag{17.51b}$$

where $\beta = k(0)$. Now, using (17.51a) and (17.51b), we obtain

$$k(x) = 2ax^3 + bx^2 + cx - A(x) + \beta. \tag{17.51c}$$

Again, from (17.51) and (17.51a), we have

$$h(x) = ax^3 + bx^2 + A(x) + d - 2\beta,$$

where $d = d_0 + \beta$. Thus we get

$$f(x) = 3ax^4 + 2bx^3 + cx^2 + dx + \alpha,$$

where $\alpha = f(0)$. The proof of the result is now complete. □

Remark. It follows easily from Theorem 17.19 that the solution of (17.45) is $g(x) = 3ax^4 + 2bx^3 + cx^2 + dx + \alpha$ and $f(x) = 12ax^3 + 6bx^2 + 2cx + d$, as predicted. Note that the general solution is obtained without any regularity assumptions of f and g.

Now we prove our main result, the solution of (17.44a).

Theorem 17.20. (Kannappan [475]). *The functions $f, g, h, \phi, \psi : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation*

$$f(x) - g(y) = (x - y)[h(x + y) + \phi(x) + \psi(y)] \tag{17.44a}$$

for all $x, y \in \mathbb{R}$ if and only if

$$\begin{aligned} f(x) &= 3ax^4 + 2bx^3 + cx^2 + dx + \alpha = g(x), \\ h(x) &= ax^3 + bx^2 + A(x) + d - 2\beta, \\ \phi(x) &= 2ax^3 + bx^2 + cx - A(x) + \beta + \gamma, \\ \psi(y) &= 2ay^3 + by^2 + cy - A(y) + \beta - \gamma, \end{aligned} \tag{17.44c}$$

where A is an additive function and $a,\ b,\ c,\ d,\ \alpha,\ \beta$, and γ are arbitrary constants.

Proof. First, letting $x = y$ in (17.44a), we see that $f = g$. Interchanging x with y in (17.44a) and using the fact that $f = g$, we get

$$f(y) - f(x) = (y - x)[h(x + y) + \phi(y) + \psi(x)]. \tag{17.52}$$

Adding (17.52) to (17.44a) and using $f = g$, we get $\psi(x) - \phi(x) = \psi(y) - \phi(y)$ for all $x, y \in \mathbb{R}$. Thus

$$\psi(x) = \phi(x) - 2\gamma,$$

where γ is an arbitrary constant. Putting this ϕ into (17.44a), we have

$$f(x) - g(y) = (x - y)[h(x + y) + \phi(x) + \phi(y) - 2\gamma]. \tag{17.48}$$

Using Theorem 17.19, we have the asserted solution (17.44c). This completes the proof.

Now we proceed to find the general solution of (17.44b) with no regularity assumption imposed on f, g, and h. The solution of this equation will be used to determine the general solution of the main functional equation (17.44) of this section.

Result 17.21. [445]. *Let s and t be real parameters. Functions $f, g, h : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation* (17.44b) *for all $x, y \in \mathbb{R}$ if and only if*

$$f(x) = \begin{cases} ax^2 + (b+d)x + c & \text{if } s = 0 = t \\ ax^2 + (b+d)x + c & \text{if } s = 0,\ t \neq 0 \\ ax^2 + (b+d)x + c & \text{if } s \neq 0,\ t = 0 \\ 3ax^4 + 2bx^3 + cx^2 + (d+2\beta)x + \alpha & \text{if } s = t \neq 0 \\ 2ax^3 + cx^2 + 2\beta x - A(x) + \alpha & \text{if } s = -t \neq 0 \\ ax^2 + (b+d)x + c & \text{if } 0 \neq s^2 \neq t^2 \neq 0, \end{cases}$$

$$g(x) = \begin{cases} ax + \frac{b}{2} & \text{if } s = 0 = t \\ ax + \frac{b}{2} & \text{if } s = 0,\ t \neq 0 \\ ax + \frac{b}{2} & \text{if } s \neq 0,\ t = 0 \\ 2ax^3 + bx^2 + cx - A(x) + \beta & \text{if } s = t \neq 0 \\ 3ax^2 + cx + \beta & \text{if } s = -t \neq 0 \\ ax + \frac{b}{2} & \text{if } 0 \neq s^2 \neq t^2 \neq 0, \end{cases}$$

$$h(x) = \begin{cases} \text{arbitrary with } h(0) = d & \text{if } s = 0 = t \\ d & \text{if } s = 0,\ t \neq 0 \\ d & \text{if } s \neq 0,\ t = 0 \\ a\left(\frac{x}{t}\right)^3 + b\left(\frac{x}{t}\right)^2 + A\left(\frac{x}{t}\right) + d & \text{if } s = t \neq 0 \\ -a\left(\frac{x}{t}\right)^2 - \frac{t}{x}A\left(\frac{x}{t}\right),\ x \neq 0 & \text{if } s = -t \neq 0 \\ d & \text{if } 0 \neq s^2 \neq t^2 \neq 0, \end{cases}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and a, b, c, d, α, and β are arbitrary real constants.

The following result is obvious from Result 17.21, and this result was established in [519] to answer a problem posed by Rudin [716] (see also Chapter 7).

Corollary. *Let s and t be real parameters. Functions $f, g, h : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation*

$$f(x) - g(y) = (x - y)h(sx + ty)$$

for all $x, y \in \mathbb{R}$ if and only if $g(x) = f(x)$ and

$$f(x) = \begin{cases} dx + c & \text{if } s = 0 = t \\ dx + c & \text{if } s = 0,\ t \neq 0 \\ dx + c & \text{if } s \neq 0,\ t = 0 \\ cx^2 + dx + \alpha & \text{if } s = t \neq 0 \\ \alpha - A(x) & \text{if } s = -t \neq 0 \\ dx + c & \text{if } 0 \neq s^2 \neq t^2 \neq 0, \end{cases}$$

$$h(x) = \begin{cases} \text{arbitrary with } h(0) = d & \text{if } s = 0 = t \\ d & \text{if } s = 0,\ t \neq 0 \\ d & \text{if } s \neq 0,\ t = 0 \\ \frac{cx}{s} + d & \text{if } s = t \neq 0 \\ -\frac{t}{x} A\left(\frac{x}{t}\right),\ x \neq 0 & \text{if } s = -t \neq 0 \\ d & \text{if } 0 \neq s^2 \neq t^2 \neq 0, \end{cases}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and c, d, and α are arbitrary constants.

17.6.1.4 Solution of the Main Functional Equation (17.44)

Now we proceed to determine the general solution of (17.44).

Theorem 17.22. [475]. *Let s and t be real parameters. Functions $f, g, h, \phi, \psi : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation* (17.44) *for all $x, y \in \mathbb{R}$ if and only if $g(x) = f(x)$ and*

$$f(x) = \begin{cases} ax^2 + (b+d)x + c & \text{if } s = 0 = t \\ ax^2 + bx + c & \text{if } s = 0,\ t \neq 0 \\ ax^2 + bx + c & \text{if } s \neq 0,\ t = 0 \\ 3ax^4 + 2bx^3 + cx^2 + (d + 2\beta)x + \alpha & \text{if } s = t \neq 0 \\ 2ax^3 + cx^2 + (2\beta - d)x - A(x) + \alpha & \text{if } s = -t \neq 0 \\ ax^2 + (b + \beta + d)x + c & \text{if } 0 \neq s^2 \neq t^2 \neq 0, \end{cases}$$

$$\phi(x) = \begin{cases} ax + \frac{(b-\delta)}{2} & \text{if } s = 0 = t \\ ax + \frac{(b+\delta)}{2} & \text{if } s = 0,\ t \neq 0 \\ ax + \frac{(b+\delta)}{2} & \text{if } s \neq 0,\ t = 0 \\ 2ax^3 + bx^2 + cx - A(x) + \beta + \frac{\delta}{2} & \text{if } s = t \neq 0 \\ 3ax^2 + cx - \frac{1}{2}A_0(x) + \beta & \text{if } s = -t \neq 0 \\ ax - A\left(\frac{stx}{t-s}\right) + \beta + \frac{b}{2} & \text{if } 0 \neq s^2 \neq t^2 \neq 0, \end{cases}$$

$$\psi(x) = \begin{cases} ax + \frac{(b+\delta)}{2} & \text{if } s = 0 = t \\ ax + \frac{(b-\delta)}{2} - h(tx) & \text{if } s = 0,\ t \neq 0 \\ ax + \frac{(b-\delta)}{2} - h(sx) & \text{if } s \neq 0,\ t = 0 \\ 2ax^3 + bx^2 + cx - A(x) + \beta - \frac{\delta}{2} & \text{if } s = t \neq 0 \\ 3ax^2 + cx + \frac{1}{2}A_0(x) + \beta - d & \text{if } s = -t \neq 0 \\ ax - A\left(\frac{t^2 x}{t-s}\right) + \frac{b}{2} & \text{if } 0 \neq s^2 \neq t^2 \neq 0, \end{cases}$$

$$h(x) = \begin{cases} \textit{arbitrary with } h(0) = d & \text{if } s = 0 = t \\ \textit{arbitrary} & \text{if } s = 0,\ t \neq 0 \\ \textit{arbitrary} & \text{if } s \neq 0,\ t = 0 \\ a\left(\frac{x}{t}\right)^3 + b\left(\frac{x}{t}\right)^2 + A\left(\frac{x}{t}\right) + d & \text{if } s = t \neq 0 \\ -a\left(\frac{x}{t}\right)^2 - \frac{t}{x}A\left(\frac{x}{t}\right) + \frac{1}{2}A_0(x),\ x \neq 0, & \text{if } s = -t \neq 0 \\ A\left(\frac{tx}{t-s}\right) + d & \text{if } 0 \neq s^2 \neq t^2 \neq 0, \end{cases}$$

where $A_0, A : \mathbb{R} \to \mathbb{R}$ are additive functions and $a,\ b,\ c,\ d,\ \alpha,\ \beta$, and δ are arbitrary real constants.

Proof. Letting $x = y$ in (17.44), we see that

$$f(x) = g(x)$$

for all $x \in \mathbb{R}$. Hence (17.44) becomes

$$f(x) - f(y) = (x - y)[h(sx + ty) + \phi(s) + \psi(y)]. \tag{17.53}$$

Interchanging x and y in the equation above and adding the resulting equation to it, we have

$$h(sx + ty) + \phi(x) + \psi(y) = h(sy + tx) + \phi(y) + \psi(x) \tag{17.54}$$

for all $x, y \in \mathbb{R}$ with $x \neq y$. But (17.54) holds even for $x = y$ also.

Now we have to consider several cases: $s = 0 = t$; $s = 0,\ t \neq 0$; and $s \neq 0,\ t \neq 0$.

The case $s = 0 = t$ yields $\phi(x) = \psi(x) = \delta$ and

$$f(x) - f(y) = (x - y)[h(sx + ty) + \psi(x) + \psi(y) - \delta].$$

Hence, by Result 17.21, we obtain the asserted solution

$$\begin{aligned} &f(x) = ax^2 + (b + d)x + c, \qquad g(x) = f(x), \\ &\phi(x) = ax + \frac{b - \delta}{2}, \qquad \psi(x) = ax + \frac{b + \delta}{2}, \\ &h(x) \text{ arbitrary with } h(0) = d, \end{aligned}$$

where $a,\ b,\ c,\ d$, and δ are arbitrary constants.

Consider the case where $s = 0$ and $t \neq 0$. (The case where $s \neq 0$ and $t = 0$ can be handled in a similar manner.) Then, for this case, from (17.54), we have

$$h(ty) + \psi(y) - \phi(y) = h(tx) + \psi(x) - \phi(x),$$

for all $x, y \in \mathbb{R}$, and using Result 17.17, we have the asserted solution

$$\begin{aligned} &f(x) = ax^2 + bx + c, \qquad && g(x) = f(x), \\ &\phi(x) = ax + \frac{b+\delta}{2}, && \psi(x) = ax + \frac{b-\delta}{2} - h(tx), \\ &h(x) \text{ arbitrary}, \end{aligned}$$

where $a,\ b,\ c,\ d$, and δ are arbitrary constants.

Now we treat the case where $s \neq 0$ and $t \neq 0$. Suppose $s \neq 0 \neq t$. Next we have to consider several subcases.

Subcase 1. Suppose $s = t$. Then, from (17.54), we get

$$h(tx + ty) + \phi(x) + \psi(y) = h(ty + tx) + \phi(y) + \psi(x).$$

Hence, we have $\phi(x) = \psi(x) - \delta$, and from (17.53) we obtain

$$f(x) - f(y) = (x - y)[h(tx + ty) + \psi(x) + \psi(y) - \delta].$$

From Result 17.21, we obtain

$$\begin{aligned} f(x) &= 3ax^4 + 2bx^3 + cx^2 + (d + 2\beta)x + \alpha = g(x), \\ \phi(x) &= 2ax^3 + bx^2 + cx - A(x) + \beta + \frac{\delta}{2}, \\ \psi(x) &= 2ax^3 + bx^2 + cx - A(x) + \beta - \frac{\delta}{2}, \\ h(x) &= a\left(\frac{x}{t}\right)^3 + b\left(\frac{x}{t}\right)^2 + A\left(\frac{x}{t}\right) + d, \end{aligned}$$

where $a,\ b,\ c,\ d,\ \alpha,\ \beta$, and δ are arbitrary constants and A an additive function.

Subcase 2. Suppose $s = -t$. From (17.54), we have

$$h(ty - tx) + \phi(x) + \psi(y) = h(tx - ty) + \phi(y) + \psi(x)$$

for all $x, y \in \mathbb{R}$. This in turn yields

$$h(tx - ty) - h(ty - tx) = H(x) - H(y),$$

where $H(x) := \phi(x) - \psi(x)$. Letting $x = 0$ in the above, we observe that

$$h(-ty) - h(ty) = d - H(y),$$

where $d = H(0)$. From these, we have

$$H(x-y)+d = H(x)+d-H(y)-d;$$

that is, $H(x)+d$ is additive on the set of reals. Hence

$$\psi(x) = \phi(x) + A_0(x) - d,$$

where $A_0 : \mathbb{R} \to \mathbb{R}$ is an additive map. Substituting this into (17.53), we get

$$f(x) - f(y) = (x-h)[h(ty-tx) + \phi(x) + \phi(y) + A_0(y) - d],$$

which is

$$F(x) - F(y) = (x-y)[K(tx-ty) + \Phi(x) + \Phi(y)],$$

where

$$F(x) = f(x) + d, \quad K(x) = h(-tx) - \frac{1}{2}A_0(x), \quad \Phi(x) = \phi(x) + \frac{1}{2}A_0(x).$$

Thus, from Result 17.21, we again have the asserted solution

$$\begin{aligned}
f(x) &= 2ax^3 + cx^2 + (2\beta - d)x - A(x) + \alpha = g(x),\\
\phi(x) &= 3ax^2 + cx - \frac{1}{2}A_0(x) + \beta,\\
\psi(x) &= 3ax^2 + cx + \frac{1}{2}A_0(x) + \beta - d,\\
h(x) &= -a\left(\frac{x}{t}\right)^2 - \frac{t}{x}A\left(\frac{x}{t}\right) + \frac{1}{2}A_0(x), \quad x \neq 0,
\end{aligned}$$

where $a,\ b,\ c,\ d,\ \alpha$, and β are arbitrary constants and $A, A_0 : \mathbb{R} \to \mathbb{R}$ are additive functions.

Subcase 3. Finally, suppose $s^2 \neq t^2$. Letting $y = 0$ in (17.54), we get

$$h(sx) + \phi(x) + \psi(0) = h(tx) + \phi(0) + \psi(x). \tag{17.55}$$

Letting (17.55) in (17.54) and simplifying, we have

$$h(sx+ty) - h(sx) - h(ty) = h(sy+tx) - h(tx) - h(sy). \tag{17.55a}$$

Interchanging tx and sx in (17.55a), we obtain

$$h(tx+ty) - h(tx) - h(ty) = h(sy+sx) - h(sx) - h(sy).$$

Replacing x by $\frac{x}{t}$ and y by $\frac{y}{t}$ in the above, we get

$$h(x+y) - h(x) - h(y) = h\left(\frac{s(x+y)}{t}\right) - h\left(\frac{sx}{t}\right) - h\left(\frac{sy}{t}\right),$$

which, by defining $H(x) := h(x) - h\left(\frac{sx}{t}\right)$, reduces to

$$H(x+y) = H(x) + H(y).$$

Thus

$$h(x) - h\left(\frac{sx}{t}\right) = A(x),$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function. From this, we have

$$h(tx) - h(sx) = A(tx). \tag{17.56}$$

Putting (17.56) into (17.55a), we get

$$h(sx+ty) - A(ty) = h(tx+sy) - A(tx). \tag{17.57}$$

Substituting $u = sx + ty$ and $v = tx + sy$ in (17.57) and simplifying, we get

$$h(u) - A\left(\frac{tu}{t-s}\right) = h(v) - A\left(\frac{tv}{t-s}\right)$$

for all $u, v \in \mathbb{R}$. Hence

$$h(u) = A\left(\frac{tu}{t-s}\right) + d, \tag{17.57a}$$

where d is a constant. From (17.55) and (17.57a), we obtain

$$\phi(x) = \psi(x) + A(tx) + \beta, \tag{17.57b}$$

where $\beta := \phi(0) - \psi(0)$. Putting (17.57a) and (17.57b) into (17.53) and simplifying, we get

$$F(x) - F(y) = (x-y)\left[\frac{G(x)+G(y)}{2}\right],$$

where

$$F(x) = f(x) - \beta x - dx \quad \text{and} \quad G(x) = 2\psi(x) + 2A\left(\frac{t^2x}{t-s}\right). \tag{17.57c}$$

Hence, by (17.57b), (17.57c), and Result 17.17, we have the asserted solution

$$\begin{aligned}
f(x) &= ax^2 + (b+\beta+d)x + c = g(x),\\
\phi(x) &= ax - A\left(\frac{stx}{t-s}\right) + \beta + \frac{b}{2},\\
\psi(x) &= ax - A\left(\frac{t^2x}{t-s}\right) + \frac{b}{2},\\
h(x) &= A\left(\frac{tx}{t-s}\right) + d,
\end{aligned}$$

where a, b, c, d, and β are arbitrary constants and $A : \mathbb{R} \to \mathbb{R}$ is an additive function.

Since no more cases are left, the proof of the theorem is now complete. □

17.6.2 Determinants

Let $X \in M_{mn}(F)$, where $F = \mathbb{R}$ or $\mathbb{C}$. Denote the jth column of X by x_j and the ith row of X by x^i, so that X can be written as

$$X = (x_{ij} = [x_1, \ldots, x_n] = (x^1, \ldots, x^m).$$

The well-known Binet-Cauchy theorem for matrices $X \in M_{mn}(F)$, $Y \in M_{nm}(F)$ takes the following forms:

(i) If $n < m$, then $\det(XY) = 0$.
(ii) If $n = m$, then $\det(XY) = \det X \det Y$.
(iii)If $n \geq m$, then

$$\det(XY) = \sum_{1 \leq k_1 \leq \cdots \leq k_m \leq n} \det[x_{k_1}, \ldots, x_{k_m}] \det(y^{k_1}, \ldots, y^{k_m}).$$

The fact that the determinant is a multiplicative function on the set of all matrices of order n turns out almost to characterize determinants; still it does not characterize them completely. It is shown in [500] that the general Binet-Cauchy theorem (iii) characterizes the determinant completely. More precisely, the following theorem is true.

Theorem 17.23. (Kucharzewski [546].) *If $f : M_m(\mathbb{R}) \to \mathbb{R}$ is a nontrivial function with the property that*

$$f(XY) = \sum_{1 \leq k_1 < \cdots < k_m \leq n} f([X_{k_1}, \ldots, X_{k_m}])\, f((Y^{k_1}, \ldots, Y^{k_m})) \tag{17.58}$$

holds for any matrix $X \in M_{mn}(\mathbb{R})$ $(m \leq n \leq m+1)$ and any matrix $Y \in M_{nm}(\mathbb{R})$, then

$$f(C) = \det C$$

is true for any $C \in M_m(\mathbb{R})$.

Proof. Two proofs are presented in [546], and we present the following one. For $n = m$, from (17.58) it follows that f is multiplicative; that is,

$$f(XY) = f(X)\, f(Y), \quad \text{for } X, Y \in M_m(\mathbb{R}).$$

According to Results 2.11 and 2.12, we get

$$f(C) = M(\det C)$$

for every matrix $C \in M_m(\mathbb{R})$, where the function M is multiplicative (M) on $\mathbb{R}$.

Now, take

$$X = \begin{bmatrix} t & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & 0 & 0 \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & s \end{bmatrix}$$

and observe that $\det([X_{k_1}, \ldots, X_{k_m}])$ $(1 \le k_1 < \cdots < k_m \le n)$ equals t if $k_1 = 1, \ldots, k_m = m$ or $k_1 = 1, \ldots, k_{m-1} = m-1$, $k_m = m+1$. Otherwise, this determinant vanishes. Using this, (M), and (17.58), we find

$$f(XY) = (m-1)M(0)^2 + M(t)M(1) + M(t)M(s).$$

On the other hand, $\det(XY) = t + ts$ together with (M) gives

$$M(t + ts) = (m-1)M(0) + M(t) + M(ts). \tag{17.59}$$

If in (17.59) we set $t = s = 0$, we get $mM(0) = 0$; i.e., $M(0) = 0$.

Replacing s by t in (17.59), we get that M is additive. M being additive and multiplicative on $\mathbb{R}$ implies (see Proposition 1.12)

$$M(t) \equiv t.$$

Thus $f(C) = M(\det C) = \det C$ for any $C \in M_m(\mathbb{R})$.

This proves the theorem. □

Corollary. *For every real matrix X of order m, we have* $\det X^t = \det X$, *where X^t denotes the transpose of X.*

Proof. For $C \in M_m(\mathbb{R})$, set

$$f(C) = \det C^t.$$

It is easy to see that f satisfies all conditions of Theorem 17.23. For example, for matrices

$$X = [x_1, \ldots, x_m, x_{m+1}], \quad Y = (y^1, \ldots, y^m, y^{m+1}),$$

we have

$$\begin{aligned} f(XY) &= \det(XY)^t \\ &= \det(Y^t X^t) \\ &= \sum_{1 \le k_1 < \cdots < k_m \le m+1} \begin{vmatrix} y_{k_1 1} & \cdots & y_{k_m 1} \\ \cdots & \cdots & \cdots \\ y_{k_1 m} & \cdots & y_{k_m m} \end{vmatrix} \cdot \begin{vmatrix} x_{1 k_1} & \cdots & x_{m k_1} \\ \cdots & \cdots & \cdots \\ x_{k_m} & \cdots & x_{m k_m} \end{vmatrix} \\ &= \sum_{1 \le k_1 < \cdots < k_m \le m+1} f([x_{k_1}, \ldots, x_{k_m}])\, f((y^{k_1}, \ldots, y^{k_m})). \end{aligned}$$

Since $f(I) = \det I' = \det I = 1$, we conclude that $f(C) = \det C$. Thus $\det C = \det C^t$ for any $C \in M_m(\mathbb{R})$.

This concludes the proof of the corollary.

Remark. The Binet-Cauchy theorem can be proved by straightforward computations using determinant functions. A determinant function $\Delta(x_1, \ldots, x_m)$ is a function of m vectors defined on an m-dimensional vector space V subject to the following conditions:

(i) Δ is linear with respect to the first argument; i.e.,

$$\Delta(ax_1 + a'x_1', x_2, \ldots, x_m) = a\Delta(x_1, x_2, \ldots, x_n) + a'\Delta(x_1', x_2, \ldots, x_m),$$

$x_i \in V$.

(ii) Δ is skew-symmetric with respect to all arguments; i.e.,

$$\Delta(\ldots, x_k, x_{k+1}, \ldots) = -\Delta(\ldots, x_{k+1}, x_k, \ldots) \quad \text{for } 1 \leq k < m.$$

For more characterizations of determinants, see Gaspar [301].

17.7 Digital Filtering

Computer vision is a relatively new and fast-growing field that primarily aids computers with artificial sensory perception. In other words, computer vision deals with image understanding. Preprocessing is a basic task in every image understanding problem. Among other things, preprocessing involves filtering. Spatial filters are used for deblurring, smoothing, sharpening, and enhancing images. Thus filtering is a technique for modifying or enhancing an image.

A digital image is a function $f : \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \to \mathbb{Z}_\ell$, where n_1, n_2, and ℓ are positive integers. Here ℓ is a parameter that indicates the number of gray levels used in representing an image. If the function f denotes an image, then a spatial filtering of f would be

(17.60)
$$\begin{aligned}\hat{f}(x, y) = {} & f(x + t, y + t) + f(x - t, y) + f(x, y - t) \\ & - f(x - t, y - t) - f(x, y + t) - f(x + t, y)\end{aligned}$$

for all $(x, y) \in \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$. The effect of this filter on a test image is shown in the figure below. The variable t is preset to 1. The filtered image $\hat{f}$ retains certain information contained in the original image.

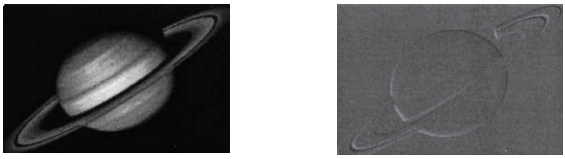


Figure: Original (left) and filtered (right) images of Saturn

A particular filter rarely produces good results for all images. For some images, a filter will produce a better result in terms of retaining certain basic information. For other images, it may not yield better results. Thus it will be useful to know on what kinds of images a particular filter gives the worst result (that is, it makes the image completely dark). To determine the class of such images with respect to the filter described in (17.60), one encounters the functional equation

$$\begin{aligned} f(x+t, y+t) + f(x-t, y) + f(x, y-t) \\ = f(x-t, y-t) + f(x, y+t) + f(x+t, y) \end{aligned} \tag{17.60a}$$

for all $x, y, t \in \mathbb{Z}_n$. The goal now is to study the functional equation in an abstract setting.

The functional equation (17.60a) also appeared in [513]. In that paper, it was shown that if the quadratic polynomial $f(x, y) = ax^2 + bx + cy^2 + dy + exy + \alpha$ with $a \neq 0$ and $c \neq 0$ is a solution of the functional differential equation

$$f(x+h, y+k) - f(x, y) = hf_x(x+\theta h, y+\theta k) + kf_y(x+\theta h, y+\theta k), \quad 0 < \theta < 1,$$

for all $x, y, h, k \in \mathbb{R}$ with $h^2 + k^2 \neq 0$, then $\theta = \frac{1}{2}$. Conversely, if a function f satisfies the equation above with $\theta = \frac{1}{2}$, then the only solution is a quadratic polynomial. Here f_x and f_y represent the first partial derivatives of f with respect to x and y, respectively.

To establish this result in [513], the authors used the functional equation

$$\begin{aligned} f(x+s, y+t) + f(x-s, y) + f(x, y-t) \\ = f(x-s, y-t) + f(x, y+t) + f(x+s, y) \end{aligned} \tag{17.60b}$$

for all $x, y, s, t \in \mathbb{R}$. Letting $s = t$ in the equation above, we have the previous equation (17.60a) holding for all $x, y, t \in \mathbb{R}$. Since (17.60b) implies (17.60a), the solution of the functional equation (17.60b) can be obtained through the solution of the equation (17.60a). The general complex-valued solution of (17.60a) when (17.60a) holds for all x, y, and t in a 2-divisible Abelian group G is given in the following proposition.

Proposition 17.24. *The function $f : G \times G \to \mathbb{C}$ satisfies the functional equation* (17.60a)*, where G is a 2-divisible Abelian group if and only if*

$$f(x, y) = B(x, y) + \phi(x) + \psi(y) + \chi(x - y) \tag{17.60c}$$

holds for all $x, y \in G$, where $B : G \times G \to \mathbb{C}$ is biadditive and $\phi, \psi, \chi : G \to \mathbb{C}$ are arbitrary functions.

17.8 Geometry

Let $\mathcal{M}_p$ be the family of all straight lines in the plane $\mathbb{R}^2$ that are parallel to the y-axis and of all curves of the form $y = p(x + \alpha) + \beta$, where p is a fixed function

and α, β run over $\mathbb{R}$. In Faber, Kuczma, and Mycielski [265], it is proved that if there exists a continuous bijection of $\mathbb{R}^2$ onto $\mathbb{R}^2$ that induces a map of the family of all straight lines onto $\mathcal{M}_p$, then p must be a polynomial of degree 2.

Proof. We present the proof in [265]. Since the set $\{(x, y) : y = p(x)\}$ is the image of some straight line, by the homeomorphism ϕ, the function p must be continuous. It is shown in [325] where this result is proved under the additional condition that p is differentiable, that there exists a constant r such that $q(x) = p(x + r)$ satisfies the functional equation

$$q(x) + q(t) - q(x + t) - q(0) = (q(1) - q(1 + t) - q(0) + q(t))x.$$

Neither p nor q can be of the form $ax + b$. In fact, if $p(x) = ax + b$, then $\mathcal{M}_p$ consists of all vertical lines and all lines with slope a. But then not every pair of points of $\mathbb{R}^2$ lies on one curve of $\mathcal{M}_p$. So, $\mathcal{M}_p$ could not be isomorphic to the family of all straight lines.

Since the left side of the equation above is symmetric in x and t, interchanging x and t on the right side and then putting $x = 1$ yields

$$q(1) - q(1 + t) - q(0) + q(t) = 2ct$$

for some constant c. Thus

$$\begin{aligned} q(x) + q(t) - q(x + t) - q(0) &= 2ctx \\ &= c[(x + t)^2 - x^2 - t^2], \end{aligned}$$

which is additive in $A(x) = q(x) - q(0) + cx^2$. Thus $q(x) = -cx^2 + ax + d$. This concludes the proof. □

Let $\mathcal{N}_p$ consist of all planes parallel to the z-axis and all surfaces of the form

$$z = p(x + \alpha, y + \beta) + \gamma,$$

where $p : \mathbb{R}^2 \to \mathbb{R}$ and α, β, and γ are real constants. In [265], the following problem was raised: Characterize those functions p for which $\mathcal{N}_p$ is continuously isomorphic to the family of all planes in $\mathbb{R}^3$. This problem is solved through a functional equation and shows that p is a polynomial of degree 2.

Let $\mathcal{P}$ denote the family of all planes in $\mathbb{R}^3$. An *isomorphism* from $\mathcal{P}$ onto $\mathcal{N}_p$ is a bijection of $\mathbb{R}^3$ that induces a bijection from $\mathcal{P}$ onto $\mathcal{N}_p$. Similarly, an *automorphism* of $\mathcal{P}$ is a bijection of $\mathbb{R}^3$ that induces a bijection from $\mathcal{P}$ onto $\mathcal{P}$.

Lemma. *Every bijection ϕ of $\mathbb{R}^3$ onto itself that induces a bijection of $\mathcal{P}$ onto itself is affine linear; that is,*

$$\phi(x, y, z) = (a_1x + b_1y + c_1z + u, a_2x + b_2y + c_2z + v, a_3x + b_3y + d_3z + w),$$

where

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \neq 0,$$

and a_i, b_i, *and* c_i $(i = 1, 2, 3)$ *are arbitrary constants.*

This result is known as the "Fundamental Theorem of Projective Geometry".

Result 17.25. *Let* $p : \mathbb{R}^2 \to \mathbb{R}$. *There exists a continuous isomorphism* θ *from* $\mathcal{P}$ *onto* $\mathcal{N}_p$ *if and only if* $p(x, y) = d_1x^2 + d_2y^2 + 2d_3xy + d_4x + d_5y + d_6$ *with* $d_1d_2 - d_3^2 \neq 0$, *where* d_i $(i = 1, 2, \ldots, 6)$ *are real constants.*

17.9 Field Homomorphisms

We treated the problem in Chapter 1 whether for $A : \mathbb{R} \to \mathbb{R}$ additive $A(x)A\left(\frac{1}{x}\right) \geq 0$ for $x \in \mathbb{R}^*$ implies the continuity of A. In [131], it is claimed that if an additive $A : K \to K$ (K a field with $\text{char}(K) \neq 2$) satisfying $A(x)A\left(\frac{1}{x}\right) = 1$ for $x \in K^*$ is a field monomorphism. Halter-Koch and Reich [337] considered the characterization of additive functions satisfying a functional equation of the form

$$A\left(\frac{ax + b}{cx + d}\right) = \frac{\alpha A(x) + \beta}{\gamma A(x) + \delta}. \tag{17.61}$$

Result 17.26. [337]. *Let* K *and* $\overline{K}$ *be fields of characteristic different from* 2, $d_1 \in K^*$, $\overline{d}_1 \in \overline{K}^*$, *and let* $A : K \to \overline{K}$ *be an additive function satisfying*

$$A\left(\frac{d_1}{x}\right) = \frac{\overline{d}_1}{A(x)} \quad \text{for all } x \in K \setminus \ker(f). \tag{17.61a}$$

Then either $A = 0$ *or* $\varepsilon = A(1) \notin 0$, $\varepsilon^{-1}A : K \to \overline{K}$ *is a field monomorphism and* $\overline{d}_1 = \varepsilon f(d)$.

Theorem 17.27. [337]. *Let* F *be a field of characteristic different from* 2, $\#F \geq 5$, *and let* K *and* $\overline{K}$ *be extension fields of* F. *Suppose that*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, K), \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, K),$$

and let $A : K \to \overline{K}$ *be an* F*-linear map satisfying the functional equation* (17.61) *for all* $x \in K$ *such that* $cx + d \neq 0$ *and* $\gamma A(x) + \delta \neq 0$. *Then one of the following three assertions holds true:*

(i) $A = 0$ *and* $\beta\delta = 0$.
(ii) $c = \gamma = 0$, $A\left(\frac{b}{d}\right) = \frac{\beta}{\delta}$, *and* $A\left(\frac{a}{d}x\right) = \frac{\alpha}{\delta}A(x)$ *for all* $x \in K$.

(iii) $c\gamma \neq 0$, $A(1) = \varepsilon \neq 0$, $\varepsilon^{-1}A : K \to \overline{K}$ *is a field monomorphism, and there exists some* $\lambda \in \overline{K}^*$ *such that*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \lambda\varepsilon A(a) & \lambda\varepsilon^2 A(b) \\ \lambda A(c) & \lambda\varepsilon A(d) \end{pmatrix}.$$

Proof. We consider the case $c\gamma \neq 0$. We may assume that $c = \gamma = 1$, and then we must prove that

$$\begin{pmatrix} \alpha & \beta \\ 1 & \delta \end{pmatrix} = \begin{pmatrix} A(a) & \varepsilon A(b) \\ 1 & A(d) \end{pmatrix}.$$

If $x \in K$ is such that $x + d \neq 0$, and $A(x) + \delta \neq 0$, then (17.61a) implies

$$A(a) - A\left(\frac{d_1}{x+d}\right) = \alpha - \frac{\overline{d}_1}{A(x)+\delta}.$$

Putting $y = x + d$, we obtain, for all $y \in K^*$ such that $A(y) - B(d) + \delta \neq 0$,

$$A(a) - A\left(\frac{d_1}{y}\right) = \alpha - \frac{d_1}{A(y) - A(d) + \delta}.$$

If $F_1 = \{\lambda \in F^* \mid \lambda A(y_0) - A(d) + \delta \neq 0\}$, then $\#F_1 \geq 3$, and if $y = \lambda y_0$ for some $\lambda \in F_1$, then

$$A(a) - \frac{1}{\lambda} A\left(\frac{d_1}{y_0}\right) = \alpha - \frac{\overline{d}_1}{\lambda A(y_0) - A(d) + \delta},$$

which implies

$$A(y_0)(A(a) - \alpha)\lambda^2 + \left[(-A(d) + \delta)(A(a) - \alpha) - A\left(\frac{d_1}{y_0}\right) A(y_0) + d_1\right]\lambda + A\left(\frac{d_1}{y_0}\right)(A(d) - \delta) = 0$$

for all $\lambda \in F_1$. Hence, we obtain

$$\begin{aligned} A(y_0)(A(a) - \alpha) &= 0, \\ (-A(d) + \delta)(A(a) - \alpha) - A\left(\frac{d_1}{y_0}\right) A(y_0) + \overline{d}_1 &= 0, \\ A\left(\frac{d_1}{y_0}\right)(A(d) - \delta) &= 0, \end{aligned}$$

and consequently, since $A(y_0) \neq 0$,

$$\begin{aligned} A(a) &= \alpha, \\ A\left(\frac{d_1}{y_0}\right) A(y_0) &= \overline{d}_1 \neq 0, \end{aligned}$$

and hence

$$A(d) = \delta.$$

Therefore A satisfies the functional equation (17.61a) for all $y \in K \setminus \ker(f)$. Hence, by Result 17.26, $\varepsilon^{-1}A : K \to \overline{K}$ is a field monomorphism, $\overline{d}_1 = \varepsilon f(d_1)$, and using $\alpha = f(a),\ \delta = f(d)$, we obtain also $\beta = \varepsilon f(b)$, which finishes the proof. □

Corollary. *If, under the conditions of Theorem* 17.27*, we assume in addition*

$$c\gamma \neq 0 \quad \textit{and} \quad A(1) = 1,$$

then an additive function $A : K \to \overline{K}$ satisfies (17.61) *if and only if A is a field monomorphism.*

A Generalization of (17.61)

Result 17.28. *Let K be a field containing $\mathbb{Q}$, $n \in \mathbb{Z}_+^*$, $n \geq 2$,*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, K),$$

and either $c = 0$ or $c \neq 0,\ (c^{-1}d)^2 = \alpha^n$ for some $\alpha \in K$. Let $A_1, A_2 : K \to K$ be additive functions such that A_1 is injective and

$$A_2\left(\frac{ax^n + b}{cx^n + d}\right) = \frac{aA_1(x)^n + b}{cA_1(x)^n + d} \tag{17.61b}$$

holds for all $x \in K$ such that $cx^n + d \neq 0$ and $cA_1(x)^n + d \neq 0$. Then $A_1(1)^{-1}A_1 : K \to K$ is a field automorphism.

17.10 Pythagorean Functional Equation

We consider now the functional equation

$$f(x^2 + y^2) = f(x)^2 + f(y)^2, \tag{17.62}$$

known as the *Pythagorean Functional Equation*, where f stands for a map of the finite field $\mathbb{F}_q$ of q elements ($q = p^n$ for some prime) into itself. As multiplication in $\mathbb{F}_q$ may be obtained from both squares and addition by polarization whenever the characteristic p of $\mathbb{F}_q$ is not 2, we obtain as nontrivial solutions of (17.62) exactly the Frobenius automorphism together with its powers if $p \neq 2$. The two functional equations required for a field isomorphism (one for addition and the other for multiplication) may be simplified to just one: functional equation (17.62). But, for the sake of completeness, we also include the solutions of (17.62) when $p = 2$, which turn out to be simpler to find out due to the fact that in this case the map $x \mapsto x^2$ is bijective (the Frobenius automorphism). However, the results for $p \neq 2$ no longer

hold if $p = 2$, as one expects from the fact that the polarization identity requires dividing by powers of 2, and thus the multiplicative structure is not recovered from squares and addition.

The solution of (17.62) is obtained via the solutions of the restricted Cauchy functional equation

$$f(x^2 + y^2) = f(x^2) + f(y^2) \tag{17.62a}$$

for $f : \mathbb{F}_q \to \mathbb{F}_q$. Equation (17.62a) is considered here because there is a trivial, but useful, connection between (17.62a) and (17.62), which allows us to proceed solving first (17.62a) and then (17.62).

Result 17.29. (Garcia-Roig and Salillas [300]). *Let p be an odd prime number and $q = p^n$, $n \geq 1$. The functional equation* (17.62a) *for $f : \mathbb{F}_q \to \mathbb{F}_q$ has exactly q^n solutions of the form*

$$f(x) = a_1 x + a_p x^p + a_{p^2} x^{p^2} + \cdots + a_{p^{n-1}} x^{p^{n-1}},$$

where $a_1, a_p, \ldots, a_{p^{n-1}}$ are in $\mathbb{F}_q$.

Theorem 17.30. [300]. *Let p be an odd prime number and $q = p^n$, $n \geq 1$. The functional equation* (17.62) *for $f : \mathbb{F}_q \to \mathbb{F}_q$ has exactly $n+2$ solutions, namely the constant functions $f \equiv 0$, $f \equiv \frac{1}{2}$ and the powers $f(x) = x^{p^i}$ $(i = 0, 1, \ldots, n-1)$ of the Frobenius automorphism of $\mathbb{F}_q$.*

Proof. Substituting x and y by zero in (17.62), we obtain $f(0) = 2f(0)^2$ so that $f(0) = 0$ or, if $p > 2$, $f(0) = \frac{1}{2}$.

Let us assume first that $f(0) = 0$. Then, setting $y = 0$ in (17.62), we see that

$$f(x^2) = f(x)^2$$

and thus (17.62) entails (17.62a), so that the solutions for (17.62) are to be found among those described in Result 17.29. Now taking a solution $f(x)$ induced by the (reduced) polynomial

$$F(T) = a_1 T + a_p T^p + a_{p^2} T^{p^2} + \cdots + a_{p^{n-1}} T^{p^{n-1}}$$

of (17.62a) with $a_1, a_p, a_{p^2}, \ldots, a_{p^{n-1}}$ in $\mathbb{F}_q$, let us impose what amounts to

$$F(T^2) = F(T)^2$$

since both polynomials are already reduced. Expanding, we get

$$\begin{aligned} &a_1 T^2 + a_p T^{2p} + \cdots + a_{p^{n-1}} T^{2p^{n-1}} \\ &\quad = a_1^2 T^2 + a_p^2 T^{2p} + \cdots + a_{p^{n-1}}^2 \cdot T^{2p^{n-1}} + 2 \sum_{0 \leq i < j < n} a_{p^i} a_{p^i} T^{p^i + p^i}, \end{aligned}$$

and from this we conclude that there cannot occur two (or more) nonzero coefficients in $p(T)$: If $i < j$ and $a_{p^i} a_{p^j} \neq 0$, then the term of degree $p^i + p^j$, which appears to the right of the equation above, does not appear on the left (observe the argument would fail in characteristic 2). Thus $p(T)$ is monomial, say $a_{p^i} T^{p^i}$ $(0 \leq i < n)$. But then, going back to the equation above again, we see that $a_{p^i}^2 = a_{p^i}$; i.e., $a_{p^i} = 0$ or 1 (which actually yield solutions of (17.62)), and this concludes this case.

Assume now that $f(0) = \frac{1}{2}$. In this case, obviously $f(x^2) = f(x)^2 + \frac{1}{4}$, and if we set $g(x) := f(x) - \frac{1}{2}$, then f being a solution of (17.62) entails that $g(x)$ satisfies (17.62a). This g may be thought to be induced by the polynomial $q(T) = a_1 T + a_p T^p + \cdots + a_{p^{n-1}} T^{p^{n-1}}$ of $\mathbb{F}_q[T]$. But now $q(T)$ is subject to the relation

$$q(T^2) = q(T)^2 + q(T)$$

since

$$\begin{aligned} g(x^2) &= f(x^2) - \frac{1}{2} = f(x)^2 + \frac{1}{4} - \frac{1}{2} = f(x)^2 - \frac{1}{4} \\ &= \left(g(x) + \frac{1}{2}\right)^2 - \frac{1}{4} = g(x)^2 + g(x). \end{aligned}$$

Expanding the equation above and proceeding in a similar way as with the case $f(0) = 0$, we immediately obtain first that $q(T)$ must be a monomial and then that it must be the zero polynomial. This proves the theorem. □

The Case of Characteristic 2

This case turns out to be simpler than that of the odd characteristic (and note that the preceding methods are still applicable). Precisely, for (17.62a) there is an immediate argument, and then the further step to get the solutions of (17.62) is straightforward.

Result 17.31. [300]. *The functional equation* (17.62a) *for* $f : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$, *where* $q = 2^n$ $(n > 0)$, *has* q^n *solutions, namely the polynomial mappings*

$$f(x) = a_1 x + a_p x^p + a_{p^2} x^{p^2} + \cdots + a_{p^{n-1}} x^{p^{n-1}},$$

where $a_1, a_p, \ldots, a_{p^{n-1}}$ *are* (*arbitrarily chosen*) *in* $\mathbb{F}_q$. *From these, the* 2^n *maps where the coefficients* $a_1, a_p, \ldots, a_{p^{n-1}}$ *are required to lie in the prime field* $\mathbb{F}_2$ *are all the solutions of the functional equation* (17.62).

17.11 Statistics

This section is devoted to characterization of some statistical distributions such as the Poisson, gamma, and normal distributions.

17.11.1 Poisson Distribution

17.11.1.1 Characterization of (Bivariate) Poisson Distributions

Suppose that items of two types have been observed a certain number of times, but these original observations have been reduced due to a destructive process that is the product of two binomial distributions and that the probabilities of these reduced numbers are the same whether damaged or undamaged. Then the original random variables had a bivariate Poisson distribution with zero mutual dependence coefficient.

Suppose m and n are the numbers of items of two types observed originally. Suppose further that these original observations are reduced to r and s due to a destructive process that is the product of two binomial distributions; i.e., the probability that (m, n) is reduced to (r, s) is

$$\binom{m}{r} p^r (1-p)^{m-r} \binom{n}{s} q^s (1-q)^{n-s}, \quad (p, q \in I_0).$$

Suppose finally that for the resulting distribution

$$P(r, s) = P(r, s \mid \text{undamaged}) = P(r, s \mid \text{damaged}).$$

Then the original random variables had a bivariate Poisson distribution with zero mutual dependence coefficient (that is, they had the distribution of two independent random variables obeying Poisson laws).

In Skitovich [748], this problem is reduced to find the solution of the functional equation

$$G(pz + 1 - p, qw + 1 - q)G(p, q) = G(pz, qw) \tag{17.63}$$

concerning the joint generating function G of the original random variables.

Theorem 17.32. [59]. *Let $G \not\equiv 0$ satisfy the functional equation* (17.63) *for all $p, q \in]0, 1[,\ z, w \in]-1, 1[$. Then, and only then,*

$$G(z, w) = \exp[A_1(z-1) + A_2(w-1)] \tag{17.63a}$$

for all $z, w \in]-1, 1[$, where A_1 and A_2 are arbitrary solutions of (A) *on* $\{(z_1, z_2) \mid -1 < z_1 < 0,\ -2 < z_2 < 0,\ -2 < 2z_1 + z_2 < 0\}$.

Further, if G is continuous in a point (of $]-1, 1[$), then, and only then,

$$G(z, w) = \exp[\lambda_1(z-1) + \lambda_2(w-1)] \tag{17.63b}$$

holds with arbitrary constants λ_1 and λ_2.

Proof. The "only then" statements are obvious by direct substitutions. In order to prove the "then" parts, introduce new variables in (17.63). Set $x = p - 1,\ y = q - 1,\ u = pz - p$, and $v = qw - q$, so that $p = x + 1,\ q = y + 1,\ z =$

$\frac{u+p}{p} = \frac{u}{x+1} + 1$, $w = \frac{v+q}{q} = \frac{v}{y+1} + 1$, ($p \neq 0$ and $q \neq 0$ by supposition, so $x \neq -1$, $y \neq -1$), and

$$pz = u + p = u + x + 1, \quad qw = v + q = v + y + 1.$$

Then (17.63) goes over into

$$F(x, y)F(u, v) = F(x + u, y + v), \tag{17.63c}$$

where

$$F(u, v) = G(u + 1, v + 1)$$

for $p, q \in I_0$, $z, w \in]-1, 1[$, $x, y \in]-1, 0[$ and u and v through $]-2, 0[$, while $2x + u = p(z + 1) - 2$ and $2y + v = q(w + 1) - 2$ are restricted to $]-2, 0[$.

The general, not identically zero, solution of (17.63c) is of the form (17.63a) (see [12, Section 5.1.1]). This proves the theorem.

17.11.2 Normal Distribution

(i) (Kac [408]). Suppose $\phi : \mathbb{R} \to \mathbb{C}$ satisfies the functional equation

$$\phi(x) = \sqrt{2\pi}\sigma\phi(ax)\phi(bx), \quad \text{for } x \in \mathbb{R}, \tag{17.64}$$

with $\int \phi(x)dx = 1$, and $\phi'(0)$ exists where $\sigma, a, b > 0$ and $a^2 + b^2 = 1$. Then ϕ is a normal distribution

$$\phi(x) = \sqrt{2\pi} \cdot \frac{1}{\sigma} e^{-x^2/\sigma^2} \quad \text{for all } x \in \mathbb{R}. \tag{17.65}$$

(ii) (Baker [93], Laha and Lukacs [593]). If we set $f(x) = \sqrt{2\pi}\sigma\phi(x)$ for $x \in \mathbb{R}$, then (17.64) is equivalent to the functional equation

$$f(x) = \prod_{j=1}^{n} [f(\beta_j x)]^{\gamma_j} \tag{17.64a}$$

studied in [593] to characterize normal distributions, with $n = 2$, $\beta_1 = a$, $\beta_2 = b$, $\gamma_1 = 1 = \gamma_2$, and $\beta_1^2\gamma_1 + \beta_2^2\gamma_2 = 1$.

Let $F : \mathbb{R} \to \mathbb{R}$ be a distribution function that is a nondecreasing, right continuous function such that $\lim_{t\to-\infty} F(t) = 0$ ($F(-\infty) = 0$) and $\lim_{t\to\infty} F(t) = 1$ ($F(+\infty) = 1$). The Fourier-Stieltjes transform

$$f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x)$$

of $F(x)$ is called the *characteristic function* of the distribution function $F(x)$. A characteristic function is always a continuous and positive definite function (in the sense of S. Bochner) and $f(0) = 1$. A characteristic function $f(t)$ is said to be infinitely divisible if for every positive integer n there exists a characteristic function $f_n(t)$ such that $f(t) = [f_n(t)]^n$.

In [593] is proved the following result.

Result 17.33. *Suppose that a characteristic function $f(x)$ satisfies the functional equation* (17.64a)*, where $0 < \beta_i < 1,\ \gamma_i > 0$ $(i = 1$ to $n)$. Then* (i) *when*

$$\sum_{i=1}^{n} \beta_i \gamma_i = 1$$

and the first moment of $f(x)$ exists, f is infinitely divisible; (ii) *the characteristic function $g(t) = f(t)\, f(-t)$ is infinitely divisible; and* (iii) *when*

$$\sum_{i=1}^{n} \beta_i^2 \gamma_i = 1,$$

f is the characteristic function of a normal distribution with zero mean.

The following results are used in proving Result 17.33 regarding (17.62a).

Proposition (i). *Suppose $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$. Then* (17.64a) *holds for all $x > 0$ if and only if there exists a function $\phi : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) = \exp[x^k \phi(\log x)] \quad \textit{for all } x > 0; \tag{17.66}$$

i.e.,

$$\phi(t) = e^{-kt} \log f(e^t) \quad \textit{for all } t \in \mathbb{R},$$

and

$$\phi(t) = \sum_{j=1}^{n} \mu_j \phi(t - p_j) \quad \textit{for all } t \in \mathbb{R}. \tag{17.66a}$$

Proposition (ii). *Suppose $\phi : \mathbb{R} \to \mathbb{R}$ (or any normal linear space),* (17.66a) *holds, and $\lim_{t \to -\infty} \phi(t)$ exists. Then ϕ is constant.*

Proposition (iii). *Suppose $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$,* (17.64a) *holds for all $x > 0$,*

$$\sum_{j=1}^{n} \gamma_j \geq 1$$

(equivalently, $k \geq 0$), and $\lim_{x \to 0_+} f(x)$ exists. Then either $f \equiv 0$ or $f(x) > 0$ for all $x \geq 0$. Moreover, assuming $f \not\equiv 0$, if $k = 0$, then f is constant, and if $k > 0$, then $\lim_{x \to 0_+} f(x) = 1$.

Theorem 17.34. [93]. *Suppose $f : \mathbb{R} \to \mathbb{R}_+$,* (17.64a) *holds for all $x \in \mathbb{R},\ k \in \mathbb{Z}_+^*$, and $f^{(k)}(0)$ exists. Then either $f \equiv 0$ or there exists $c \in \mathbb{R}$ such that*

$$f(x) = \exp[cx^k] \quad \textit{for all } x \in \mathbb{R}. \tag{17.67}$$

Proof. The proof is based on induction. Since $f'(0)$ exists, f is continuous at 0. By Proposition (iii) (applied to the restriction of f to $(0, +\infty)$), either $f(x) = 0$ for all $x \geq 0$ or $f(0) = 1$ and $f(x) > 0$ for all $x > 0$. Assume the latter.

Let $F(x) = \log f(x)$ for $x \in \mathbb{R}$. Choose $\phi : \mathbb{R} \to \mathbb{R}$ such that (17.66) and (17.66a) hold.

Then $F(x) = x^k \phi(\log x)$ for all $x > 0$, $F(0) = 0$, and $F'(0) = f'(0)/f(0) = f'(0)$.

Suppose $k = 1$. Let

$$c = F'(0) = \lim_{x \to 0_+} \frac{F(x) - F(0)}{x} = \lim_{x \to 0_+} \phi(\log x) = \lim_{t \to -\infty} \phi(t).$$

By Proposition (ii), $\phi(t) = c$ for all $t \in \mathbb{R}$. Hence $F(x) = cx$ for all $x > 0$ and thus $f(x) = \exp[cx]$ for all $x > 0$. The continuity of f at 0 thus implies that $f(x) = \exp[cx]$ for all $x \geq 0$. Let $g(x) = f(-x)$ for $x \in \mathbb{R}$. Then

$$g(x) = f(-x) = \prod_{j=1}^{n} f(\beta_j(-x))^{\gamma_j} = \prod_{j=1}^{n} g(\beta_j x)^{\gamma_j}$$

for all $x \in \mathbb{R}$ and $g'(0) = -f'(0) = -c$. Hence $g(x) = e^{-cx}$ for all $x > 0$. That is, $f(x) = e^{cx}$ for all $x < 0$. Thus our assertion is true when $k = 1$.

Now suppose that $k \geq 2$. By assumption, there exists $\varepsilon > 0$ such that $f^{(k-1)}(x)$ is defined for all $x \in (-\varepsilon, \varepsilon)$. Hence $\phi^{(k-1)}(t)$ is defined for all $t \in (-\infty, \log \varepsilon)$ and hence, by (17.66a), for all $t \in \mathbb{R}$. It then follows from (17.66) that $f^{(k-1)}(x)$ and $F^{(k-1)}(x)$ are defined for all $x > 0$. Now, for all $x > 0$,

$$F(x) = x^k \phi(\log x)$$

and

$$F'(x) = kx^{k-1}\phi(\log x) + x^{k-1}\phi'(\log x)$$

or

$$F'(x) = x^{k-1}\{\phi'(\log x) + k\phi(\log x)\}.$$

For $\alpha \in \mathbb{R}$, let $\frac{d}{dx} + \alpha$ denote the linear differential operator defined by

$$\left(\left[\frac{d}{dt} + \alpha\right]\psi\right)(t) = \psi^1(t) + \alpha\psi(t), \quad \text{for all } t \in \mathbb{R},$$

whenever $\psi : \mathbb{R} \to \mathbb{R}$ is differentiable. Then

$$F'(x) = x^{k-1}\left(\left[\frac{d}{dx} + k\right]\phi\right)(\log x) \quad \text{for all } x > 0.$$

Similarly, if $k \geq 3$, we have, for all $x > 0$,

$$F''(x) = x^{k-2}\left(\left[\frac{d}{dx}+k-1\right]\left[\frac{d}{dx}+k\right]\phi\right)(\log x)$$

$$\vdots$$

$$F^{(k-1)}(x) = x\left(\left[\frac{d}{dx}+2\right]\cdots\left[\frac{d}{dx}+k\right]\phi\right)(\log x).$$

Let $\psi = \left[\frac{d}{dx}+2\right]\cdots\left[\frac{d}{dx}+k\right]\phi$. Then

$$\psi = \phi^{(k-1)} + a_{k-2}\phi^{k-2} + \cdots + a_1\phi' + a_0\phi,$$

where $a_0, a_1, \ldots, a_{k-2} \in \mathbb{R}$, are such that, for all $z \in \mathbb{C}$,

$$z^{k-1} + a_{k-2}z^{k-2} + \cdots + a_1 z + a_0 = (z+2)\cdots(z+k).$$

Notice that $a_0 = k!$ It then follows from (17.66a) that

$$\psi(t) = \sum_{j=1}^{N} \mu_j \psi(t - p_j) \quad \text{for all } t \in \mathbb{R}. \tag{17.66b}$$

We also know that

$$F^{(k-1)}(x) = x\psi(\log x) \quad \text{for all } x > 0. \tag{17.68}$$

Now

$$F(x) = \log f(x) = \sum_{j=1}^{n} \gamma_j \log f(\beta_j x) = \sum_{j=1}^{n} \gamma_j F(\beta_j x) \quad \text{for all } x \in \mathbb{R},$$

so that

$$F'(x) = \sum_{j=1}^{n} \beta_j \gamma_j F'(\beta_j x) \quad \text{for all } x \in \mathbb{R}.$$

Hence

$$F'(0) = \left\{\sum_{j=1}^{n} \beta_j \gamma_j\right\} F'(0).$$

But

$$\sum_{j=1}^{n} \beta_j \gamma_j > \sum_{j=1}^{n} \beta_j^k \gamma_1 = 1$$

since $k \geq 2$, so $F'(0) = 0$. If $k \geq 3$, then

$$F''(x) = \sum_{j=1}^{n} \beta_j^2 \gamma_j F''(\beta_j x) \quad \text{for all } x \in \mathbb{R},$$

so that $F''(0) = 0$ since

$$\sum_{j=1}^{N} \beta_j^2 \gamma_j > \sum_{j=1}^{N} \beta_j^k \gamma_j = 1.$$

Continuing this argument, we find that $0 = F^{(m)}(0)$ for $m = 1, 2, \ldots, k-1$.

But $F^{(k)}(0)$ exists since $f^{(k)}(0)$ exists and we know that $F^{(k-1)}(0) = 0$. Hence, by (17.68),

$$\begin{aligned} F^{(k)}(0) &= \lim_{x \to 0^+} \frac{F^{(k-1)}(x) - F^{(k-1)}(0)}{x} \\ &= \lim_{x \to 0^+} \psi(\log x). \end{aligned}$$

By (17.66b) and Proposition (ii), ψ is constant, say $\psi(t) = \lambda \in \mathbb{R}$ for all $t \in \mathbb{R}$. That is, for all $t \in \mathbb{R}$,

$$\phi^{(k-1)}(t) + a_{k-2}\phi^{(k-2)}(t) + \cdots + a_1\phi^1(t) + (k!)\phi(t) = \lambda.$$

Hence there exist real constants $c_2, \ldots, c_k$ such that

$$\phi(t) = c_2 e^{-2t} + \cdots + c_k e^{-kt} + (\lambda/k!) \quad \text{for all } t \in \mathbb{R}.$$

Thus

$$\begin{aligned} F(x) &= x^k \phi(\log x) \\ &= x^k \left[\frac{c_2}{x^2} + \cdots + \frac{c_k}{x^k} + \left(\frac{\lambda}{k!} \right) \right] \\ &= \left(\frac{\lambda}{k!} \right) x^k + c_k + \cdots + c_2 x^{k-2} \quad \text{for all } x > 0. \end{aligned}$$

Since $0 = F^{(m)}(0)$ for $m = 1, 2, \ldots, k-1$, we have $c_2 = \cdots = c_k = 0$. Thus, if $c = \lambda/k!$, then $F(x) = cx^k$ for all $x > 0$ and hence, for all $x > 0$,

$$f(x) = \exp[cx^k]. \tag{17.67}$$

Arguing as we did in the case where $k = 1$, we find that (17.67) holds for all real x. This proves the theorem. □

(iii) [144, 12]. This is characterization of the normal distribution by an important property. We try to find a probability density function of the form $f(x - a)$, where a is the parameter, such that the maximum likelihood estimate of "a" is the mean value of the sample. This implies that the maximum of the functions

$$\prod_{i=1}^{n} f(x_i - a) \Longleftrightarrow \sum_{i=1}^{n} \log[f(x_i - a)]$$

must be attained at the sample mean. Thus, we must have

$$\sum_{i=1}^{n} h'\left(x_i - \frac{x_1 + x_2 + \cdots + x_n}{n}\right) = 0 \Longleftrightarrow \sum_{i=1}^{n} h'(y_i) = 0, \quad \sum_{i=1}^{n} y_i = 0,$$

where $h(x) = \log[f(x)]$.

Substituting $y_i = 0$ for all $i = 1, 2, \ldots, n$, we get $h'(0) = 0$, and substituting $y_i = 0$ for $i = 2, 3, \ldots, n-1$, we get $h'(y) = -h'(-y)$. Therefore, we have

$$\begin{aligned} h'(y_1 + y_2 + \cdots + y_{n-1}) &= h'(-y_n) \\ &= -h'(y_n) \\ &= h'(y_1) + h'(y_2) + \cdots + h'(y_{n-1}), \end{aligned}$$

which is equation (A). Thus, $h'(x) = cx$ and $h(x) = cx^2/2 + b$. Consequently, we have $f(x) = \exp(cx^2/2 + b)$. If we now impose the normalization condition

$$\int_{-\infty}^{\infty} f(x - a)dx = 1,$$

we get

$$f(x - a) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-a)^2}{2\sigma^2}\right], \quad -\infty < x < \infty, \tag{17.65a}$$

which is a normal family of distributions. Note that $c = -1/\sigma^2$.

(iv) Now we turn our attention to the functional equation

$$f(x)g(y) = h(ax + by)k(cx + dy) \tag{17.69}$$

for $x, y \in \mathbb{R}$, $f, g, h, k : \mathbb{R} \to \mathbb{C}$, $a, b, c, d \in \mathbb{R}_+^*$, and its generalization

$$f(x)g(y) = \prod_{i=1}^{n} h_i(a_i x + b_i y), \tag{17.70}$$

where $f, g, h_i : \mathbb{R} \to \mathbb{C}$, $a_i, b_i \in \mathbb{R}_+^*$ with $a_i b_j - a_j b_i \neq 0$, i, j (1 to n) $i \neq j$. The functional equation (17.69) has applications to the quantum mechanical three-body problem and to the characterization of normal distributions.

At the eighth annual meeting on functional equations held at Oberwolfach in 1970, the following two problems were raised that are special cases of (17.69).

Problem (R.H. Hall). Find the general, even, square-integrable real solutions of the functional equation

$$f(x)g(y) = f\left(\frac{x}{2} + \frac{\sqrt{3}}{2}y\right) g\left(\frac{\sqrt{3}}{2}x - \frac{y}{2}\right). \tag{17.69i}$$

This has applications to the quantum mechanical three-body problem.

Problem (J. Aczél). Find the general continuous real solution of the functional equation

$$f(x)g(y) = f(ax+by)g(cx+dy) \tag{17.69ii}$$

(a, b, c, and d real constants).

At the same meeting A. Lundberg found the most general logarithmically differentiable solution of (17.69i). In [334], (17.69ii) was solved assuming f and g to be twice continuously differentiable real functions and assuming $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be a rotation matrix. Lundberg also noted that if f and g were continuous solutions of (17.69ii) that were not identically zero, then they were never zero, again assuming $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be a rotation matrix.

Now we consider the functional equation (17.69) where f, g, h, and k are complex-valued functions of a real variable and a, b, c, and d are fixed nonzero real numbers such that $ad - bc \neq 0$. Let $P(x, y) = f(x)g(y)$ for $x, y \in \mathbb{R}$.

If, in addition, P is Lebesgue measurable, then f, g, h, and k are continuous. Using this and a theorem of Kemperman [529], we determine (Theorem 17.36) the continuous solutions of (17.69).

The functional equation (17.34) was treated in [524], and this is also solved under weaker regularity conditions in [91] than were assumed in [524].

17.11.2.1 The Equation $f(x)g(y) = h(ax+by)k(cx+dy)$

Result 17.35. (Baker [91]). *Let f, g, h, k satisfy* (17.69). *If there exists a subset V of $\mathbb{R}^2$ of positive Lebesgue measure such that $P(x, y) = f(x)g(y) \neq 0$ for all $(x, y) \in V$, then $f(x)g(y)h(u)k(v) \neq 0$ for all $x, y, u, v \in \mathbb{R}$. If, in addition, P is measurable, then f, g, h, and k are continuous.*

Theorem 17.36. [91]. *If f, g, h, and k satisfying* (17.69) *are measurable and not almost everywhere zero, then there exist complex polynomials p, q, r, and s, of degree at most 2, such that $f(x) = \exp(p(x))$, $g(x) = \exp(q(x))$, $h(x) = \exp(r(x))$, and $k(x) = \exp(s(x))$ for all $x \in \mathbb{R}$.*

Proof. By Result 17.35, f, g, h, and k are continuous and nowhere zero.

Put $x = 0$ to obtain

$$f(0)g(0) = h(0)k(0).$$

Hence

$$\frac{f(x)}{f(0)} \cdot \frac{g(y)}{g(0)} = \frac{h(ax+by)}{h(0)} \cdot \frac{k(cx+dy)}{k(0)}$$

for all $x, y \in \mathbb{R}$.

Letting log denote the principal branch of the logarithm in the right half plane and defining

$$F(x) = \log(f(x)/f(0)),$$
$$G(y) = \log(g(y)/g(0)),$$
$$H(u) = \log(h(u)/h(0)),$$
$$K(v) = \log(k(v)/k(0)),$$

for sufficiently small $x, y, u, v \in \mathbb{R}$, we deduce from (17.69) that

$$F(x) + G(y) = H(ax + by) + K(cx + dy) \tag{17.69a}$$

for all x and y in some neighbourhood of 0.

Choose $\varepsilon > 0$ such that (17.69a) holds whenever $|x| < \varepsilon$ and $|y| < \varepsilon$. Integrating over $[-\varepsilon, \varepsilon]$, we find, for $|x| < \varepsilon$,

$$2\varepsilon F(x) + \int_{-\varepsilon}^{\varepsilon} G(y)dy = \int_{-\varepsilon}^{\varepsilon} \{H(ax + by) + K(cx + dy)\}dy,$$

from which it follows, using the Fundamental Theorem of Calculus, that F is continuously differentiable in a neighbourhood of 0. Similarly, G, H and K are continuously differentiable in a neighbourhood of 0.

Differentiating (17.69a) with respect to x, we find that

$$F'(x) = aH'(ax + by) + cK'(cx + dy)$$

for all sufficiently small x and y. Integrating this equation with respect to y over a suitably small neighbourhood of 0, we find that F'' exists and is continuous in a neighbourhood of 0. Similarly, G, H, and K have continuous second derivatives in a neighbourhood of 0.

Differentiating (17.69a) with respect to x and y, we find

$$0 = abH''(ax + by) + cdK''(cx + dy)$$

for sufficiently small x and y. Hence

$$0 = abH''(u) + cdK''(v)$$

for sufficiently small u and v so that H'' and K'' are constant in a neighbourhood of 0. Similarly, F'' and G'' are constant in some neighbourhood of 0.

In particular, there is a polynomial p such that $F(x) = p(x)$, and hence

$$f(x) = \exp(p(x)) \tag{17.71}$$

for all sufficiently small x.

Now, for any fixed $x_0 \in \mathbb{R}$,

$$f(x_0)g(0) = h(ax_0)k(cx_0),$$

so that

$$\frac{f(x_0+x)}{f(x_0)}\cdot\frac{g(y)}{g(0)}=\frac{h(ax_0+ax+by)}{h(ax_0)}\cdot\frac{k(cx_0+cx+dy)}{k(cx_0)}$$

for all $x, y \in \mathbb{R}$. Now it follows from the equation above that f is analytic in a neighbourhood of x_0. Thus, from the theory of analytic functions, we conclude that (17.71) is valid for all real x. Similarly, we obtain the required representations of g, h, and k. □

Remark. The continuous solutions of (17.69) are one of the following forms:

(i) $f \equiv 0,\ h \equiv 0;\ g$ and k arbitrary.
(ii) $f \equiv 0,\ k \equiv 0;\ g$ and h arbitrary.
(iii) $g \equiv 0,\ h \equiv 0;\ f$ and k arbitrary.
(iv) $g \equiv 0,\ k \equiv 0;\ f$ and h arbitrary.
(v) $f(x) = a' \exp(\alpha x^2 + \beta x)$,
$g(x) = b' \exp(\gamma x^2 + \delta x)$,
$h(x) = c' \exp(\mu x^3 + \nu x)$,
$k(x) = d' \exp(\rho x^3 + \sigma x)$,

for all $x \in \mathbb{R}$, where the constants $a, b, c, d, \alpha, \beta, \ldots$, are such that

$$a'b' = c'd' \neq 0, \qquad a^2\mu + c^2\rho = \alpha, \qquad b^2\mu + d^2\rho = \gamma,$$
$$ab\mu + cd\rho = 0, \qquad a\nu + c\sigma = \beta, \qquad b\nu + d\sigma = \delta.$$

Remark. There are measurable solutions of (17.69) that are almost everywhere zero but not identically zero. Indeed, let

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases}$$

and let $f = g = h = k$.

The general solution of (17.69) is given in the following result.

Result 17.37. (Lajko [600]). *Suppose that the functions $f, g, h, k : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation* (17.69)*, where $a, b, c, d \in \mathbb{R}_+^*$ are arbitrary constants with $\Delta = ad - bc \neq 0$ and there exists a subset $V \subset \mathbb{R}^2$ of positive Lebesgue measure such that $f(x)g(y) \neq 0$ for all $x, y \in V$. Then f, g, h, and k have the form*

$$f(x) = \alpha_1 \exp[A_1(x) + n_1(x)], \quad x \in \mathbb{R},$$
$$g(x) = \alpha_2 \exp[A_2(x) + n_2(x)], \quad x \in \mathbb{R},$$
$$h(x) = \beta_1 f\left(\frac{d}{\Delta}x\right) g\left(-\frac{c}{\Delta}x\right), \quad x \in \mathbb{R},$$
$$k(x) = \beta_2 f\left(-\frac{b}{\Delta}x\right) g\left(\frac{a}{\Delta}x\right), \quad x \in \mathbb{R},$$

where

$$\alpha_1\beta_1\alpha_2\beta_2 = 1,$$

$$n_1(x) + n_2(y) = n_1\left(\frac{ad}{\Delta}x + \frac{bd}{\Delta}y\right) + n_1\left(\frac{bc}{\Delta}x + \frac{bd}{\Delta}y\right) + n_2\left(\frac{ac}{\Delta}x + \frac{bc}{\Delta}y\right) + n_2\left(\frac{ac}{\Delta}x + \frac{ad}{\Delta}y\right)$$

hold and the functions $A_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ *satisfy* (A) *and* $n_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ *satisfy the functional equation* (Q) (*quadratic*).

17.11.2.2 The Equation $f(x)g(y) = \prod_{i=1}^{n} h_i(a_i x + b_i y)$

Now we treat the generalization (17.71).

Result 17.38. (Baker [91]). *If there exists a subset* V *of positive planar Lebesgue measure such that* $P(x, y) \neq 0$ *whenever* $(x, y) \in V$, *then*

$$f(x)g(y)h_i(u) \neq 0 \quad \textit{for all } x, y, u \in \mathbb{R} \textit{ and } 1 \leq i \leq n.$$

If, in addition, f, g, *and* h_i $(1 \leq i \leq n)$ *are measurable, then there exist polynomials* p, q, r_i $(1 \leq i \leq n)$ *of degree at most* n *such that*

$$f(x) = \exp(p(x)), \quad g(y) = \exp(q(y)), \quad \textit{and} \quad h_i(u) = \exp(r_i(u))$$

for all $x, y, u \in \mathbb{R}$ *and all* $i = 1, 2, \ldots, n$.

Remark 17.39. If we set $f(x)$ by $f(x)g(x)$ and $g(y)$ by $f(y)g(-y)$, $h_1 = f$, $a_1 = b_1 = 1$, $h_2 = g$, $a_2 = 1 = -b$, $n = 2$ (17.71) can be written as

$$f(x + y)g(x - y) = f(x)f(y)g(x)g(-y). \tag{17.72}$$

Now we consider (17.72) with motivation.

(v) Suppose that u and v are independent random variables. If $u + v$ and $u - v$ are independent, then u and v are normally distributed. Functional equations are derived from this, and then we determine the most general solution of these equations, the regular solutions of which are normal distributions.

At an international conference on "Inequalities and Functional Equations" held in Debrecen some years ago, in a conversation it was suggested by I. Olkin to C.J. Eliezer that the solution of the functional equation (17.72) would be of interest, as the equation arose in a problem on statistical distributions. Subsequently, the continuous solution of (17.72) was found in [254] (see also Van der Lyn [810], Olkin [654]). Here we are determining the most general solution of (17.72) and a connected equation (17.73) again connected to a normal distribution without assuming any regularity condition whatsoever. First we give the motivation for treating (17.72).

After obtaining the solution, Professors Eliezer and Olkin were contacted in April 1991, and here is the gist of the reply from Professor Olkin in May 1991. The problem you refer to is this: Suppose that u and v are independent random variables. If $u+v$ and $u-v$ are independent, then u and v are normally distributed.

The known proofs are based on characteristic functions (Kapur [525]). Here we obtain the solution of (17.72) by a simple and direct method without any assumptions.

Remark. The following assertion holds. If u_i $(i = 1, 2, \ldots, n)$ are independent random variables and $\sum a_i u_i$ and $\sum b_i u_i$ are independent linear statistics with $a_i b_i \neq 0$, then u_i's are normally distributed (see Skitovich [749] and Kapur [525, p. 89]).

The functional equation in the present case is derived as follows. Suppose that u and v have densities, and let $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$. The joint probability density function p of u, v is

$$p(u, v) = f_1(u)\, f_2(v)$$

(this is the independence property); that is, $p(x, y) = \text{constant}\ f_1(x+y)\, f_2(x-y)$. By hypothesis, x and y are independent, which says that (forgetting the constant)

$$f_1(x+y)\, f_2(x-y) = \phi_1(x)\phi_2(y), \tag{17.73}$$

which will be treated in (vi).

From (17.73), for $x = 0$,

$$\phi_2(y) = \frac{f_1(y)\, f_2(-y)}{\phi_1(0)},$$

and for $y = 0$,

$$\phi_1(x) = \frac{f_1(x)\, f_2(x)}{\phi_2(0)}.$$

Consequently, (17.73) becomes

$$f_1(x+y)\, f_2(x+y) = \frac{f_1(x)\, f_2(x)\, f_1(y)\, f_2(-y)}{\phi_1(0)\phi_2(0)},$$

which can be written as

$$f(x+y)g(x-y) = f(x)\, f(y)g(x)g(-y), \tag{17.72}$$

where

$$f(x) = \frac{f_1(x)}{\phi_1(0)}, \quad g(y) = \frac{f_2(y)}{\phi_2(0)}.$$

17.11.2.3 Solution of the Functional Equation (17.72)

We first determine the most general solution of (17.72).

Theorem 17.40. (Kannappan [473]). *Suppose that* $f, g : \mathbb{R} \to \mathbb{R}_+^*$ *satisfy the functional equation* (17.72) *for all* $x, y \in \mathbb{R}$. *Then* f *and* g *are given by*

$$f(x) = ce^{q(x)}E(x), \quad g(x) = \frac{1}{c}e^{q(x)}E'(x), \tag{17.72s}$$

where E *and* E' *are exponentials satisfying* (E), q *is a quadratic functional satisfying* (Q), *and* c *is an arbitrary constant.*

Remark. The general solution of (17.72) is obtained without assuming any regularity condition on f or g.

Proof. Note that neither f nor g can take zero value. With $x = 0 = y$, (17.72) gives $f(0)g(0) = 1$. So, without loss of generality, assume that $f(0) = 1$.

Change y to $-y$ in (17.72) to get

$$f(x-y)g(x+y) = f(x)f(-y)g(x)g(y) \quad \text{for } x, y \in \mathbb{R}. \tag{17.72a}$$

Notice the symmetric role played by f and g. Interchange x and y in (17.72a) and divide by the resultant equation to obtain

$$E_1(x-y) = \frac{E_1(x)}{E_1(y)}, \tag{E}$$

where

$$E_1(x) = \frac{f(x)}{f(-x)}. \tag{17.74}$$

From (E) there results $E_1(x+y) = E_1(x)E_1(y)$; that is, E_1 is exponential.

Now, dividing (17.72) by (17.72a), we have

$$\frac{E_2(x+y)}{E_2(x-y)} = \frac{E_2(y)}{E_2(-y)} \quad \text{for } x, y \in \mathbb{R},$$

where

$$E_2(x) = \frac{f(x)}{g(x)}. \tag{17.74a}$$

Changing x to $x+y$ in the equation above, we get

$$\frac{E_2(x+2y)}{E_2(x)} = \frac{E_2(y)}{E_2(-y)}.$$

With $x = 0$, the equation above, gives

$$E_2(2y) = \frac{E_2(y)}{E_2(-y)}$$

(note that $E_2(0) = 1$) so that

$$E_2(x + 2y) = E_2(x)E_2(2y).$$

Thus E_2 is also exponential (E). Now, from (17.74a), we have

$$\frac{f(x)}{g(x)} \cdot \frac{f(-x)}{g(-x)} = E_2(x)E_2(-x) = E_2(0) = 1;$$

that is,

$$f(x)\,f(-x) = g(x)g(-x) \quad \text{for } x \in \mathbb{R}.$$

Set

$$T(x) = f(x)\,f(-x) = g(x)g(-x) \quad \text{for } x \in \mathbb{R}. \tag{17.75}$$

Changing x to $-x$ and y to $-y$ in (17.72) and multiplying (17.72) side by side by the resultant equation and using (17.75), we obtain

$$T(x + y)T(x - y) = T(x)^2T(y)^2 \quad \text{for } x, y \in \mathbb{R}. \tag{17.76}$$

Noticing that T is positive and taking the logarithm on both sides of this equation, we have (Q) satisfied by q for $x, y \in \mathbb{R}$, where $q(x) = \log T(x)$.

So, q is a quadratic functional and

$$T(x) = f(x)\,f(-x) = e^{q(x)} \quad \text{for } x \in \mathbb{R}.$$

From (17.74) and the equation above, it follows that

$$f(x)^2 = e^{q(x)}E_1(x) \quad \text{for } x \in \mathbb{R}.$$

Since f and E_1 are positive, taking the square root on both sides, we have

$$f(x) = e^{\frac{q(x)}{2}}E_1(x)^{\frac{1}{2}},$$

which is the same as f in (17.72s). From (17.74a), we have

$$g(x) = f(x)E_2(-x) = e^{\frac{q(x)}{2}}E_1(x)^{\frac{1}{2}}E_2(-x),$$

which is the same as g in (17.72s). This proves the theorem. □

Corollary 17.41. *If f and g are measurable or continuous in the theorem, then the solutions of* (17.72) *are given by*

$$f(x) = ce^{ax^2+b_1x}, \quad g(x) = \frac{1}{c}e^{ax^2+b_2x}, \tag{17.77}$$

where a, b_1, b_2, and c are arbitrary constants (*refer to* [254]).

Proof. Since f and g are regular, so are E, E', and q, and $E(x) = e^{b_1 x}$, $E'(x) = e^{b_2 x}$, and $q(x) = ax^2$. The result follows.

For f and g to be probability density functions, we must have

$$\int_{-\infty}^{\infty} f(x)dx = 1 = \int_{-\infty}^{\infty} g(x)dx.$$

From

$$\int_{-\infty}^{\infty} f(x)dx = 1,$$

we have

$$\frac{c}{\sqrt{-a}} e^{-\frac{b_1^2}{a}} \cdot \sqrt{\pi} = 1.$$

With $m_1 = -\frac{b_1}{2a}$, $\sigma = \sqrt{-\frac{1}{2a}}$, f can be written as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m_1)^2}{2\sigma^2}},$$

which is a normal distribution. Similarly, g has the form

$$g(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m_2)^2}{2\sigma^2}},$$

with $m_2 = -\frac{b_2}{2a}$.

We now state the result regarding (17.72). Let F be a field and $F(+)$ a dyadic group (that is, the equation $2x = x + x = a$ has a solution in F). Suppose $f, g : F \to F^*$ satisfy (17.72). Then f and g are given by

$$f(x) = a\psi(x)E(x), \quad g(x) = \frac{1}{a}\psi(x)E'(x),$$

where ψ satisfies $\psi(x+y)\psi(x-y) = \psi(x)^2\psi(y)^2$ (see (17.76)) and E, and E' are exponential.

(iv) Let X and Y be independent random variables with density functions f and g, respectively. If the random variables $X + Y$ and $X - Y$ are independent, then X and Y are normal distributions with the same variance (see [12, p. 109], [144], [600], [473], and Theorem 17.40).

Proof. From Theorem 17.40, we see that

$$\phi_1(x)\phi_2(y) = f_1(x+y)\, f_2(x-y), \quad \text{for } x, y \in \mathbb{R}, \tag{17.73}$$

holds.

Proof (i). From Theorem 17.40, it follows that the general solution of (17.73) is given by

$$f_1(x) = c_1 e^{q(x)} E(x), \quad f_2(x) = c_2 e^{q(x)} E'(x),$$

and that

$$f(x) = c_3 e^{q(x)} E(x), \quad q(x) = c_4 e^{q(x)} E'(x). \tag{17.72s}$$

The measurable (continuous) solutions are given by

$$f(x) = ce^{ax^2+b_1x}, \quad g(x) = \frac{1}{c} e^{ax^2+b_2x}, \tag{17.77}$$

and from Corollary 17.41, we obtain f and g as normal distributions.

Proof (ii). Equation (17.73) is of the form (17.69) with $f = \phi_1,\ g = \phi_2,\ h = f_1$ with $a = 1 = b,\ k = f_2, c = 1 = -d$.

From Theorem 17.36 (see also [144]), we obtain (17.77), and thus f and g are normal distributions.

17.11.3 Gamma Distribution

The associated functional equation is

$$f(x)g(y) = p(x+y)q\left(\frac{x}{y}\right) \quad \text{for } x, y > 0. \tag{17.78}$$

This equation arises in statistics in the characterization of distributions as follows.

Let X and Y be two nondegenerate and positive random variables, and suppose they are independently distributed. Then the random variables $X + Y$ and X/Y are independently distributed if and only if X and Y have gamma distributions with the same scale parameter.

If one assumes the existence of density functions $f,\ g,\ p$, and q for the positive random variables $X,\ Y,\ X + Y$, and X/Y, respectively, and that X and Y are independent and $X + Y$ is independent of X/Y, then it follows that

$$f(x)g(y) = p(x+y)q(x/y), \quad x, y > 0. \tag{17.78}$$

Result 17.42. (Baker [91]). *The functions $f,\ g,\ p$, and q from $\mathbb{R}_+^* \to \mathbb{R}^*$ satisfy* (17.78) *if and only if there exist real constants $a,\ b,\ c$, and d and functions $A : \mathbb{R} \to \mathbb{R}$ and $L_1, L_2 : \mathbb{R}_+^* \to \mathbb{R}$ such that $ab = cd \neq 0$, A is additive, L_1, and L_2 are logarithmic, and*

$$f(x) = ae^{A(x)+L_1(x)}, \qquad g(x) = be^{A(x)+L_2(x)},$$
$$p(x) = ce^{A(x)+L_1(x)+L_2(x)}, \quad \text{and} \quad q(x) = de^{L_1\left(\frac{x}{1+x}\right)+L_2\left(\frac{1}{1+x}\right)},$$

for all $x > 0$.

The continuous solutions are given by

$$f(x) = ax^\alpha e^{\gamma x}, \qquad g(x) = bx^\beta e^{\gamma x},$$
$$p(x) = cx^{\alpha+\beta} e^{\gamma x}, \quad \text{and} \quad q(x) = dx^\alpha (1+x)^{-\alpha-\beta}.$$

17.12 Information Theory

17.12.1 Bose-Einstein Entropy

Introduction

Boltzmann's "Vorlesungen über Gastheorie", published in 1896, laid the foundation of thermodynamical entropy. However, this notion of entropy was only limited to physics prior to 1948. Shannon's "A mathematical theory of communication" (Shannon [732]), published in 1948, is the Magna Carta of the information age. Shannon's discovery of the fundamental laws of data compression and transmission marks the birth of information theory. A key feature of Shannon's information theory is the discovery that the colloquial term information can be given a mathematical meaning as a numerically measurable quantity based on probability theory. Shannon defined this measure of information based on a discrete probability distribution $(p_1, p_2, \ldots, p_n)$ as (SE) (Shannon entropy) (see Chapter 10).

Over the last fifty years, several other information measures have been discovered. One such measure of information, known as Bose-Einstein entropy (Kannappan and Sahoo [515]), is given by

$$\hat{H}_n(p_1, p_2, \ldots, p_n) = -\sum_{k=1}^{n} p_k \log p_k + \sum_{k=1}^{n}(1 + p_k)\log(1 + p_k). \tag{17.79}$$

In Kapur [524], this entropy is used for deriving the well-known Bose-Einstein distribution in statistical mechanics.

Today, information theory (see Cover and Thomas [177]) interacts through the notion of information measure (or entropy) with many scientific disciplines such as probability theory (e.g., Central Limit Theorem, large deviations, random processes and divergence, queueing theory), statistical inference (e.g., minimum description length, hypothesis testing, parameter estimation, density estimation, spectral estimation, Bayesian statistics, the inverse problem, pattern recognition, neural networks, and speech recognition), computer science (e.g., algorithmic complexity, data structures, hashing, cryptology, computational complexity, and quantum computing), mathematics (e.g., ergodic theory and dynamical systems, combinatorics, graph theory, inequality, harmonic analysis, differential geometry, number theory, and control theory), physics (e.g., thermodynamics, statistical mechanics, quantum information theory, and chaos), economics (e.g., portfolio theory and econometrics), and biology (e.g., molecular biology, DNA sequencing, and sensory processing). These interactions suggest that the notion of information measure is a fundamental concept. Many researchers have studied this notion extensively in the last fifty years. Today there is a mathematical theory concerning various information measures (see [43] and [245]).

Measures of information are one of the sources that provide a large class of functional equations for investigation (see Chapter 10). While characterizing Shannon entropy by the properties of symmetry and recursivity, one encounters the *fundamental equation of information* (FEI) *theory*. The Shannon function $s(x)$, given by (SF), is a solution of this fundamental equation.

The Bose-Einstein entropy $\hat{H}_n$ admits a sum form; that is,

$$\hat{H}_n(p_1, p_2, \ldots, p_n) = \sum_{k=1}^{n} F(p_k),$$

where the generating function $F : I \to \mathbb{R}$ is given by

$$F(x) = -x \log x + (1+x)\log(1+x). \tag{17.80}$$

We refer to the function F as the *Bose-Einstein* function. Like the Shannon function, $s(x)$, the Bose-Einstein function, F, satisfies the functional equation

$$f(x) + (1+x) f\left(\frac{y}{1+x}\right) = f(y) + (1+y) f\left(\frac{x}{1+y}\right) \tag{17.81}$$

for all $x, y \in I$. A generalization of (17.81) is the functional equation

$$f(x) + (1+x) g\left(\frac{y}{1+x}\right) = h(y) + (1+y) k\left(\frac{x}{1+y}\right) \tag{17.81a}$$

for all $x, y \in I$. Another generalization of (17.81) is

$$\begin{aligned} &G(x,y) + (1+x) G\left(\frac{u}{1+x}, \frac{v}{1+y}\right) \\ &\quad = G(u,v) + (1+u) G\left(\frac{x}{1+u}, \frac{y}{1+v}\right) \end{aligned} \tag{17.81b}$$

for all $x, y, u, v \in I$.

Now we determine the *locally integrable* solutions of these functional equations.

17.12.1.1 Solution of Equations (17.81) and (17.81a)

In this section, the letter F will be used exclusively to denote the Bose-Einstein function. The following theorem gives the locally integrable solution of (17.81).

Theorem 17.43. [515]. *Let $f : I \to \mathbb{R}$ satisfy* (17.81) *and be locally integrable. Then f is given by*

$$f(x) = -cF(x) + ax, \quad \text{for } x \in I, \tag{17.82}$$

where a and c are constants.

Proof. First we prove that local integrability implies the differentiability of the function f. Integrating (17.81) with respect to y from c to d ($(c,d) \subseteq [0,1]$), we have

$$\begin{aligned} (d-c) f(x) &= \int_c^d f(y)dy + \int_c^d (1+y) f\left(\frac{x}{1+y}\right) dy - (1+x) \int_c^d f\left(\frac{y}{1+x}\right) dy \\ &= \int_c^d f(y)dy - \frac{1}{x^2} \int_{\frac{x}{1+c}}^{\frac{x}{1+d}} s^2 f(s)ds - (1+x)^2 \int_{\frac{c}{1+x}}^{\frac{d}{1+x}} f(t)dt. \end{aligned}$$

Since f is integrable, the right side is a continuous function of x, and so, from the left side, f is continuous. Now using the continuity of f, the right side and thus the left side imply the differentiability of f. Then, again, f is twice differentiable.

Differentiating both sides of (17.81) first with respect to x and then with respect to y, we obtain

$$\frac{y}{(1+x)^2} f''\left(\frac{y}{1+x}\right) = \frac{x}{(1+y)^2} f''\left(\frac{x}{1+y}\right).$$

Setting $u = \frac{y}{1+x}$, $v = \frac{x}{1+y}$, and using $\frac{1+u}{1+v} = \frac{1+y}{1+x}$, we get

$$\frac{u}{1+x} f''(u) = \frac{v}{1+y} f''(v);$$

that is,

$$u(1+u) f''(u) = v(1+v) f''(v) = c,$$

where c is a constant. From the last equation, we have the second-order differential equation

$$f''(u) = \frac{c}{u(1+u)},$$

so

$$f'(u) = c[\log u - \log(1+u)] + a.$$

Solving the first-order differential equation above, we obtain

$$f(u) = c[u \log u - (1+u)\log(1+u)] + au + b.$$

Setting $y = 0$ in (17.81), we see that $f(0) = 0$, so $b = 0$. This proves the theorem. □

Now we give the solution of the functional equation (17.81a), which will be used to solve (17.81b).

Theorem 17.44. [515]. *Suppose* $f, g, h, k : I \to \mathbb{R}$ *satisfy* (17.81a) *and one of them is locally integrable. Then*

$$\text{(17.82a)} \qquad \begin{cases} f(x) = -cF(x) + (a_2 - b_1)x + b_3, \\ g(y) = -cF(y) + a_1 y + b_1, \\ h(u) = -cF(u) + (a_1 - b_2)u + b_4, \\ k(v) = -cF(v) + a_2 v + b_2, \end{cases}$$

with $b_1 + b_3 = b_2 + b_4$, *where* a_1 *and* a_2 *are arbitrary constants.*

Proof. It is easy to see that if any one of the functions f, g, h, k is locally integrable, so are the others. We also have, by letting $y = 0$ and $x = 0$ respectively in (17.81a),

$$\text{(17.83)} \qquad f(x) + (1+x)g(0) = h(0) + k(x),$$

$$\text{(17.83a)} \qquad f(0) + g(y) = h(y) + (1+y)k(0).$$

As in Theorem 17.43, it can be shown that f and hence g, h, and k are twice differentiable. Differentiating first with respect to x and then by y, we get

$$\frac{y}{(1+x)^2}g''\left(\frac{y}{1+x}\right) = \frac{x}{(1+y)^2}k''\left(\frac{x}{1+y}\right).$$

As before, taking $u = \frac{y}{1+x}$, $v = \frac{x}{1+y}$, we have

$$u(1+u)g''(u) = v(1+v)k''(v) = c,$$

where c is a constant. Thus

$$g(u) = -cF(u) + a_1u + b_1, \quad k(v) = -cF(v) + a_2v + b_2.$$

Now we use (17.83) and (17.83a) to get the asserted solution. This completes the proof. □

17.12.1.2 Solution of Equation (17.81b)

In this section, we prove the following theorem.

Theorem 17.45. [515]. *Let $G : I \times I \to \mathbb{R}$ satisfy* (17.81b) *and G be locally integrable in its first variable. Then G has the form*

$$G(x,y) = -cF(x) + xL(y) - (1+x)L(1+y) + ax, \tag{17.82b}$$

where a and c are arbitrary constants and L is logarithmic (L) *with* $0\log 0 = 0$.

Proof. If we temporarily fixed y, v in (17.81b), then (17.81b) becomes (17.81a) with $f(x) = G(x,y)$, $g(x) = G\left(x, \frac{v}{1+y}\right)$, etc. Since G is locally integrable in its first variable, applying Theorem 17.44 we get

$$G(x,y) = -c(y,v)F(x) + a(y,v)x + b(y,v),$$

where a, b, and c are functions of y and v. Since the left side is a function of y only, a, b, and c are functions of y only, say $a(y)$, $b(y)$, and $c(y)$. Since c occurs in g, h, and k, c is a constant. Thus

$$G(x,y) = -cF(x) + a(y)x + b(y). \tag{17.84}$$

To determine $a(y)$ and $b(y)$, we substitute the form of G in (17.84) into (17.81b) and obtain

$$\begin{aligned} &a(y)x + b(y) + ua\left(\frac{v}{1+y}\right) + (1+x)b\left(\frac{v}{1+y}\right) \\ &\quad = a(v)u + b(v) + xa\left(\frac{y}{1+v}\right) + (1+u)b\left(\frac{y}{1+v}\right). \end{aligned}$$

Equating the coefficients of x and the constant terms, we have

$$a(y) + b\left(\frac{v}{1+y}\right) = a\left(\frac{y}{1+v}\right), \tag{17.85}$$

$$b(y) + b\left(\frac{v}{1+y}\right) = b(v) + b\left(\frac{y}{1+v}\right). \tag{17.85a}$$

To find $a(y)$ and $b(y)$ (cf. [450]), we subtract (17.85a) from (17.85) to get

$$\alpha(y) = \alpha\left(\frac{y}{1+v}\right) - b(v), \tag{17.85b}$$

where $\alpha(y) = a(y) - b(y)$. Letting $y = 1 + v$, we obtain $\alpha(1 + v) = \alpha(1) - b(v)$. Next, setting $\frac{y}{1+v} = t$ in (17.85b), we get

$$\alpha(t(1 + v)) = \alpha(t) + \alpha(1 + v) - \alpha(1);$$

that is,

$$L(xy) = L(x) + L(y),$$

where

$$L(x) = \alpha(x) - \alpha(1) = \alpha(x) + c \quad \text{(say)}. \tag{17.85c}$$

From (17.85b) and (17.85c), we see that

$$L(y) - c_1 = L\left(\frac{y}{1+v}\right) - c_1 - b(v)$$

or

$$b(v) = -L(1 + v). \tag{17.86}$$

Using this and (17.85), we have

$$a(y) - L\left(\frac{1+y+v}{1+y}\right) = a\left(\frac{y}{1+v}\right).$$

Next, we set $\frac{y}{1+v} = t$. Then

$$a(y) - L\left(\frac{y}{1+y} + \frac{y}{t(1+y)}\right) = a(y) - L\left(\frac{y}{1+y}\left(1 + \frac{1}{t}\right)\right) = a(t)$$

or

$$a(y) - L\left(\frac{y}{1+y}\right) = a(t) - L\left(\frac{t}{1+t}\right) = a_1, \tag{17.86a}$$

where a_1 is a constant. Now (17.84), (17.86), and (17.86a) yield the asserted solution (17.82b), and the proof of the theorem is complete. □

The general solutions of these equations without any regularity assumptions will appear elsewhere.

17.12.2 Sums of Powers

A method is introduced to find the sum of powers on arithmetic progressions by using Cauchy's equation and obtain a general formula. Then we apply these results to show how to determine some other sums of powers and sums of products. These results are more general. Finally, we discuss the sum of powers on arithmetic progressions in commutative rings with characteristic 2 and find "full polynomials".

Since the time of James Bernoulli (1655–1705) [122], finding the sum of powers of integers

$$S_k(n) = 1^k + 2^k + \cdots + n^k, \quad k \in \mathbb{Z}_+,\ n \in \mathbb{Z}_+^*, \tag{17.87}$$

has been fascinating mathematicians (see Chapter 1, Section 1.8 application), and several methods have been developed to find the sum. One of the methods to find the sum in (17.87) uses the theory of functional equations. In particular, in Kannappan and Zhang [523], the sum $S_k(n)$ can be determined by applying the known recurrent formula (Snow [758], Au-Yeung [836]), grouping terms of the same power and reducing them to Cauchy's additive equation. This method is also applicable in combinatorics and genetics [523]. Inspired by [55], we introduce this method to find the sum of powers on arithmetic progressions

$$S_k(n; a, h) = a^k + (a+h)^k + \cdots + [a+(n-1)h]^k, \quad k \in \mathbb{Z}_+,\ n \in \mathbb{Z}_+^*, \tag{17.88}$$

where a and h are given real constants, and obtain a general formula for $S_k(n; a, h)$. Applying our results, we determine some other sums of powers and sums of products such as

$$P_k(n; a, h) = \prod_{j=0}^{k-1}(a+j) + \prod_{j=0}^{k-1}(a+j+h) + \cdots + \prod_{j=0}^{k-1}(a+j+(n-1)h). \tag{17.89}$$

Results in [523] will be generalized here. Finally we discuss the sum of powers on arithmetic progressions in commutative rings, especially in commutative rings with characteristic 2, and find full polynomials.

17.12.2.1 A General Result

Theorem 17.46. [523]. *For each integer $k > 0$,*

$$S_k(n; a, h) = n(\mu_{k,k}n^k + \mu_{k,k-1}n^{k-1} + \mu_{k,k-2}n^{k-2} + \cdots + \mu_{k,1}n + \mu_{k,0}), \tag{17.88a}$$

where $\mu_{k,j} = \mu_{k,j}(a, h)$, $j = 0, 1, 2, \ldots, k$, are (homogeneous) polynomials of degree k in a and h and are given by

$$\begin{cases} \mu_{k,k}(a,h) = \dfrac{1}{k+1}h^k, \\ \mu_{k,k-i}(a,h) = \dfrac{1}{k+1-i}\binom{k}{i}h^{k-i}\left(a^i - \displaystyle\sum_{j=1}^{i}\mu_{i,j}\right), \end{cases} \tag{17.88b}$$

for $i = 1, 2, \ldots, k$.

Proof. The proof of this theorem is based on induction. First, for $k = 1$, we easily see that (17.88a) holds for all natural numbers n (see [836]). Assume that, for all natural numbers $k \le t$, the formula (17.88a) holds for all natural numbers n. We prove that (17.88a) holds for $k = t + 1$ and all natural numbers n. For $n = 1$, and $t + 1$ for k in the right-hand side of (17.88a),

$$\mu_{t+1,t+1}+\mu_{t+1,t}+\mu_{t+1,t-1}+\cdots+\mu_{t+1,1}+\mu_{t+1,0} = a^{t+1} = S_{t+1}(1; a, h) \tag{17.90}$$

because

$$\mu_{t+1,0} = a^{t+1} - \mu_{t+1,1} - \cdots - \mu_{t+1,t} - \mu_{t+1,t+1}$$

by (17.88b). Furthermore, suppose that (17.88a) holds for $k = t + 1$ and n; that is, $S_{t+1}(n; a, h)$ holds for all natural numbers $n \le m$, and then consider the next case $m + 1$. By $(**)$ in Chapter 1, we have

$$S_{t+1}(m + 1; a, h) - S_{t+1}(m; a, h) = a^{t+1} + \sum_{i=1}^{t+1} \binom{t+1}{i} S_{t+1-i}(m; a, h)h^i.$$

By induction hypotheses for k and for n,

$$\begin{aligned} S_{t+1}(m + 1; a, h) &= S_{t+1}(m; a, h) + \sum_{i=1}^{t+1} \binom{t+1}{i} S_{t+1-i}(m; a, h)h^i + a^{t+1} \\ &= \sum_{j=0}^{t+1} \mu_{t+1,t+1-j} m^{t+2-j} \\ &\quad + \sum_{i=1}^{t+1} \binom{t+1}{i} \sum_{j=0}^{t+1-i} \mu_{t+1-i,t+1-i-j} m^{t+2-i-j} h^i + a^{t+1} \\ &= \sum_{i=0}^{t+1} \left(\sum_{j=0}^{i} \binom{t+1}{j} \mu_{t+1-j,t+1-i} h^j \right) m^{t+2-i} + a^{t+1}, \end{aligned} \tag{17.91}$$

written as a polynomial in m. Observe that

$$\begin{aligned} &\sum_{j=0}^{t+1} \mu_{t+1,t+1-j} (m + 1)^{t+2-j} \\ &\quad = \sum_{j=0}^{t+1} \mu_{t+1,t+1-j} \sum_{i=0}^{t+2-j} \binom{t+2-j}{i} m^{t+2-j-i} \\ &\quad = \sum_{i=0}^{t+1} \left(\sum_{j=0}^{i} \binom{t+2-i+j}{j} \mu_{t+1,t+1-i+j} \right) m^{t+2-i} + \sum_{j=0}^{t+1} \mu_{t+1,t+1-j}. \end{aligned} \tag{17.91a}$$

By comparing coefficients of powers of m between (17.91) and (17.91a), it suffices to prove that, for $i = 0, 1, \ldots, t + 1$,

$$\sum_{j=0}^{i}\binom{t+2-i+j}{j}\mu_{t+1,t+1-i+j}=\sum_{j=0}^{i}\binom{t+1}{j}\mu_{t+1-j,t+1-i}h^j, \tag{17.92}$$

$$\sum_{j=0}^{t+1}\mu_{t+1,t+1-j}=a^{t+1}. \tag{17.92a}$$

Actually, (17.92a) is (17.88b). Equality (17.92) also holds because, for each $j = 0, 1, \ldots, i$,

$$\begin{aligned}
&\binom{t+2-i+j}{j}\mu_{t+1,t+1-i+j}\\
&\quad=\binom{t+2-i+j}{j}\frac{1}{t+2-i+j}\binom{t+1}{i-j}h^{t+1-i+j}\\
&\qquad\left(a^{i-j}-\sum_{\ell=1}^{i-j}\mu_{i-j,\ell}\right)\\
&\quad=\frac{(t+1)!}{j!(i-j)!(t+2-i)!}\left(a^{i-j}-\sum_{\ell=1}^{i-j}\mu_{i-j,\ell}\right)h^{t+1-i+j}\\
&\quad=\binom{t+1}{j}\frac{1}{t+2-i}\binom{t+1-j}{i-j}h^{t+1-i}\left(a^{i-j}-\sum_{\ell=1}^{i-j}\mu_{i-j,\ell}\right)h^j\\
&\quad=\binom{t+1}{j}\mu_{t+1-j,t+1-i}h^j
\end{aligned}$$

by (17.88a). This proves that (17.88a) holds for $k = t + 1$ and all n, and this completes the proof. □

Aczél [21] obtained the general sum as

$$S_k(n;a,h)=\frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j+1}n^{j+1}c_{k-j}h^j,\quad n\in\mathbb{Z}_+^*, \tag{17.88c}$$

where constants c_ℓ safisfy

$$\sum_{j=1}^{\ell}\binom{\ell}{j}c_{\ell-j}h^{j-1}=\ell a^{\ell-1},\quad \ell=1,2,\ldots.$$

These formulas are the same as (17.88a) with (17.88b). For example, we identify the cases for $k = 1, 2$. For $k = 1$, from (17.88c) we get

$$c_0=1,\quad c_1=a-h/2,\quad \text{and}\quad S_1(n;a,h)=\frac{1}{2}[n(2a-h)+n^2h],$$

while from (17.88a) and (17.88b) we have

$$\mu_{1,1} = h/2, \quad \mu_{1,0} = a - \frac{h}{2}, \quad \text{and} \quad S_1(n; a, h) = n\left[\left(a - \frac{h}{2}\right) + \frac{h}{2}n\right].$$

Both results are the same. For $k = 2$, from (17.88c) we get

$$c_0 = 1, \quad c_1 = a - \frac{h}{2}, \quad c_3 = a^2 - \left(a - \frac{h}{2}\right)h - \frac{h^2}{3},$$

and

$$S_2(n; a, h) = \frac{1}{3}\left\{3n\left(a^2 - \left(a - \frac{h}{2}\right)h - \frac{h^2}{3}\right) + 3n^2\left(a - \frac{h}{2}\right) + n^3h^2\right\}.$$

On the other hand, from (17.88a) and (17.88b), we have

$$\mu_{2,2} = \frac{h^2}{3}, \quad \mu_{2,1} = h\left(a - \frac{h}{2}\right), \quad \mu_{2,0} = a^2 - h\left(a - \frac{h}{2}\right) - \frac{h^2}{3},$$

and

$$S_2(n; a, h) = n\left[\frac{h^2}{3}n^2 + h\left(a - \frac{h}{2}\right)n + \left(a^2 - h\left(a - \frac{h}{2}\right) - \frac{h^2}{3}\right)\right].$$

We also get the same results.

Now we find the sum

$$t_k(n) = 1^k + 5^k + \cdots + (4n-3)^k, \quad k = 1, 2, 3, \ n \in \mathbb{Z}_+^*,$$

given in [472] by our formulas.

We can consider sums of powers of integers with positive and negative terms, too. For example, consider the sum

$$s_k(n, -1) = -1^k + 2^k + \cdots + (-1)^n n^k, \quad k \in \mathbb{Z}_+, \ n \in \mathbb{Z}_+^*.$$

It is easy to see that

$$s_k(n, -1) = S_k\left(\frac{n}{2}; 2, 2\right) - S_k\left(\frac{n}{2}; 1, 2\right)$$

for n even and that

$$s_k(n, -1) = S_k\left(\frac{n-1}{2}; 0, 2\right) - S_k\left(\frac{n-1}{2}; 1, 2\right)$$

for n odd. In particular, $s_2(2n, -1) = S_2(n; 2, 2) - S_2(n; 1, 2) = 2n^2 + n$ and $s_2(2n+1, -1) = S_2(n; 0, 2) - S_2(n; 1, 2) = -2n^2 + n$. As a second example, consider

$$
\begin{aligned}
&-1^2-2^2+3^2+\cdots-(3n-2)^2-(3n-1)^2+(3n^2)\\
&\quad=-S_n(n;1,3)-S_2(n;2,3)+S_2(n;3,3)\\
&\quad=-3n^3+\frac{9}{2}n^2+\frac{5}{2}n \quad \text{for } n\in\mathbb{Z}_+^*.
\end{aligned}
$$

Some Generalizations

Now we consider the sum $S_k(n;a,h)$ on a divisible commutative ring $(\Gamma,+,\cdot)$ (that is, with $a,h\in\Gamma$ and $k\in\mathbb{Z}_+$, $n\in\mathbb{Z}_+^*$) and generalize our results. Here "divisible" is meant in its traditional sense that for all $n\in\mathbb{Z}_+^*$ and $b\in\Gamma$ there exists a $c\in\Gamma$ such that $b=nc$ (no uniqueness required). Obviously, all basic results discussed above are valid on such a ring.

Furthermore, if we consider the sum $S_k(n;a,h)$ on a divisible commutative ring $(\Gamma,+,\cdot)$ of characteristic $m\in\mathbb{Z}_+^*$ (that is, $mb=0$ for all $b\in\Gamma$,) then we observe some interesting phenomena.

Theorem 17.47. [523]. *For a, h in a divisible commutative ring $(\Gamma,+,\cdot)$ of characteristic $m\in\mathbb{Z}_+^*$,*

$$
S_k(n;a,h)=\begin{cases}
\beta\displaystyle\sum_{j=0}^{m-1}\sum_{i=1}^{k}\xi_i(j)a^{k-i}h^i & \text{if } n=\beta m,\ \beta\in\mathbb{Z}_+,\\
\gamma\,a^k+(\beta+1) & \text{if } n=\beta m+\gamma,\\
\displaystyle\sum_{j=0}^{\gamma-1}\sum_{i=1}^{k}\xi_i(j)a^{k-i}h^i & \beta\in\mathbb{Z}_+,\\
\quad+\beta\displaystyle\sum_{j=\gamma}^{m-1}\sum_{i=1}^{k}\xi_k(j)a^{k-i}h^i, & \gamma=1,\dots,m-1,
\end{cases}
\tag{17.93}
$$

where $\xi_i(j)=\binom{k}{i}j^i \pmod m\in\{0,1,\dots,m-1\}$.

Proof. Let $n=\beta m+\gamma$, where $\beta\in\mathbb{Z}_+$ and $\gamma=0,1,\dots,m-1$. Since the ring $(\Gamma,+,\cdot)$ is of characteristic m, we see $mh=0$. Notice that

$$
(a+jh)^k=a^k+\sum_{i=1}^{k}\xi(j)a^{k-i}h^i,\quad 0\le j\le m-1,
$$

where $\xi_k(j)$ is defined in the theorem. It follows from (17.88) that, when $\gamma=0$,

$$
\begin{aligned}
S_k(n;a,h)&=\sum_{j=0}^{n-1}(a+jh)^k=\beta\sum_{j=0}^{m-1}(a+jh)^k\\
&=na^k+\beta\sum_{j=0}^{m-1}\sum_{i=1}^{k}\xi_i(j)a^{k-i}h^i,
\end{aligned}
$$

and when $\gamma \neq 0$,

$$S_k(n;a,h) = \sum_{j=0}^{n-1}(a+jh)^k = (\beta+1)\sum_{j=0}^{\gamma-1}(a+jh)^k + \beta\sum_{j=\gamma}^{m-1}(a+jh)^k$$
$$= na^k + (\beta+1)\sum_{j=0}^{\gamma-1}\sum_{i=1}^{k}\xi_i(j)a^{k-i}h^i + \beta\sum_{j=\gamma}^{m-1}\sum_{i=1}^{k}\xi_i(j)a^{k-i}h^i.$$

This proves the theorem.

More concretely, with the characteristic $m = 2$, by Theorem 17.47 we see

$$S_2(n;a,h) = \begin{cases} \frac{n}{2}h^2 & (\text{even } n), \\ a^2 + \frac{n-1}{2}h^2 & (\text{odd } n), \end{cases}$$
$$S_3(n;a,h) = \begin{cases} \frac{n}{2}[a^2h + ah^2 + h^3] & (\text{even } n), \\ a^3 + \frac{n-1}{2}[a^2h + ah^2 + h^3] & (\text{odd } n), \end{cases}$$

and so on. As is known from the proof of Theorem 17.47, the scale of representations (17.93) is clearly decided by the expansion of $(a+jh)^k$. The expression (17.93) is said to be *full* if all coefficients of powers of a and h are nonzero. For $m = 2$, expanding $(a+jh)^k$ patiently up to $k = 15$, we see that the full expression occurs for $k = 3, 7, 15$; i.e.,

$$(a+h)^3 = a^3 + a^2h + ah^2 + h^3,$$
$$(a+h)^7 = a^7 + a^6h + a^5h^2 + a^4h^3 + a^3h^4 + a^2h^5 + ah^6 + h^7,$$
$$(a+h)^{15} = a^{15} + a^{14}h + a^{13}h^2 + a^{12}h^3 + a^{11}h^4 + a^{10}h^5 + a^9h^6 + a^8h^7$$
$$+ a^7h^8 + a^6h^9 + a^5h^{10} + a^4h^{11} + a^3h^{12} + a^2h^{13} + ah^{14} + h^{15}.$$

We refer to such polynomials as *full polynomials.* It will be interesting to find k's for which we have a full polynomial of degree k. Trying it with *Maple V5.1* software, we found that the fourth time for all terms to occur or for full polynomials to occur is when $k = 31$, and the fifth time is when $k = 63$, where the lists of integer coefficients are given respectively in the following:

```
[31, 465, 4495, 31465, 169911, 736281, 2629575,
7888725, 20160075, 44352165, 84672315, 141120525,
206253075, 265182525, 300540195, 300540195,
265182525, 206253075, 141120525, 84672315, 44352165,
20160075, 7888725, 2629575, 736281, 169911,
31465, 4495, 465, 31]
```

and

```
[63, 1953, 39711, 595665, 7028847, 67945521,
553270671, 3872894697, 23667689815, 127805525001,
61590256823, 2668424446233, 10468434365991,
37387265592825, 122131734269895, 366395202809685,
1012974972473835, 2588713818544245, 6131164307078475,
13488561475572645, 27619435402363035,
52728013040874885, 93993414551124795,
156655690918541325, 244382877832924467,
357174975294274221, 489462003181042451,
629308289804197437, 759510004936100355,
860778005594247069, 916312070471295267,
916312070471295267, 860778005594247069,
759510004936100355, 629308289804197437,
489462003181042451, 357174975294274221,
244382877832924467, 156655690918541325,
93993414551124795, 52728013040874885,
27619435402363035, 13488561475572645,
6131164307078475, 2588713818544245,
1012974972473835, 366395202809685,
122131734269895, 37387265592825,
10468434365991, 2668424446233, 615790256823,
127805525001, 23667689815, 3872894697,
553270671, 67945521, 7028847, 595665,
39711, 1953, 63].
```

Trying some more for $k = 127, 255, \ldots$, we observe that full polynomials appear regularly for $k = k_i,\ i = 1, 2, \ldots$, where $k_1 = 3$ and $k_{i+1} = 2k_i + 1$. For convenience, we say that those integers are *totally odd*. Obviously, for $k = k_0 := 1$, the corresponding polynomial is also full.

Result 17.48. [523]. *For $a,\ h$ in a divisible commutative ring $(\Gamma, +, \cdot)$ of characteristic 2, the sum $S_k(n; a, h)$ is a full polynomial in a and h if, and only if, k is totally odd.*

Remark. It will be interesting to see how the characteristic m affects the sum $S_k(n; a, h)$ when $m \neq 2$.

17.13 Behavioural Sciences

Functional equations are useful in the experimental sciences because they offer a tool for narrowing the possible models for a phenomenon. A model can be formulated by one or more not very restrictive equations, which when paired with an empirical or logical constraint of a general character lead—via functional equation techniques—to precise quantitative relationships. The article by Aczél, Falmagne,

and Luce [47] reviews various applications of functional equations in some areas of the behavioural sciences such as sensory psychology (psychophysics), utility theory under uncertainty, and aggregation of inputs and outputs in an economic or social context.

The purpose of this section is to describe some of the uses of functional equations in the behavioural and social sciences. Applications of functional equations in three empirical fields are discussed: examples in sensory psychology (in particular, psychophysics) resembling the Plateau example but more complex and covering different situations; individual decision making under uncertainty (utility theory); and aggregation of inputs and outputs in a social or economic context. We start with a summary of some classic types of functional equations, solved long ago (for details, see [32]), that have proved useful in the analysis of behavioural models from a functional equation viewpoint. In all cases that we consider, the functional equations are numerical ones, generated by a mathematical model or a numerical representation of the phenomenon under consideration, in which the modelling is incomplete and at least some of the functions are unspecified. A substantial body of literature on representational measurement theory describes classes of qualitative systems that give rise to numerical representations.

17.13.1 A Behavioural Example

Functional equations arise in the sciences because they allow researchers to formulate mathematical models in general terms, via some not very restrictive equations that only stipulate basic properties of functions appearing in these equations, without postulating the exact forms of such functions. However, the data or the experimental situation itself may exhibit regularities or symmetries that can be captured by some other equation(s) involving the same functions, thereby constraining their forms and specifying the model.

Functional equations enter the picture in at least three distinct ways. One occurs when some invariance property holds. An example of such an invariance is some type of homogeneity of one of the functions. Other examples of invariance involve utility and weighting functions from utility theory.

The second example arises when two independent measurement schemes give rise to distinct measures of the same underlying attribute. An example in physics is mass. It can be measured by concatenating objects on the pans of a pan balance (in a vacuum) that is used to determine the order of greater mass. The resulting mass measure is additive over concatenation. It can also be measured fundamentally by varying the container size and the choice of homogeneous substances within them on a pan balance. This measure of mass exhibits a product structure with corresponding measures of volume and density. Clearly, because both mass measures represent the same ordering, they are related by a strictly increasing function. Any additional empirical law linking the two measurement structures imposes a constraint on that function in the sense that it must satisfy a functional equation. In the physical case, the link is a qualitative distribution law.

The third example concerns what may be thought of as consistency principles. One such principle is the commutative property used in modelling aggregation. Suppose we have variables x_{ij} that can be aggregated over either i or j and, once done, over the remaining subscript. An economic example concerns several types of inputs, such as raw materials, energy, and labour, to some production. Often one wishes to speak of an aggregated result over each type of input and over the outputs. The question is under what circumstances the order of aggregation does not matter.

Let $m(x, y)$ be the luminance of a disk appearing midway between the disks x and y. Plateau's data can then be formalized by the homogeneity equation

$$m(\lambda x, \lambda y) = \lambda m(x, y), \quad \lambda, x, y > 0,$$

where the value of λ reflects the differing conditions of illumination. (Realistically, the domain of m should be restricted to a suitable positive region near the origin). There is some sensory scale u mapping the lux scale into the reals, such that

$$u[m(x, y)] = \frac{u(x) + u(y)}{2}.$$

A function m satisfying this equation for some continuous strictly monotonic function u is called a *quasiarithmetic mean* (see Chapters 15 and 17). It is assumed that the scale u is defined on $\mathbb{R}_+^*$. The equation above leads to the functional equation

$$u^{-1}\left[\frac{u(\lambda x) + u(\lambda y)}{2}\right] = \lambda u^{-1}\left[\frac{u(x) + u(y)}{2}\right], \quad \lambda, x, y > 0, \tag{17.94}$$

which has only two families of solutions for the function u. In addition to Cauchy, Pexider, and Jensen equations, u satisfies their generalizations, such as

$$u(\lambda x) = m(\lambda)u(x) + n(\lambda), \quad \lambda, x \in \mathbb{R}_+^*,$$

which is a special case of (1.94b) and (1.94a). Here we need only $u(x)$. From Result 1.61, we have either

$$u(x) = aL(x) + b \quad \text{or} \quad u(m(x, y)) = \frac{aL(xy) + 2b}{2},$$

where L satisfies (L). Since u is continuous, we have

$$u(x) = a \log x + b, \quad m(x, y) = (xy)^{\frac{1}{2}},$$

or

$$u(x) = c(M(x) - 1) + b, \quad u(m(x, y)) = \frac{c(M(x) + M(y) - 2) + 2b}{2},$$

where M is multiplicative (M); that is, since u is continuous,

$$u(x) = cx^{\beta} + d, \quad m(x, y) = \left(\frac{x^{\beta} + y^{\beta}}{2}\right)^{\beta},$$

where $a, c, \beta \neq 0$ and b and c are constants. For strictly increasing solutions, $a > 0$, $c, \beta > 0$.

17.13.2 Psychophysics

Many applications of functional equations in psychophysics are in the style illustrated by Plateau's situation discussed earlier. On the experimental side, one starts with some homogeneity equation that seems to capture an important regularity in the data. In the case of Plateau, the function m is homogeneous of degree 1. Next, one introduces some reasonable model, involving one or more unknown functions, formalizing a possible mechanism for explaining the behaviour of the individuals in the study.

The cases reviewed here all involve a binary discrimination situation in which an individual must decide which of the two stimuli presented has more of some sensory attribute. For instance, the individual compares the loudness of pure tones differing only in their intensities, which is measured in standard ratio scale units (such as sound pressure level).

In a classic nineteenth century case involving E.E. Weber and G.T. Fechner, the individual is presented with a pair of stimuli (x, y) and the researcher's task is to estimate experimentally the probability $P(x, y)$ that a stimulus of intensity x is judged as louder (or brighter, longer, etc.) than a stimulus of intensity y. In mathematical terminology, the so-called *Weber Law* states that the function $P : \mathbb{R}_+^2 \to]0, 1[$ is homogeneous of degree 0:

$$P(\lambda x, \lambda y) = P(x, y), \quad x, y, \lambda > 0.$$

Fechner's contribution consisted of proposing that the individual's judgment resulted from a computation of the differences between the two sensations produced by x and y, this difference being evaluated in terms of some unknown sensation scale. Formally, Fechner's idea is expressed by the equation

$$P(x, y) = F(u(x) - u(y)),$$

where the functions u and F are real-valued, strictly increasing, and continuous, but otherwise not specified a priori. In the context of Weber's Law, the possible forms of u and F above are severely constrained. With the two equations above, we get the functional equation

$$F(u(\lambda x) - u(\lambda y)) = F(u(x) - u(y)), \quad x > 0,\ y > 0,\ \lambda > 0,$$

which, since F is strictly increasing, is equivalent to

$$u(\lambda x) - u(\lambda y) = u(x) - u(y), \quad x > 0,\ y > 0,\ \lambda > 0;$$

that is,

$$u(\lambda x) - u(x) = u(\lambda y) - u(y) = b(\lambda), \quad \text{say},$$

or

$$u(\lambda x) = u(x) + b(\lambda),$$

which is logarithmic Pexider. So

$$u(x) = aL(x) + b \quad \text{and} \quad P(x, y) + F\left(aL\left(\frac{x}{y}\right)\right), \quad x, y \in \mathbb{R}_+^*,$$

where L satisfies (L).

Since u is continuous,

$$u(x) = a \log x + b \quad \text{and} \quad F(x) = P\left(e^{\frac{x}{a}}, 1\right).$$

17.13.2.1 The Conjoint Weber Law

Suppose that an individual wearing earphones is presented with a 1000 Hz tone delivered simultaneously, with different intensities, to the two auditory channels. The impression created by such a stimulus is that of a single sensation, the loudness of which depends on the combination of two inputs. (The location of the sensation inside the individual's head also depends on the combination of the two intensities, but this aspect of the phenomenon is not relevant here.) We write (a, x) for such a two-dimensional stimulus, where a and x stand for positive real numbers denoting the sound pressures in the left and right auditory channels, respectively. Extending our earlier notation, we write $P(a, x; b, y)$ for the probability that the individual judges the two-dimensional stimulus (a, x) to be louder than the two-dimensional stimulus (b, y). Various concepts concerning the function P are gathered in the definitions below.

Definitions

We suppose that $P : \mathbb{R}_+^4 \to]0, 1[$ is continuous, with $P(a, x; b, y)$ strictly increasing in a and strictly decreasing in b, and strictly contramonotonic in x and y (that is, $P(a, x; b, y)$ is either strictly increasing in x and strictly decreasing in y or vice versa). We will call such a function a *discrimination probability*. Note that the hypothesis that P is strictly comonotonic in the second and fourth variables is weaker than what is suggested by the binaural loudness summation example but makes sense because the formalism is then also applicable to other empirical cases. For instance, a pair (a, x) could represent a stimulus of intensity a presented over a "noisy" background of intensity x.

A discrimination probability P satisfies the *Conjoint Weber Law* if it is homogeneous of degree zero; that is,

$$P(\lambda a, \lambda x; \lambda b, \lambda y) = P(a, x; b, y), \quad \lambda, a, x, b, y > 0. \tag{17.95}$$

It is easily shown that (17.95) holds if and only if there is a function $T : \mathbb{R}_+^3 \to I_0$ such that

$$P(a, x; b, y) = T(a/x, x/y, b/y), \tag{17.95a}$$

with T strictly increasing in the first variable and strictly decreasing in the third variable.

Result 17.49. [47]. *Suppose that $P : \mathbb{R}_+^4 \to I_0$ is a simply scalable discrimination probability satisfying the Conjoint Weber Law* (17.95)*. Then, one of the two following possibilities holds:*

(a) *There are some real-valued, continuous, strictly monotonic functions $T_1 > 0$ and c satisfying*

$$P(a, x; b, y) = c\left(\frac{T_1(a/x)x}{T_1(b/y)y}\right), \quad a, x, b, y > 0.$$

(b) *There is a continuous function Q_0, strictly increasing in the first argument and strictly decreasing in the second argument, such that*

$$P(a, x; b, y) = Q_0(a/x, b/y).$$

Result 17.50. [47]. *Suppose that $P : \mathbb{R}_+^4 \to I_0$ is a discrimination probability satisfying the Conjoint Weber Law* (17.95)*, together with the component additivity equation*

$$P(a, x; b, y) = H(f(a) + r(x),\ f(b) + r(y)), \quad a, x, b, y > 0,$$

where f, r, and H are continuous functions, with f strictly increasing, r strictly monotonic, and H strictly increasing in the first variable and strictly decreasing in the second variable. Then one of the three equations

$$P(a, x; b, y) = G\left(\frac{a^\beta + \delta x^\beta}{b^\beta + \delta y^\beta}\right),$$
$$P(a, x; b, y) = G\left(\frac{a^\beta x^\gamma}{b^\beta y^\gamma}\right),$$
$$P(a, x; b, y) = Q_0\left(\frac{a}{x}, \frac{b}{y}\right),$$

must hold for $a, x, b, y > 0$, where G is continuous and strictly increasing, and $\beta > 0$, $\delta \neq 0$, and $\gamma \neq 0$ are constants and Q_0 is continuous, strictly increasing in its first argument, and strictly decreasing in its second argument.

Thus the function P is necessarily decreasing in its second argument and increasing in its fourth argument.

For more details on the applications above and utility theory, see [47].

17.13.3 Binocular Vision

A general theory of the relationship of binocular visual space to physical space is formulated within a conjoint measurement framework. Psychophysically motivated invariance relations induce various functional equations. Another class of functional equations arises if we additionally assume the validity of a different psychophysical theory that is based on a formula suggested by A.A. Blank. We interpret most of

these equations. The results obtained lead to a measurement-theoretic foundation of the psychophysical assumptions underlying the Luneburg theory of binocular vision. They also contribute to clarifying the relationship between the general theory presented and Blank's approach.

We perceive that we are located in a unitary and stable perceptual space. The so-called visual space is distinguished from other perceptual spaces by its compelling phenomenological reality and its undoubted spatial character. In a highly sophisticated approach, Luneburg [617] embodied visual space in a geometrically formulated perceptual theory. The Luneburg theory of binocular vision not only provides a geometrical characterization of the internal structure of visual space but also describes its dependence on physical space. It accounts for classical empirical phenomena, such as the Helmholtz frontal horopter (Helmholtz) or the visual alleys (Hillebrand, Blumenfeld) that arise under viewing conditions that isolate the effect of binocularity by excluding monocular cues and experiential factors. These conditions are usually realized by presenting the observer with point-like light sources of low intensity in complete darkness. The position of the head is kept fixed, while the eyes are allowed to move freely. The main achievement of Luneburg was to relate the above-mentioned phenomena to a non-Euclidean geometrical structure of visual space.

17.13.3.1 The Luneburg Theory of Binocular Vision

The following two postulates are at the core of the Luneburg theory of binocular vision:

(L1) Visual space, which may be coordinatized with respect to perceived egocentric distance and perceived direction, is a Riemannian space of constant Gaussian curvature. Its metric determines the perceived interpoint distances, while its geodesics are the perceptually straight lines.
(L2) The points lying on the Vieth-Müller circle are perceived as being equidistant from the observer, and the hyperbolas of Hillebrand are perceived as radial lines of constant direction.

By assumption (L2), the preimages of the visual coordinate lines under the psychophysical function are the Vieth-Müller circles and the hyperbolas of Hillebrand, respectively. This psychophysical relationship is not most elegantly formulated by the Cartesian coordinates of the physical space, where the rotation centres of the eyes are located symmetric to the origin on the y-axis and the stimuli are the points lying in the horizontal positive half-plane $x > 0$. Instead, introduced in [617] were bipolar coordinates $\alpha,\ \beta$, which are angular coordinates describing the monocular directions relative to the rotation centres of the eyes.

The most often used modified bipolar coordinates are then given by the linear combinations

$$\gamma = \alpha - \beta \quad \text{and} \quad \phi = \frac{\alpha + \beta}{2}. \tag{17.96}$$

The bipolar parallax γ is thus the difference of the monocular directions, and the bipolar latitude ϕ is their average.

The reason for referring to the modified bipolar coordinates becomes obvious if we notice that the trajectories γ = constant are the Vieth-Müller circles, while the trajectories ϕ = constant are the hyperbolas of Hillebrand. Luneburg's postulate (L2) is then equivalent to the statement that the perceived egocentric distance only depends on γ and the perceived direction only on ϕ.

There is an impressive body of experimental data that tests the empirical adequacy of the Luneburg theory. The theoretical predictions for individual observers were found to be qualitatively in accord with the data. The exact quantitative predictions, however, are often not within experimental error. In looking for possible reasons that might be responsible for this situation, we want to focus on postulate (L2) without regard to the assumption concerning the geometrical structure of visual space as formulated in (L1).

17.13.3.2 A Conjoint Representation Generalizing the Luneburg Theory

Let $S = \{(\alpha, \beta) \in V^2 \mid \alpha > \beta\}$ be the set of stimuli with $V =]-\pi/2, \pi/2[$, where $\alpha,\ \beta$ are bipolar coordinates.

We start by characterizing the loci of perceived equidistance and constant perceived direction, respectively, as equivalence classes that are induced by ordering relations. Let us introduce binary relations $\preceq_\rho,\ \preceq_\theta$ on S that describe an observer's qualitative order judgments with respect to the perceptions considered. By $(\alpha, \beta) \preceq_\rho (\alpha', \beta')$ we express that the perceived egocentric distance of the stimulus (α, β) is judged not to exceed that of (α', β'). Similarly, $(\alpha, \beta) \preceq_\theta (\alpha', \beta')$ means that (α, β) is not perceived as lying to the right of (α', β'). The relation $\preceq_\rho$ thus orders the stimuli from near to far, and $\preceq_\theta$ from left to right. Let $\sim_\rho,\ \sim_\theta$ and $\prec_\rho,\ \prec_\theta$ denote the symmetric and asymmetric parts of $\preceq_\rho,\ \preceq_\theta$, respectively. Note that these orderings are assumed to remain invariant under fixation changes.

Using equation (17.96), we may now reformulate Luneburg's psychophysical assumptions concerning the relations $\preceq_\rho$ and $\preceq_\theta$ in the format of representational measurement theory. For all $(\alpha, \beta), (\alpha', \beta') \in S$,

$$\begin{aligned} &(\alpha, \beta) \preceq_\rho (\alpha', \beta') \quad \text{iff } \alpha - \beta \geq \alpha' - \beta', \\ &(\alpha, \beta) \preceq_\theta (\alpha', \beta') \quad \text{iff } \alpha + \beta \geq \alpha' + \beta'. \end{aligned} \tag{17.96a}$$

As an immediate consequence of (17.96a), we observe that

$$(\alpha, \beta) \preceq_\rho (\alpha', \beta') \quad \text{iff } (\alpha, \beta') \preceq_\theta (\alpha', \beta)$$

holds, whenever $(\alpha, \beta), (\alpha', \beta'), (\alpha, \beta'), (\alpha', \beta) \in S$. We call binary relations $\preceq_\rho$, $\preceq_\theta$ *dual* if they have the property above. Interestingly, the duality condition is related to the ambiguity problem in stereopsis, which refers to the identity of the stereograms representing the points $(\alpha, \beta), (\alpha', \beta')$ or $(\alpha, \beta'), (\alpha', \beta)$, respectively.

Our generalization is obtained in a straightforward manner by assuming that an observer's judgments are not based on a direct comparison of the respective bipolar

coordinates as in (17.96a) but by comparing some transformation or evaluation of these coordinates. In other words, we assume that there exist *strictly increasing functions* f, g, *mapping* V *into real intervals* (which are necessarily of *positive length*), such that for all $(\alpha, \beta), (\alpha', \beta') \in S$

$$(\alpha, \beta) \preceq_\rho (\alpha', \beta') \quad \text{iff } f(\alpha) - g(\beta) \geq f(\alpha') - g(\beta'), \tag{17.96b}$$

$$(\alpha, \beta) \preceq_\theta (\alpha', \beta') \quad \text{iff } f(\alpha) + g(\beta) \geq f(\alpha') + g(\beta'). \tag{17.96c}$$

Heller [363] provides a set of sufficient conditions that warrants a representation according to (17.96b) and (17.96c) with strictly increasing functions f, g. The empirical relational structure $\langle S, \preceq_\rho, \preceq_\theta \rangle$ satisfying these prerequisites is called a *binocular coordinate structure.* Available empirical evidence supports the proposed representation (see [363]). Since for the following functions f, g have to map I onto real intervals, we will use the term binocular coordinate structure only if this additional requirement is met by a relational structure $\langle S, \preceq_\rho, \preceq_\theta \rangle$. This continuity assumption does not pose problems in the present application.

The approach presented amounts to a psychologically significant recoordinatization of physical space. Generalizing (17.96), we may define

$$\Gamma = f(\alpha) - g(\beta) \quad \text{and} \quad \Phi = \frac{f(\alpha) + g(\beta)}{2}. \tag{17.96d}$$

The coordinates Γ, Φ of the horizontal plane at eye level are satisfying some of the fundamental properties that Luneburg originally intended when he introduced the modified bipolar coordinates γ, ϕ.

17.13.4 Functional Equations Resulting from Psychophysical Invariances

A psychophysical invariance is a physical transformation of the stimuli that leaves perception invariant. The psychophysical invariance can be translated into a functional equation that has to be satisfied by the psychophysical function and thereby restricts its possible form (cf. Falmagne [268]). In the present case, we may derive restrictions for the functions f and g appearing in the representations (17.96b) and (17.96c).

The Luneburg theory as expressed in (17.96a) suggests the following definition of psychophysical invariances.

Definition 17.51. A binary relation $\preceq$ on S is called

γ-shift invariant if

$$(\alpha, \beta) \preceq (\alpha', \beta') \quad \text{iff } (\alpha + \tau, \beta - \tau) \preceq (\alpha' + \tau, \beta' - \tau),$$

ϕ-shift invariant if

$$(\alpha, \beta) \preceq (\alpha', \beta') \quad \text{iff } (\alpha + \tau, \beta + \tau) \preceq (\alpha' + \tau, \beta' + \tau),$$

α-shift invariant if

$$(\alpha, \beta) \preceq (\alpha', \beta') \quad \text{iff } (\alpha + \tau, \beta) \preceq (\alpha' + \tau, \beta'),$$

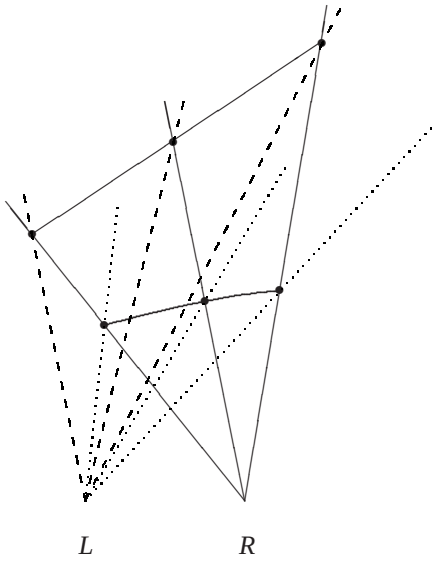
and
β-shift invariant if

$$(\alpha, \beta) \preceq (\alpha', \beta') \quad \text{iff } (\alpha, \beta + \tau) \preceq (\alpha', \beta' + \tau),$$

whenever the pairs involved are in S.

The significance of the transformations in Definition 17.51 (see the figure below) is due to the fact that they keep invariant the parallax differences between stimuli, which are commonly identified with the disparities inducing binocular depth perception.

In what follows, we show how functional equations arise from assuming that the transformations introduced above constitute automorphisms of the binocular coordinate structure $\langle S, \preceq_\rho, \preceq_\theta \rangle$.



Graphical illustration of a β-shift as introduced in Definition 17.51 applied to three collinear points. Keeping their α-coordinates (solid lines emerging from R) fixed, the β-coordinates are transformed by adding an (in this case negative) amount τ, which results in a rotation around L that moves the fan of dashed lines into the fan of dotted lines.

By the representation formulated in (17.96b), γ-shift invariance of $\preceq_\rho$ translates into

$$f(\alpha) - g(\beta) \geq f(\alpha') - g(\beta') \quad \text{iff } f(\alpha + \tau) - g(\beta - \tau) \geq f(\alpha' + \tau) - g(\beta' - \tau).$$

The real-valued function H given by

$$\text{(i)} \qquad H[f(\alpha) - g(\beta), \tau] = f(\alpha + \tau) - g(\beta - \tau)$$

is thus well defined and strictly increasing in its first variable; i.e., $H(z, \tau)$ strictly increases in z. (It is also strictly increasing in τ, but that is incidental.) The functional equation (i) provides a complete characterization of the γ-shift invariance of $\preceq_\rho$ in the context of the proposed conjoint representation. The table below summarizes the functional equations obtained similarly in each case considered. The functions H in these eight equations are in general different.

TABLE

Functional Equations Induced by the Invariance Properties Introduced in Definition 17.51

Invariance	Applied to	Equation	No.
γ-shift	$\preceq_\rho$	$H[f(\alpha) - g(\beta), \tau] = f(\alpha + \tau) - g(\beta - \tau)$	(i)
	$\preceq_\theta$	$H[f(\alpha) + g(\beta), \tau] = f(\alpha + \tau) + g(\beta - \tau)$	(i′)
ϕ-shift	$\preceq_\rho$	$H[f(\alpha) - g(\beta), \tau] = f(\alpha + \tau) - g(\beta + \tau)$	(ii)
	$\preceq_\theta$	$H[f(\alpha) + g(\beta), \tau] = f(\alpha + \tau) + g(\beta + \tau)$	(ii′)
α-shift	$\preceq_\rho$	$H[f(\alpha) - g(\beta), \tau] = f(\alpha + \tau) - g(\beta)$	(iii)
	$\preceq_\theta$	$H[f(\alpha) + g(\beta), \tau] = f(\alpha + \tau) + g(\beta)$	(iii′)
β-shift	$\preceq_\rho$	$H[f(\alpha) - g(\beta), \tau] = f(\alpha) - g(\beta + \tau)$	(iv)
	$\preceq_\theta$	$H[f(\alpha) + g(\beta), \tau] = f(\alpha) + g(\beta + \tau)$	(iv′)

Solutions of the Functional Equations (i)–(iv) and (i′)–(iv′)

Result 17.52. (Aczél, Boros, Heller, and Ng [39]). *The general strictly increasing solutions f, g of* (i) *or* (i′) *for all $(\alpha, \beta), (\alpha + \gamma, \beta - \phi) \in S$, which map V onto real intervals, are given by*

$$f(x) = ax + b, \quad g(\beta) = c\beta + d, \quad a, c > 0,$$

and by

$$f(\alpha) = ae^{C\alpha} + b, \quad g(\beta) = -ce^{-C\beta} + d, \quad a, c, C > 0 \text{ or } a, c, C < 0,$$

where $\alpha, \beta \in V$. Except for the sign restrictions, a, b, c, d, and C are arbitrary constants.

Result 17.53. [39]. *The general strictly increasing solutions* $f,\ g$ *of* (ii) *or* (ii′) *for all* $(\alpha, \beta) \in S,\ (\alpha + \tau, \beta + \tau) \in S$, *which map* V *onto real intervals, are given by*

$$f(\alpha) = a\alpha + b, \quad g(\beta) = c\beta + d, \quad a > 0,\ c > 0$$

and

$$f(\alpha) = ae^{C\alpha} + b, \quad g(\beta) = ce^{C\beta} + d, \quad a, c, C > 0 \text{ or } a, c, C < 0,$$

where $a, \beta \in V$. *The constants* $a,\ b,\ c,\ d$, *and* C *are arbitrary up to the positivity and negativity restrictions.*

Result 17.54. [39]. *The general strictly increasing solutions* f *and* g *of* (iii) *for all* $(\alpha, \beta) \in S,\ (\alpha + \tau, \beta) \in S$, *which map* V *onto a real interval, are given by*

$$f(\alpha) = a\alpha + b, \quad g \text{ arbitrary}, \ a > 0,$$

$\alpha \in V$. *Under the same conditions on* f *and* g, *the general solution of* (iv) *for all* $(\alpha, \beta) \in S,\ (\alpha, \beta + \tau) \in S$, *is given by*

$$f \text{ arbitrary}, \quad g(\beta) = c\beta + d, \quad c > 0,$$

where $\beta \in V$.

For more details of these and other applications, see [39].

Symbols

Z – the set of all integers
Z_+ – the set of all nonnegative integers
Z_+^* – the set of all positive integers
Z_- – the set of all negative integers
Q – the set of all rational numbers
$Q^* = Q - \{0\}$
$\mathbb{R}$ – the set of all real numbers
$\mathbb{R}_+$ – the set of all nonnegative reals (≥ 0)
$\mathbb{R}^* = \mathbb{R}\backslash\{0\},\ \mathbb{R}_+^* = \mathbb{R}_+ - \{0\}$ – the set of all positive real numbers
$\mathbb{C}$ – the set of all complex numbers
$\mathbb{C}^* = \mathbb{C} - \{0\}$
$nZ = \{nk :\ k \in Z\},\ n \in Z$
$I = [0, 1]$
$I_0 =]0, 1[,\ I_1 = I$ or I_0
$\langle C\rangle$ – the subgroup generated by a nonempty subset C of a group G
$f|S$ – a function f restricted to S
For subsets K, C of a field F, $K + C = \{k + c : k \in K$ and $c \in C\}$, $K - C = \{k - c :\ k \in K,\ c \in C\},\ \alpha K = \{\alpha k : \alpha \in F,\ k \in K\}$

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \end{cases} \quad \text{for } x \in \mathbb{R},$$

$\cdot, +, 0, *$ are symbols of binary operations
(f, g, h) is a homotopism means $f, g, h : G(o) \to H(*)$, where G and H are groupoids such that $f(x \circ y) = g(x) * h(y)$
For $C \subseteq X \times X,\ C_x = \{x \in X | (x, y) \in C$ for some $y \in X\}$
$C_y = \{y \in X | (x, y) \in C$ for some $x \in X\}$
In general, A is used for additive function, E for exponential function, L for logarithmic function, M for multiplicative function, B for biadditive function, D for derivation.
$\langle\ ,\ \rangle$ – symbol for inner product

$M_{mn}(F)$ – stands for all matrices with m rows and n columns over a field F, which could be $\mathbb{R}$ or $\mathbb{C}$
$M_n(F)$ – all square matrices of order n with entries from F
$\det(c)$ – determinant of a square matrix c
Z_n – ring of integers modulo n
$\ker(f)$ – kernel of a mapping f
$GL_n(F)$ – linear group of $n \times n$ matrices over a field F
$F[x]$ – set of all polynomials over a field or ring F
$\Gamma_n = \{P = (p_1, p_2, \ldots, p_n) : p_k \geq 0, \ \sum_{k=1}^{n} p_k = 1\}$
$\Gamma_n^0 = \{P = (p_1, p_2, \ldots, p_n) : p_k > 0, \ \sum_{k=1}^{n} p_k = 1\}$
$\Delta_n = \Gamma_n$ or Γ_n^0
$s(x) = -x \log x - (1 - x) \log(1 - x)$, Shannon function

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